Birth, transition and maturation of canard cycles in PWL systems

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  - canard explosion, canard regime
  - excitability threshold, transitory canard
- PWL slow-fast dynamic.
  - stated of the problem
  - simplifying the model
  - transition map
  - main results

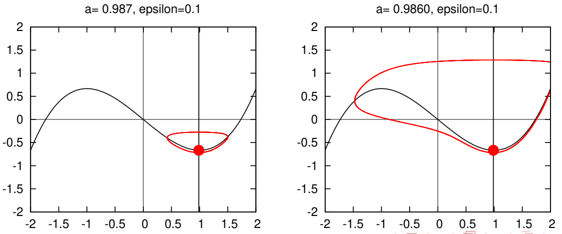
Slow-fast dynamics

- Dynamic behaviour of planar slow-fast systems

$$\begin{cases} \dot{x} = f(x, y, a, \varepsilon), \\ \dot{y} = \varepsilon g(x, y, a, \varepsilon), \end{cases} \qquad f, g \in \mathcal{C}^k, \;\; k \geq 3, \;\; a \in \mathbb{R}, \;\; 0 < \varepsilon \ll 1$$

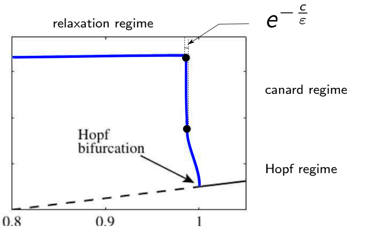
- Canard explosion: Very fast growth in the amplitude of a one parametric family of limit cycles upon a small variation of the parameter.
- It explains the very fast transition from a small amplitude limit cycle to a relaxation oscillation in the VdP system.

$$\begin{aligned} \dot{x} &= x - \tfrac{1}{3}x^3 - y \\ \dot{y} &= \varepsilon(a - x) \end{aligned}$$



Slow-fast dynamics

- Limit cycles organize along a curve which starts at Hopf bif. and ends at relaxation oscillation, into: Hopf regime, canard regime and relaxation regime.
- Canard explosion: parameter variation $O(e^{-\frac{c}{\varepsilon}})$.



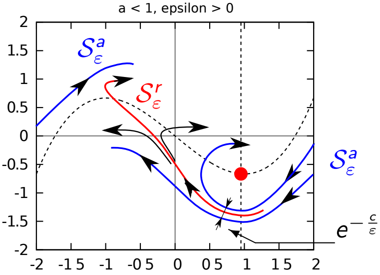
Amplitude "y" vs parameter "a"

- It is explained through the interplay of the Fenichel manifolds $\mathcal{S}_\varepsilon^a$, $\mathcal{S}_\varepsilon^r$, perturbing from the critical manifold

$$\mathcal{S}_0 = \{f(x,y,a,0) = 0\}$$

$$\begin{cases} \dot{x} = f(x,y,a,\varepsilon) \\ \dot{y} = \varepsilon g(x,y,a,\varepsilon) \end{cases} \qquad \begin{cases} \varepsilon x' = f(x,y,a,\varepsilon) \\ y' = g(x,y,a,\varepsilon) \end{cases}$$

$$t = \frac{\tau}{\varepsilon} \qquad\qquad \tau = \varepsilon t$$



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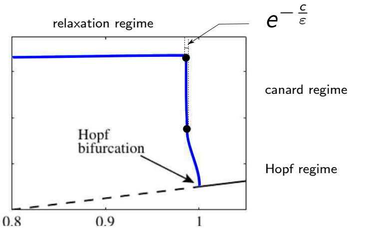

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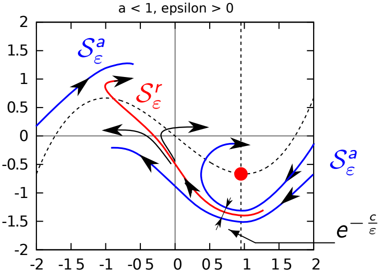

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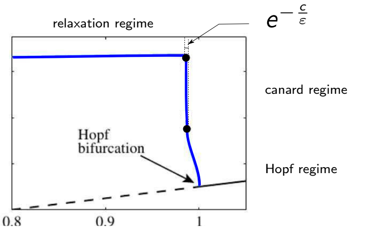

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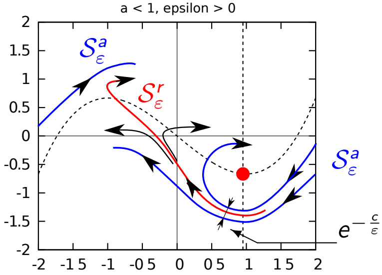
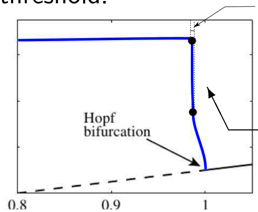


Slow-fast dynamics

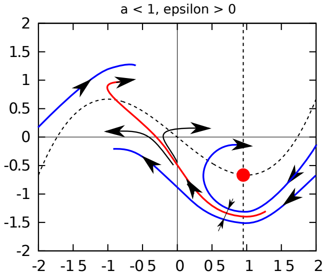
- In applications, the canard explosion models the excitability threshold.



