Monomial multisummability through Borel-Laplace transforms. Applications to singularly perturbed differential equations and Pfaffian systems

DDays, Salou, 10 de Noviembre de 2016

Sergio Alejandro Carrillo Torres.
Advisors: Jorge Mozo Fernández
David Blázquez Sanz.

Departamento de Álgebra, Análisis Matemático, Geometría y Topología
Universidad de Valladolid
The notion of *monomial summability* was introduced in the paper:


in order to study the formal solutions of the *doubly singular equation*

$$
\varepsilon^q x^{p+1} \frac{dy}{dx} = F(x, \varepsilon, y).
$$

The method combines the variables $x$ and $\varepsilon$ in the new one $t = x^p \varepsilon^q$, corresponding to the source of divergence of the solutions.
We work in the \( \mathbb{C} \)-algebra \( \mathbb{C}[[x, \varepsilon]] \) of formal power series in two variables with complex coefficients.
Formal setting

We work in the $\mathbb{C}$–algebra $\mathbb{C}[[x, \varepsilon]]$ of formal power series in two variables with complex coefficients.

Given a monomial $x^p \varepsilon^q$ and a formal power series $\hat{f}$ we can write it uniquely as

$$\hat{f}(x, \varepsilon) = \sum_{n=0}^{\infty} f_n(x, \varepsilon) (x^p \varepsilon^q)^n.$$
Formal setting

We work in the $\mathbb{C}$–algebra $\mathbb{C}[[x, \varepsilon]]$ of formal power series in two variables with complex coefficients.

Given a monomial $x^p \varepsilon^q$ and a formal power series $\hat{f}$ we can write it uniquely as

$$\hat{f}(x, \varepsilon) = \sum_{n=0}^{\infty} f_n(x, \varepsilon)(x^p \varepsilon^q)^n.$$ 

The series $\hat{f} = \sum a_{n,m} x^n \varepsilon^m$ is $s$–Gevrey in the monomial $x^p \varepsilon^q$ if and only if there are positive constants $C, A$ satisfying

$$|a_{n,m}| \leq C A^{n+m} \min\{n!^{s/p}, m!^{s/q}\},$$

for all $n, m \in \mathbb{N}.$
A sector in the monomial $x^p \varepsilon^q$ is a set defined as

$$\Pi_{p,q}(a, b, r) = S_{p,q}(d, b - a, r)$$

$$= \left\{ (x, \varepsilon) \in \mathbb{C}^2 \mid 0 < |x|^p, |\varepsilon|^q < r, \ a < \text{arg}(x^p \varepsilon^q) < b \right\},$$

where $a, b \in \mathbb{R}$ with $a < b$ and $r > 0$. The number $r$ is called the radius, $b - a$ the opening and $d = (b + a)/2$ the bisecting direction of the sector, respectively.
Figure: $\Pi_{p,q}(\pi/2, 3\pi/2, r)$ for $p = 2$, $q = 3$. 
Asymptotic expansions in a monomial

Definition
Let \( f \in O(\Pi_{p,q}) \), \( \Pi_{p,q} = \Pi_{p,q}(a, b, r) \) and \( \hat{f} \in \mathcal{C} \) with
\[
\hat{T}_{p,q} \hat{f} = \sum f_n t^n \in \mathcal{E}_{r'}^{(p,q)}[[t]] \text{ for some } 0 < r' \leq r.
\]

We say that \( f \) has \( \hat{f} \) as asymptotic expansion in \( x^p \varepsilon^q \) on \( \Pi_{p,q} \) (\( f \sim^{(p,q)} \hat{f} \) on \( \Pi_{p,q} \)) if for every subsector \( \tilde{\Pi}_{p,q} \) and \( N \in \mathbb{N} \) there is a positive constant \( C_N \) such that for \( (x, \varepsilon) \in \tilde{\Pi}_{p,q} \) we have:

\[
\left| f(x, \varepsilon) - \sum_{n=0}^{N-1} f_n(x, \varepsilon)(x^p \varepsilon^q)^n \right| \leq C_N |x^p \varepsilon^q|^N. \tag{1}
\]
Asymptotic expansions in a monomial

Definition
Let \( f \in \mathcal{O}(\Pi_{p,q}) \), \( \Pi_{p,q} = \Pi_{p,q}(a,b,r) \) and \( \hat{f} \in \mathcal{C} \) with \( \hat{T}_{p,q} \hat{f} = \sum f_n t^n \in \mathcal{E}_{r'}^{(p,q)}[[t]] \) for some \( 0 < r' \leq r \).

