Perturbation theory, KAM theory and Celestial Mechanics

6. Two models of Celestial Mechanics

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1. Rotational dynamics

2. Conservative/Dissipative spin–orbit problem
   2.1 Conservative spin-orbit problem
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3. Delaunay action–angle variables

4. Mean, eccentric anomaly and Kepler’s equation

5. The restricted three–body problem
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Rotational dynamics

- Rotational dynamics: different shapes
  From round bodies (Moon, Mercury), to irregular bodies (Hyperion), to dumbbell satellite (4179 Toutatis, 216 Kleopatra)
- The **Moon** always points the same face to the Earth. All evolved satellites of the Solar System always point the same hemisphere to the host planet.
- **Mars:** Phobos, Deimos. **Jupiter:** Io, Europa, Ganymede, Callisto. **Saturn:** Titan, Rhea, Enceladus, Dione. **Uranus:** Ariel, Umbriel, Titania. **Neptune:** Triton, Proteus. **Pluto:** Charon.
Rotational dynamics: consequences of its study

- Moon: physical librations due to earth tides, study of the internal composition (SMART 1)
- Mercury: study of the gravitational field, the variation of obliquity and libration provide constraints on the internal structure of the planet, such as the existence of a solid surface and a liquid core, thus provoking a dynamo effect responsible of Mercury’s magnetic field (BepiColombo)
- Europa: mass distribution, rotation eventually compatible with a liquid ocean which could explain the tectonics (Voyager - Galileo)
- Enceladus: resonance conditions can be responsible of the heat excess and surface geysers
- Hyperion: example of chaotic rotation (in orbital resonance with Titan)
- Titan: an anomalous obliquity might be due to an internal ocean (Cassini–Huygens)
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Model: satellite $S$, ellipsoid rotating about an internal spin–axis and revolving around a central body $\mathcal{P}$:

(i) $S$ moves on a Keplerian orbit;
(ii) the spin–axis coincides with the smallest physical axis (principal rotation);
(iii) the spin–axis is perpendicular to the orbital plane (zero obliquity);
(iv) dissipative forces: tidal torque $\mathcal{T}$ depending linearly on the angular velocity of rotation.
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• Notation:
  $A < B < C$ principal moments of inertia; $n = \frac{2\pi}{T_{\text{rev}}} \equiv 1$ mean motion; $a$ semimajor axis; $e$ eccentricity; $r$ orbital radius; $f$ true anomaly; $x$ angle between pericenter line and major axis of the ellipsoid.
Conservative spin-orbit problem

- Neglecting the dissipation:

\[ \ddot{x} + \frac{3}{2} \frac{B - A}{C} \left( \frac{a}{r} \right)^3 \sin(2x - 2f) = 0. \]  

(1)
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(i) \( \varepsilon \equiv \frac{3}{2} \frac{B - A}{C} \) perturbing parameter; Moon–Mercury: \( \varepsilon \approx 10^{-4} \); if \( \varepsilon = 0 \) the system is integrable.

(ii) \( r \) and \( f \) are known Keplerian functions of the time:

\[
\begin{align*}
    r &= a(1 - e \cos u) \\
    f &= 2 \arctan \left( \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \right). 
\end{align*}
\]

(iii) \( r, f \) depend on \( e \) and for \( e = 0 \) one has \( r = a, f = t + t_0 \) for a suitable constant \( t_0 \); hence, for circular orbits one gets the integrable equation \( \ddot{x} + \varepsilon \sin(2x - 2t - 2t_0) = 0 \).

(iv) Considering the lift of the angle \( x \) on \( \mathbb{R} \), a \( p : q \) spin–orbit resonance for \( p, q \in \mathbb{Z} \) with \( q > 0 \) is a periodic solution for the conservative equation, say \( t \in \mathbb{R} \to x = x(t) \in \mathbb{R} \), such that

\[
x(t + 2\pi q) = x(t) + 2\pi p \quad \text{for any } t \in \mathbb{R}.
\]
• Expanding in power series of $e$ and Fourier series, the spin-orbit eq. is

$$
\ddot{x} + \varepsilon \sum_{m \neq 0, m = -\infty}^{+\infty} W(\frac{m}{2}, e) \sin(2x - mt) = 0, \quad (2)
$$

where the coefficients $W(\frac{m}{2}, e)$ decay as power series of $e$.