Parameter value at which the system passes from rest to a excitable response. Involves, the starting and ending of the canard regime.

Canard cycle acting as a threshold.

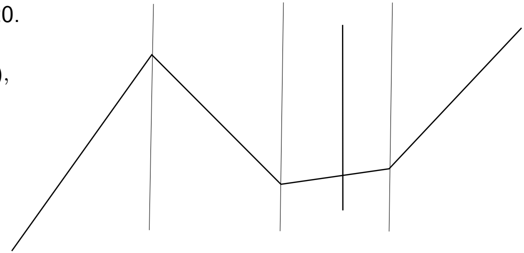
- Transitory canard: boundary between headless canard cycles and canard cycles with head.
- Maximal canard: canard cycle at the connection.
- Maximal period canard: canard cycle with maximal period.



PWL slow-fast dynamics

- PWL systems exhibit dynamical richness comparable with that exhibited by nonlinear systems.
- PWL slow-fast systems offer some advantatges: canonical slow manifold formed by segments.
- V. Carmona, S. Fernández-García and A. E. Teruel, *Saddle-node canard cycles in planar PWL differential systems* arXiv:2003.14112v2, 2020.

$$\begin{cases} x' = y - f(x,a,k,m,\varepsilon), \\ y' = \varepsilon(a - x), \end{cases}$$



$$f(x,a,k,m,\varepsilon) = \begin{cases} x + 1 - k(\sqrt{\varepsilon} - 1) - m(\sqrt{\varepsilon} + a), & \text{if } x < -1 \\ -k(x + \sqrt{\varepsilon}) - m(\sqrt{\varepsilon} + a), & \text{if } -1 < x \leq -\sqrt{\varepsilon}, \\ m(x - a), & \text{if } |x| \leq \sqrt{\varepsilon}, \\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a), & \text{if } x > \sqrt{\varepsilon}, \end{cases}$$

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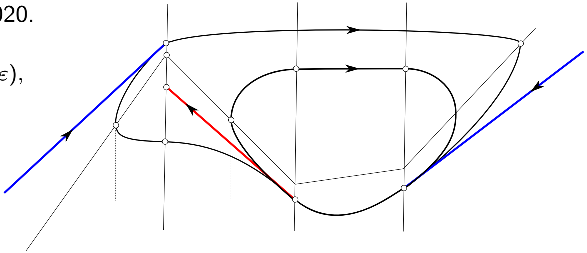

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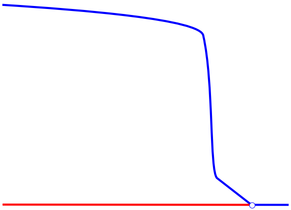

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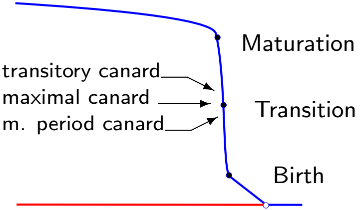

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PWL slow-fast dynamics

Flat slow manifold in an aircraft ground dynamics model.

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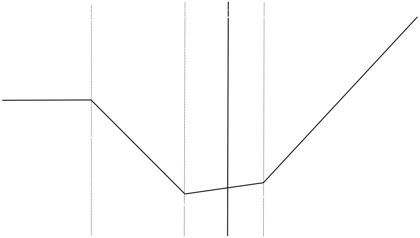

$$f(x,a,\varepsilon) = \begin{cases} 1+\sqrt{\varepsilon}(a-1)+\varepsilon, & \text{if } x<-1, \\ -x+\sqrt{\varepsilon}(a-1)+\varepsilon, & \text{if } -1<x\le-\sqrt{\varepsilon}, \\ \sqrt{\varepsilon}(a-x), & \text{if } |x|\le\sqrt{\varepsilon}, \\ x+\sqrt{\varepsilon}(a-1)-\varepsilon, & \text{if } x>\sqrt{\varepsilon}. \end{cases}$$

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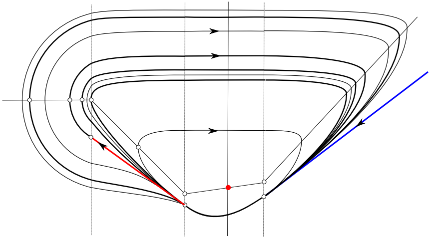