We say that \( f \) has \( \hat{f} \) as asymptotic expansion in \( x^p \varepsilon^q \) on \( \Pi_{p,q} \) (\( f \sim^{(p,q)} \hat{f} \) on \( \Pi_{p,q} \)) if for every subsector \( \widetilde{\Pi}_{p,q} \) and \( N \in \mathbb{N} \) there is a positive constant \( C_N \) such that for \( (x,\varepsilon) \in \widetilde{\Pi}_{p,q} \) we have:

\[
\left| f(x,\varepsilon) - \sum_{n=0}^{N-1} f_n(x,\varepsilon)(x^p \varepsilon^q)^n \right| \leq C_N |x^p \varepsilon^q|^N. \tag{1}
\]

The asymptotic expansion is said to be of \( s-\text{Gevrey type} \) (\( f \sim^{(p,q)}_s \hat{f} \) on \( \Pi_{p,q} \)) if it is possible to choose \( C_N = CA^N N!^s \) for some \( C, A \) independent of \( N \). In this case \( \hat{f} \in \mathcal{C}[[x,\varepsilon]]_{s}^{(p,q)} \).
Monomial summability

Definition
Let $k > 0$ and $\hat{f} \in C$ be given. We say that $\hat{f}$ is $k$–summable in the monomial $x^p \varepsilon^q$ in the direction $d$ if there is a sector $\Pi_{p,q}(a, b, r)$ bisected by $d$ with opening $b - a > \pi/k$ and $f \in \mathcal{O}(\Pi_{p,q}(a, b, r))$ with $f \sim_{1/k} (p,q) \hat{f}$ on $\Pi_{p,q}(a, b, r)$.

We simply say that $\hat{f}$ is $k$–summable in the monomial $x^p \varepsilon^q$ if it is $k$–summable in the monomial $x^p \varepsilon^q$ in every direction $d$, with finitely many exceptions mod. $2\pi$.

- $\mathbb{C}\{x, \varepsilon\}_{1/k, d}^{(p,q)}$: $k$–summable series in $x^p \varepsilon^q$ in the direction $d$,
- $\mathbb{C}\{x, \varepsilon\}_{1/k}^{(p,q)}$: $k$–summable series in $x^p \varepsilon^q$. 
Consider the charts of the classical blow-up of the origin of $\mathbb{C}^2$, given by

$$
\pi_1(x, \varepsilon) = (x\varepsilon, \varepsilon), \quad \pi_2(x, \varepsilon) = (x, x\varepsilon).
$$
Consider the charts of the classical blow-up of the origin of $\mathbb{C}^2$, given by

$$
\pi_1(x, \varepsilon) = (x\varepsilon, \varepsilon), \quad \pi_2(x, \varepsilon) = (x, x\varepsilon).
$$

**Proposition**

Let $\hat{f} \in \mathbb{C}\{x, \varepsilon\}_{\frac{1}{k,d}}^{(p,q)}$ with sum $f$. Then $\hat{f} \circ \pi_1 \in \mathbb{C}\{x, \varepsilon\}_{\frac{1}{k,d}}^{(p,p+q)}$, $\hat{f} \circ \pi_2 \in \mathbb{C}\{x, \varepsilon\}_{\frac{1}{k,d}}^{(p+q,q)}$ and have sums $f \circ \pi_1$, $f \circ \pi_2$, respectively.
Proposition

If $\hat{f} \in \mathbb{C}\{x, \varepsilon\}^{(p,q)}_{1/k}$ has no singular directions then $\hat{f} \in \mathbb{C}\{x, \varepsilon\}$. 
Proposition
If \( \hat{f} \in \mathbb{C}\{x, \varepsilon\}^{(p,q)}_{1/k} \) has no singular directions then \( \hat{f} \in \mathbb{C}\{x, \varepsilon\} \).