• Up to the order 4 in $e$, one obtains

$$
\ddot{x} + \varepsilon \left[ \frac{e^4}{24} \sin(2x + 2t) + \frac{e^3}{48} \sin(2x + t) + \left( -\frac{e}{2} + \frac{e^3}{16} \right) \sin(2x - t) + \\
+ \left( 1 - \frac{5}{2}e^2 + \frac{13}{16}e^4 \right) \sin(2x - 2t) + \left( \frac{7}{2}e - \frac{123}{16}e^3 \right) \sin(2x - 3t) + \\
+ \left( \frac{17}{2}e^2 - \frac{115}{6}e^4 \right) \sin(2x - 4t) + \frac{845}{48}e^3 \sin(2x - 5t) + \\
+ \frac{533}{16}e^4 \sin(2x - 6t) \right] = 0.
$$
The previous equation can be written in compact form as

\[ \ddot{x} + \varepsilon V_x(x, t) = 0, \]

for a suitable periodic function \( V = V(x, t) \). Such equation corresponds to that of a pendulum subject to a forcing term, depending periodically upon time.

In Hamiltonian form it is:

\[ H(y, x, t) = \frac{1}{2}y^2 + \varepsilon V(x, t). \]

The Hamiltonian is integrable for \( \varepsilon = 0 \), nearly-integrable for \( \varepsilon \neq 0 \).
Dissipative spin-orbit problem

- Tidal torque $\mathcal{T}$ due to internal non–rigidity: as in [Correia–Laskar] average over one orbital period:

$$\langle \mathcal{T} \rangle = -\mu(e, K) \left[ \dot{x} - \eta(e) \right] ,$$

with

$$\mu(e, K) = K \frac{1 + 3e^2 + \frac{3}{8}e^4}{(1 - e^2)^{9/2}} , \quad \eta(e) = \frac{1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6}{(1 + 3e^2 + \frac{3}{8}e^4)(1 - e^2)^{3/2}} .$$

- The quantity $K \equiv 3n \frac{k_2}{\xi Q} \left( \frac{R_e}{a} \right)^3 \frac{M}{m}$, where $n = \text{mean motion}$, $k_2 = \text{Love number}$ (depending on the structure of the body), $Q = \text{quality factor}$ (which compares the frequency of oscillation of the system to the rate of dissipation of energy), $\xi$ is a structure constant such that $I_3 = \xi m R_e^2$, $R_e = \text{equatorial radius}$, $M = \text{mass of the central body}$, $m = \text{mass of the satellite}$.
- $K \simeq 10^{-8}$ for Moon–Mercury depending on the physical and orbital characteristics
• We are led to consider the following equation of motion for the dissipative spin–orbit problem:

\[ \ddot{x} + \frac{3}{2} \frac{B - A}{C} \left( \frac{a}{r} \right)^3 \sin(2x - 2f) = -\mu [\dot{x} - \eta] \]

or

\[ \ddot{x} + \varepsilon V_x(x, t) = -\mu [\dot{x} - \eta]. \] (3)

• The tidal torque vanishes provided

\[ \dot{x} \equiv \eta(e) = \frac{1 + \frac{15}{2} e^2 + \frac{45}{8} e^4 + \frac{5}{16} e^6}{(1 - e^2)^{3/2} (1 + 3e^2 + \frac{3}{8} e^4)}. \]

• It is readily shown that for circular orbits the angular velocity of rotation corresponds to the synchronous resonance, being \( \dot{x} = 1 \). For Mercury’s eccentricity \( e = 0.2056 \), it turns out that \( \dot{x} = 1.256 \).
Poincaré sections in the plane $(x, y)$, conservative and dissipative settings, different values of the eccentricity.

Figure: (a) $e = 0.0549, \varepsilon = 10^{-3}, K = 0$; (b) $e = 0.0549, \varepsilon = 10^{-3}, K = 10^{-3}$; (c) $e = 0.2056, \varepsilon = 10^{-3}, K = 0$; (d) $e = 0.2056, \varepsilon = 10^{-3}, K = 10^{-3}$. 
• SM corresponds to the Poincaré map at times $2\pi$, obtained integrating the conservative spin–orbit problem with a leap–frog method.
• DSM corresponds to the Poincaré map at times $2\pi$, obtained integrating the dissipative spin–orbit problem with a leap–frog method.

\[ \ddot{x} + \varepsilon V_x(x, t) = -\mu [\dot{x} - \eta] . \]

is equivalent to

\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= -\varepsilon V_x(x, t) - \mu [y - \eta] ,
\end{align*} \]

which can be integrated through a leap–frog method with time-step $T$ as

\[ \begin{align*}
y' &= (1 - \mu T)y + \mu \eta T - \varepsilon V_x(x, t) T \\
x' &= x + y' .
\end{align*} \]
Outline

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5. The restricted three–body problem
• Action–angle variables for the two–body problem $\mathcal{P}_1 - \mathcal{P}_2$ are known as Delaunay variables.
• Let $(r, \vartheta)$ be the polar coordinates and let $(p_r, p_\vartheta)$ be the conjugated momenta. It is readily seen that $p_\vartheta = h$, $h$ being the angular momentum.

Figure: Geometrical configuration of Kepler’s problem.
The Hamiltonian function governing the two–body motion is given by
\( \kappa = G(m_1 + m_2) \)
\[
\mathcal{H}(p_r, p_\vartheta, r, \vartheta) = \frac{1}{2} \left( p_r^2 + \frac{p_\vartheta^2}{r^2} \right) - \frac{\kappa}{r} .
\]

Being \( \vartheta \) a cyclic variable, we introduce the effective potential (see Figure 3) as
\[
V_e(r) = \frac{p_\vartheta^2}{2r^2} - \frac{\kappa}{r} .
\] (4)

Figure: Graph of the effective potential \( V_e(r) \) given in (4) for \( p_\vartheta = 0.4025 \) and \( \kappa = 1 \).
• The Hamiltonian can be written as the one–dim. Hamiltonian:

\[ \mathcal{H}(p_r, r) = \frac{p_r^2}{2} + V_e(r) . \]

• Taking into account that \( \vartheta \) is cyclic, let us define the Delaunay action variables \( L_0, G_0 \) as

\[
L_0 \equiv \sqrt{\kappa a} \\
G_0 \equiv p_\vartheta = h = \sqrt{\kappa a(1 - e^2)} = L_0 \sqrt{1 - e^2} .
\]

• Notice that one can express the elliptic elements \( a, e \) in terms of the Delaunay action variables as

\[
a = \frac{L_0^2}{\kappa} , \quad e = \sqrt{1 - \frac{G_0^2}{L_0^2}} .
\]
• The Hamiltonian function expressed in terms of the action variables becomes

$$\mathcal{H} = \mathcal{H}(L_0) = -\frac{\kappa^2}{2L_0^2}. \quad (5)$$

• The Delaunay angle variables are the mean anomaly

$$\ell_0 \equiv n(t - t_0) = \frac{2\pi}{T}(t - t_0)$$

and the argument of perihelion $g_0$.

![Figure: The argument of perihelion $g_0$.](image)
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Mean, eccentric anomaly and Kepler’s equation

- We introduce as follows a quantity \( u \) called the *eccentric anomaly*:

**Figure:** The eccentric anomaly \( u \).

- It follows that

\[
\begin{align*}
    r &= a(1 - e \cos u) \\
    \tan \frac{f}{2} &= \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2} \\
    \ell_0 &= u - e \sin u,
\end{align*}
\]

the latter known as *Kepler’s equation.*
• Solve this equation to get $u$ as a function of the time, being $\ell_0 = n(t - t_0)$ as well as $u = u(t)$; insert it in the previous relations to obtain $r = r(t), f = f(t)$.

• An approximate solution can be computed as far as $e$ is small. Indeed, the inversion of Kepler’s equation provides $u$ as a function of $\ell_0$ as a series of $e$:

\[
\begin{align*}
   u &= \ell_0 + e \sin u \\
   &= \ell_0 + e \sin(\ell_0 + e \sin u) \\
   &= \ell_0 + e \sin(\ell_0 + e \sin(\ell_0 + e \sin u)) \\
   &= \ell_0 + (e - \frac{e^3}{8}) \sin \ell_0 + \frac{1}{2} e^2 \sin(2\ell_0) + \frac{3}{8} e^3 \sin(3\ell_0) + \mathcal{O}(e^4),
\end{align*}
\]

where $\mathcal{O}(e^4)$ denotes a quantity of order $e^4$. 
The complete solution can be expressed as

\[ u = \ell_0 + e \sum_{k=1}^{\infty} \frac{1}{k} \left[ J_{k-1}(ke) + J_{k+1}(ke) \right] \sin(k\ell_0) , \]  

where \( J_k(x) \) are the Bessel’s functions of order \( k \) and argument \( x \); they are defined by the relation

\[ J_k(x) \equiv \frac{1}{2\pi} \int_0^{2\pi} \cos(kt - x \sin t) dt . \]