Theorem
Let \( k, l > 0 \) be positive real numbers and let \( x^p \varepsilon^q \) and \( x^{p'} \varepsilon^{q'} \) be two monomials. Then \( \mathbb{C}\{x, \varepsilon\}^{(p,q)}_{1/k} \cap \mathbb{C}\{x, \varepsilon\}^{(p',q')}_{1/l} = \mathbb{C}\{x, \varepsilon\} \), except in the case \( p/p' = q/q' = l/k \) where \( \mathbb{C}\{x, \varepsilon\}^{(p,q)}_{1/k} = \mathbb{C}\{x, \varepsilon\}^{(p',q')}_{1/l} \).
Borel transform

Definition
Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$. The $k$–Borel transform associated to the monomial $x^p \epsilon^q$ with weight $(s_1, s_2)$ of a function $f$ is defined by the formula

$$B_{k,(s_1,s_2)}^{(p,q)}(f)(\xi,\nu) = \frac{(\xi^p \nu^q)^{-k}}{2\pi i} \int_{\gamma} f(\xi u^{-s_1/p^k}, \nu u^{-s_2/q^k}) e^u du,$$

where $\gamma$ denotes a Hankel path.
Borel transform

Definition
Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$. The $k$–Borel transform associated to the monomial $x^p \varepsilon^q$ with weight $(s_1, s_2)$ of a function $f$ is defined by the formula

$$
B_{k,(s_1,s_2)}^{(p,q)}(f)(\xi, \upsilon) = \frac{(\xi^p \upsilon^q)^{-k}}{2\pi i} \int_{\gamma} f(\xi u^{-s_1/p^k}, \upsilon u^{-s_2/q^k}) e^{u} du,
$$

where $\gamma$ denotes a Hankel path.

The formula is adapted from the papers:

Laplace transform

Definition
Let \( s_1, s_2 > 0 \) such that \( s_1 + s_2 = 1 \) and \( |\alpha| < \pi/2 \). The \( k \)-Laplace transform associated to the monomial \( x^p \varepsilon^q \) with weight \( (s_1, s_2) \) in direction \( \alpha \) of a function \( f \) is defined by the formula

\[
\mathcal{L}_{k,\alpha,(s_1,s_2)}^{(p,q)}(f)(x, \varepsilon) = (x^p \varepsilon^q)^k \int_0^{e^{i\alpha}\infty} f(xu^{s_1/pk}, \varepsilon u^{s_2/qk})e^{-u} du.
\]
Laplace transform

**Definition**

Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $|\alpha| < \pi/2$. The $k-$Laplace transform associated to the monomial $x^p \varepsilon^q$ with weight $(s_1, s_2)$ in direction $\alpha$ of a function $f$ is defined by the formula

$$L_{k, \alpha, (s_1, s_2)}^{(p, q)}(f)(x, \varepsilon) = (x^p \varepsilon^q)^k \int_0^{e^{i\alpha} \infty} f(xu^{s_1/pk}, \varepsilon u^{s_2/qk}) e^{-u} du.$$  

We assume that $f$ has an exponential growth of the form

$$|f(\xi, \upsilon)| \leq Ce^{B \max\{|\xi|^{p/s_1}, |\upsilon|^{q/s_2}\}}.$$  

(2)
Monomial Borel-Laplace summation methods

Definition
Let \( \hat{f} \) be a \( 1/k \)-Gevrey series in \( x^p \varepsilon^q \) and set \( \hat{\varphi}_{s_1,s_2} = \hat{B}_{k,(s_1,s_2)}^{(p,q)}((x^p \varepsilon^q)^k \hat{f}) \).

We will say that \( \hat{f} \) is \( k-(s_1,s_2) \)-Borel summable in the monomial \( x^p \varepsilon^q \) in direction \( d \) if:

1. \( \hat{\varphi}_{s_1,s_2} \) can be analytically continued, say as \( \varphi_{s_1,s_2} \), to a monomial sector of the form \( S_{p,q}(d,2\varepsilon,+\infty) \),
2. \( \varphi_{s_1,s_2} \) has exponential growth as in (2).
Monomial Borel-Laplace summation methods

Definition
Let \( \hat{f} \) be a \( 1/k \)-Gevrey series in \( x^p \varepsilon^q \) and set \( \hat{\varphi}_{s_1,s_2} = \hat{\mathcal{B}}_{k,(s_1,s_2)}((x^p \varepsilon^q)^k \hat{f}) \).

We will say that \( \hat{f} \) is \( k -(s_1,s_2) \)-Borel summable in the monomial \( x^p \varepsilon^q \) in direction \( d \) if:

1. \( \hat{\varphi}_{s_1,s_2} \) can be analytically continued, say as \( \varphi_{s_1,s_2} \), to a monomial sector of the form \( S_{p,q}(d, 2\varepsilon, +\infty) \),
2. \( \varphi_{s_1,s_2} \) has exponential growth as in (2).