The functions \( J_k(x) \) can be developed as follows:

\[ J_0(x) \equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left( \frac{x}{2} \right)^{2m} \]

\[ J_k(x) \equiv \left( \frac{x}{2} \right)^k \frac{1}{k!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \prod_{j=1}^{m} (k+j)} \left( \frac{x}{2} \right)^{2m} . \]
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The restricted three–body problem

- Consider a particle (i.e. an asteroid) under the influence of 2 primaries $P_1, P_2$ with masses $m_1, m_2$ (i.e. Sun and Jupiter). Assume that
  - all bodies move on the same plane;
  - the mass of the particle is so small that it does not influence the primaries;
  - the primaries move on circular Keplerian orbits.

- This problem is named the restricted, circular, planar 3–body problem (RCPTBP) described by a 2 d.o.f. Hamiltonian:
  \[ H(L, G, \ell, g; \epsilon) = -\frac{1}{2} L^2 - G + \epsilon F(\ell, G, \ell, g; \epsilon). \]

- Angle variables:
  - $\ell$ is the mean-anomaly,
  - $g = g_0 - \psi$ with $g_0$ = argument of the perihelion,
  - $\psi$ = longitude of $P_2$, coinciding with time if the common frequency of the primaries is 1 and if $m_1 + m_2 = 1$.

- Action variables:
  - $L = \sqrt{\kappa a}$ and $G = L \sqrt{1 - e^2}$.

- Perturbative parameter $\epsilon = m_2 / (m_1 + m_2)$.
The restricted three–body problem

- Consider a particle (i.e. an asteroid) under the influence of 2 primaries $\mathcal{P}_1$, $\mathcal{P}_2$ with masses $m_1$, $m_2$ (i.e. Sun and Jupiter). Assume that
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- This problem is named the restricted, circular, planar 3–body problem (RCPTBP) → described by a 2 d.o.f. Hamiltonian:

$$H(L, G, \ell, g; \varepsilon) = -\frac{1}{2L^2} - G + \varepsilon F_\varepsilon(L, G, \ell, g; \varepsilon).$$

- Angle variables: $\ell$ is the mean-anomaly, $g = g_0 - \psi$ with $g_0 =$ argument of the perihelion, $\psi =$ longitude of $\mathcal{P}_2$, coinciding with time if the common frequency of the primaries is 1 and if $m_1 + m_2 = 1$.
- Action variables: $L = \sqrt{\kappa a}$ and $G = L\sqrt{1 - e^2}$.
- Perturbative parameter $\varepsilon = m_2/(m_1 + m_2)$. 

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• About the perturbation $F_\varepsilon(L, G, \ell, g; \varepsilon)$.

• Setting $x^{(2)}$ the Jupiter–Sun vector, $x^{(A)}$ the asteroid–Sun vector, the perturbation is

$$F_\varepsilon = x^{(A)} \cdot x^{(2)} - \frac{1}{|x^{(A)} - x^{(2)}|},$$

expressed in terms of the Delaunay variables, with $x^{(2)}$ being the relative circular motion of $P_1$: $x^{(2)} = (\cos(t_0 + t), \sin(t_0 + t))$.

• Expanding in Fourier-Taylor series:

$$F_\varepsilon(L, G, \ell, g) = -(1 + \frac{a^2}{4} + \frac{9}{64} a^4 + \frac{3}{8} a^2 e^2)$$

$$+ \left(\frac{1}{2} + \frac{9}{16} a^2\right) a^2 e \cos \ell - \left(\frac{3}{8} a^3 + \frac{15}{64} a^5\right) \cos(\ell + g)$$

$$+ \left(\frac{9}{4} + \frac{5}{4} a^2\right) a^2 e \cos(\ell + 2g) - \left(\frac{3}{4} a^2 + \frac{5}{16} a^4\right) \cos(2\ell + 2g)$$

$$- \frac{3}{4} a^2 e \cos(3\ell + 2g) - \left(\frac{5}{8} a^3 + \frac{35}{128} a^5\right) \cos(3\ell + 3g)$$

$$- \frac{35}{64} a^4 \cos(4\ell + 4g) - \frac{63}{128} a^5 \cos(5\ell + 5g).$$