In this case the \( k -(s_1,s_2) \)-Borel sum of \( \hat{f} \) in direction \( d \) is defined as

\[
\hat{f}(x, \varepsilon) = \frac{1}{(x^p \varepsilon^q)_k} \mathcal{L}_{k,(s_1,s_2)}^{(p,q)}(\varphi_{s_1,s_2})(x, \varepsilon).
\]
Theorem

Let \( \hat{f} \in \mathbb{C}[[x, \varepsilon]]_{1/k}^{(p,q)} \) be a \( 1/k \)-Gevrey series in the monomial \( x^p \varepsilon^q \). Then it is equivalent:

1. \( \hat{f} \in \mathbb{C}\{x, \varepsilon\}_{1/k,d}^{(p,q)} \),

2. There are \( s_1, s_2 > 0 \) with \( s_1 + s_2 = 1 \) such that \( \hat{f} \) is \( k - (s_1, s_2) \)-Borel summable in the monomial \( x^p \varepsilon^q \) in direction \( d \).

3. For all \( s_1, s_2 > 0 \) such that \( s_1 + s_2 = 1 \), \( \hat{f} \) is \( k - (s_1, s_2) \)-Borel summable in the monomial \( x^p \varepsilon^q \) in direction \( d \).

In all cases the corresponding sums coincide.
Applications
Doubly singular equations

Theorem

Consider the singularly perturbed differential equation

$$\varepsilon^q x^{p+1} \frac{dy}{dx} = F(x, \varepsilon, y),$$

where $y \in \mathbb{C}^l$, $p, q \in \mathbb{N}^*$, $F$ analytic in a neighborhood of $(0, 0, 0)$ and $F(0, 0, 0) = 0$.

If $\partial F/\partial y(0, 0, 0)$ is invertible then the previous equation has a unique formal solution $\hat{y}$. Furthermore it is $1-$summable in $x^p \varepsilon^q$. 
Consider the problem

\[ x^p \varepsilon^q \left( \frac{s_1}{p} x \frac{\partial y}{\partial x} + \frac{s_2}{q} \varepsilon \frac{\partial y}{\partial \varepsilon} \right) = C(x, \varepsilon)y(x, \varepsilon) + \gamma(x, \varepsilon), \]  

(3)

where \( p, q \in \mathbb{N}^* \), \( s_1, s_2 > 0 \) satisfy \( s_1 + s_2 = 1 \) and \( C \in \text{Mat}(l \times l, \mathbb{C}\{x, \varepsilon\}) \), \( \gamma \in \mathbb{C}\{x, \varepsilon\}^l \).

**Theorem**

If \( C(0, 0) \) is invertible then equation (3) has a unique formal solution \( \hat{y} \) and it is 1–summable in \( x^p \varepsilon^q \). Its possible singular directions are the directions passing through the eigenvalues of \( C(0, 0) \).
Consider the following the system of PDEs:

\[
\begin{align*}
\varepsilon^p x^{p+1} \frac{\partial y}{\partial x} &= f_1(x, \varepsilon, y), \quad (4a) \\
\varepsilon^{q'} x^{q'+1} \frac{\partial y}{\partial \varepsilon} &= f_2(x, \varepsilon, y), \quad (4b)
\end{align*}
\]

where \( p, q, p', q' \in \mathbb{N}^* \), \( y \in \mathbb{C}^l \), and \( f_1, f_2 \) are analytic in a neighborhood of \((0, 0, 0)\).
Consider the following the system of PDEs:

\[
\begin{align*}
\varepsilon^q x^{p+1} \frac{\partial y}{\partial x} &= f_1(x, \varepsilon, y), \\
\varepsilon^{q'} x^{p'} \frac{\partial y}{\partial \varepsilon} &= f_2(x, \varepsilon, y),
\end{align*}
\]

where \( p, q, p', q' \in \mathbb{N}^* \), \( y \in \mathbb{C}^l \), and \( f_1, f_2 \) are analytic in a neighborhood of \((0, 0, 0)\).

It is called *completely integrable* if \( f_1(x, \varepsilon, 0) = f_2(x, \varepsilon, 0) = 0 \) and the functions \( f_1, f_2 \) satisfy the following identity on their domains of definition:

\[
\begin{align*}
\frac{\partial}{\partial \varepsilon} \left( \frac{1}{x^{p+1} \varepsilon^q} \right) f_1 + \frac{1}{x^{p+1} \varepsilon^q} \left( \frac{\partial f_1}{\partial \varepsilon} + \frac{\partial f_1}{\partial y} \frac{f_2}{x^{p'} \varepsilon^{q'+1}} \right) &= \\
\frac{\partial}{\partial x} \left( \frac{1}{x^{p'} \varepsilon^{q'+1}} \right) f_2 + \frac{1}{x^{p'} \varepsilon^{q'+1}} \left( \frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial y} \frac{f_1}{x^{p+1} \varepsilon^q} \right).
\end{align*}
\]
If the system is completely integrable, \( f_1 = Ay + h.o.t. \) and \( f_2 = By + h.o.t. \), then \( A \) and \( B \) satisfy

\[
x^{p'} \varepsilon^{q'} \left( \varepsilon \frac{\partial A}{\partial \varepsilon} - qA \right) - x^p \varepsilon^q \left( x \frac{\partial B}{\partial x} - p'B \right) + AB - BA = 0.
\]
If the system is completely integrable, \( f_1 = Ay + h.o.t. \) and \( f_2 = By + h.o.t. \) then \( A \) and \( B \) satisfy

\[
x^{p'} \varepsilon^{q'} \left( \varepsilon \frac{\partial A}{\partial \varepsilon} - qA \right) - x^p \varepsilon^q \left( x \frac{\partial B}{\partial x} - p'B \right) + AB - BA = 0.
\]

From this equation we have deduced that:

1. If \( p' < p \) or \( q' < q \) then \( A(0,0) \) is nilpotent.
2. If \( p < p' \) or \( q < q' \) then \( B(0,0) \) is nilpotent.
3. If \( p = p' \) and \( q = q' \), for every eigenvalue \( \mu \) of \( B(0,0) \) there is an eigenvalue \( \lambda \) of \( A(0,0) \) such that \( q\lambda = p\mu \). The number \( \lambda \) is an eigenvalue of \( A(0,0) \), when restricted to its invariant subspace \( E_\mu = \{ v \in \mathbb{C}^n | (B(0,0) - \mu I)^k v = 0 \text{ for some } k \in \mathbb{N} \} \).
Theorem (Gérard-Sibuya)

Consider the completely integrable Pffafian system (4a), (4b), with $q = p' = 0$. If $\frac{\partial f_1}{\partial y}(0, 0, 0)$ and $\frac{\partial f_2}{\partial y}(0, 0, 0)$ are invertible then the Pfaffian system admits a unique analytic solution $y$ at the origin such that $y(0, 0) = 0$. 
Convergence of solutions for different monomials

Theorem (Gérard-Sibuya)

Consider the completely integrable Pfaffian system (4a), (4b), with \( q = p' = 0 \). If \( \frac{\partial f_1}{\partial y}(0, 0, 0) \) and \( \frac{\partial f_2}{\partial y}(0, 0, 0) \) are invertible then the Pfaffian system admits a unique analytic solution \( y \) at the origin such that \( y(0, 0) = 0 \).

Theorem

Consider the system (4a), (4b). Suppose the system has a formal solution \( \hat{y} \). If \( \frac{\partial f_1}{\partial y}(0, 0, 0) \) and \( \frac{\partial f_2}{\partial y}(0, 0, 0) \) are invertible and \( x^p \varepsilon^q \neq x^{p'} \varepsilon^{q'} \) then \( \hat{y} \) is convergent.
Convergence of solutions for the same monomial

For the same monomial, in the linear case we have

\[
\begin{align*}
\varepsilon^q x^{p+1} \frac{\partial y}{\partial x} &= A(x, \varepsilon)y(x, \varepsilon) + a(x, \varepsilon), \\
 x^p \varepsilon^{q+1} \frac{\partial y}{\partial \varepsilon} &= B(x, \varepsilon)y(x, \varepsilon) + b(x, \varepsilon),
\end{align*}
\]

(5a) \hspace{1cm} (5b)

**Corollary**

*If the system has a formal solution \( \hat{y} \) and there are \( s_1, s_2 > 0 \) such that \( s_1 + s_2 = 1 \) and \( s_1/pA(0, 0) + s_2/qB(0, 0) \) is invertible, then \( \hat{y} \) is 1-summable in \( x^p \varepsilon^q \). Its possible singular directions are the directions passing through the eigenvalues of \( s_1/pA(0, 0) + s_2/qB(0, 0) \).*
Towards Monomial Multisummability
Examples of series not $k$—summable in any monomial

**Theorem**

Let $\hat{f}_j \in \mathbb{C}\{x, \varepsilon\}^{(p_j, q_j)} \setminus \mathbb{C}\{x, \varepsilon\}$ be $k_j$—summable in $x^{p_j} \varepsilon^{q_j}$, for $j = 1, \ldots, r$, respectively.

Then $\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_r$ is $k_0$—summable in $x^{p_0} \varepsilon^{q_0}$ if and only if $k_0 p_0 = k_j p_j$ and $k_0 q_0 = k_j q_j$ for all $j = 1, \ldots, r$. 
Monomial acceleration operators

Following the same idea as in the one variable case, we formally compute the composition of a Borel and Laplace transform for different indexes. Indeed, we see that

$$B_{l,(s_1',s_2')}^{(p',q')} \circ \mathcal{L}_{k,d,(s_1,s_2)}^{(p,q)}(f)(\xi,\upsilon) = \left(\frac{\xi^p \upsilon^q}{\xi'^{p'} \upsilon'^{q'}}\right)^k \int_0^{e^{i\theta} \infty} f(\xi \tau^{s_1/p^k}, \upsilon \tau^{s_2/q^k}) C_{\Lambda l/k}(\tau) \, d\tau.$$ 

where the parameters satisfy the relations

$$\Lambda := \frac{s_1'}{s_1} \frac{p'}{p} = \frac{s_2'}{s_2} \frac{q'}{q}, 

s_1(p'q - pq') > \frac{p}{l} (qk - q'l).$$
Let $I = (p', q', l, s'_1, s'_2, p, q, k, s_1, s_2)$, with parameters as before. The acceleration operator associated to $I$ in direction $\theta$ is given by

$$
\mathcal{A}_{I, \theta}(f)(\xi, \upsilon) = \frac{(\xi p \upsilon q)^k}{(\xi p' \upsilon q')^l} \int_0^{e^{i\theta} \infty} f(\xi \tau^{s_1/pk}, \upsilon \tau^{s_2/qk}) C_{\Lambda l/k}(\tau) d\tau,
$$

and it is defined for functions $f$ with exponential growth

$$
|f(\xi, \upsilon)| \leq Ce^M \max\{|\xi|^\kappa_1, |\upsilon|^\kappa_2\}, \quad (6)
$$

$$
\frac{1}{\kappa_1} := \frac{s_1}{pk} - \frac{s'_1}{p'l}, \quad \frac{1}{\kappa_2} := \frac{s_2}{qk} - \frac{s'_2}{q'l}.
$$
Monominal multisummability for two levels

Definition
We say that \( \hat{f} \) is \( I \)-multisummable in the multidirection \( (d_1, d_2) \) if the following conditions are satisfied

1. \( \hat{f} \) is \( 1/k \)-Gevrey in the monomial \( x^p \varepsilon^q \),

2. \( \hat{B}^{(p,q)}_{k,(s_1,s_2)}(\hat{f}) \) can be analytically continued, say \( \varphi \), to some \( S_{p,q}(d_1, \theta_1, +\infty) \), with exponential growth as in (6).

3. \( \mathcal{A}_I(\varphi) \) can be analytically continued, say \( \psi \), to \( S_{p,q}(d_1, \theta'_1, +\infty) \cap S_{p',q'}(d_2, \theta'_2, +\infty) \), with exponential growth as in (2).
Monomial multisummability for two levels

Definition
We say that \( \hat{f} \) is \( I \)-multisummable in the multidirection \((d_1, d_2)\) if the following conditions are satisfied

1. \( \hat{f} \) is \( 1/k \)-Gevrey in the monomial \( x^p \varepsilon^q \),

2. \( \mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(\hat{f}) \) can be analytically continued, say \( \varphi \), to some \( S_{p,q}(d_1, \theta_1, +\infty) \), with exponential growth as in (6).

3. \( \mathfrak{A}_I(\varphi) \) can be analytically continued, say \( \psi \), to \( S_{p,q}(d_1, \theta_1', +\infty) \cap S_{p',q'}(d_2, \theta_2', +\infty) \), with exponential growth as in (2).

The \( I \)-multisum of \( \hat{f} \) in the multidirection \((d_1, d_2)\) is defined as

\[
f(x, \varepsilon) = \mathcal{L}_{l,(s_1',s_2')}(\psi)(x, \varepsilon),
\]

and it is analytic on some \( S_{p,q}(d_1, \theta_1'' + \pi/l\Lambda, r) \cap S_{p',q'}(d_2, \theta_2'' + \pi/l, r) \), where \( \theta_1'' < \theta_1', \theta_2'' < \theta_2' \) and \( r \) is small enough.
The results of this thesis are contained in the following papers:


Thanks for your attention.