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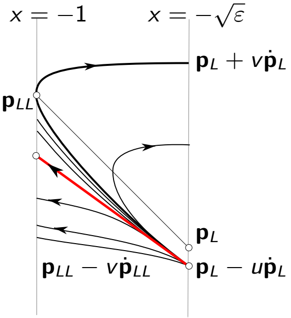
Canard regime: repelling slow manifold



- In this region the system is linear $\dot{\mathbf{x}} = A_L\mathbf{x} + \mathbf{b}_L$.
  $Tr(A_L) = 1, \quad Det(A_L) = \varepsilon,$
  $\lambda^s = \varepsilon + O(\varepsilon^2), \quad \lambda^q = 1 - \lambda^s.$
- In the base $x\mathbf{p}_L + y\dot{\mathbf{p}}_L$ function is a first integral:

$$H(x,y) = \frac{|x + y\lambda^s|^{\lambda^q}}{|x + y\lambda^q|^{\lambda^s}}$$

- $\mathbf{p}_L = \sqrt{\varepsilon}\begin{pmatrix} -1 \\ \sqrt{\varepsilon}+a \end{pmatrix} \quad \dot{\mathbf{p}}_L = \varepsilon\begin{pmatrix} 0 \\ \sqrt{\varepsilon}+a \end{pmatrix} \quad r = \left(\frac{1+a}{\sqrt{\varepsilon}+a}\right)$
  $\mathbf{p}_{LL} = r\mathbf{p}_L \qquad \dot{\mathbf{p}}_{LL} = r\dot{\mathbf{p}}_L.$
- Transition map around $\mathcal{S}^r_\varepsilon$, implicit expression

$$\begin{cases} H(1,-u) = H(1,v), & 0 \le u \le u_t \\ H(1,-u) = r^{\lambda^q - \lambda^s} H(1,-v), & u_t \le u. \end{cases}$$

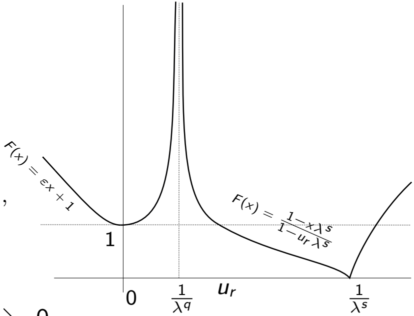
- Since $H(1,-u_t) = r^{\lambda^q-\lambda^s} H(1,0) = r^{\lambda^q-\lambda^s}$.

$$u_t = \frac{1}{\lambda^q} - \frac{1}{\lambda^q} e^{\frac{1}{2\lambda^s}\ln(\varepsilon)}, \qquad u_s = \frac{1}{\lambda^q}.$$

Canard regime: Transition map

- Set $F(x) = H(1, -x)$, therefore

$$F(x) = \begin{cases} \dfrac{(1 - x\lambda_L^s)^{\lambda_L^q}}{(1 - x\lambda_L^q)^{\lambda_L^s}}, & \text{if } x < 1/\lambda_L^q, \\ \dfrac{(1 - x\lambda_L^s)^{\lambda_L^q}}{(x\lambda_L^q - 1)^{\lambda_L^s}}, & \text{if } 1/\lambda_L^q < x < 1/\lambda_L^s, \\ \dfrac{(x\lambda_L^s - 1)^{\lambda_L^q}}{(x\lambda_L^q - 1)^{\lambda_L^s}}, & \text{if } x > 1/\lambda_L^s, \end{cases}$$



- $F(x)$ converges puntually to 1 as $\varepsilon \searrow 0$.
- $F(x)$ quadratic tangency at $x = 0$, monotonic decreasing in $(\frac{1}{\lambda^q}, \frac{1}{\lambda^s})$.
- Transition map is implicitely given by

$$\begin{array}{ll} F(u) = F(-v) & u \in (0, u_t) \\ F(u) = r^{\lambda^q - \lambda^s} F(v) & u \in (u_t, +\infty) \end{array}$$

- For ε small enough and u close $\frac{1}{\lambda^q}$

$$0 < u < \frac{1}{\lambda^q}, \qquad F(u) \approx \frac{1}{(1-u\lambda_L^q)^{\lambda_L^s}} \rightarrow u = \frac{1}{\lambda^q} - \frac{1}{\lambda^q} e^{-\frac{1}{\lambda^s}\ln(F(u))}$$

$$u > \frac{1}{\lambda^q}, \qquad F(u) \approx \frac{1}{(u\lambda_L^q-1)^{\lambda_L^s}} \rightarrow u = \frac{1}{\lambda^q} + \frac{1}{\lambda^q} e^{-\frac{1}{\lambda^s}\ln(F(u))}$$

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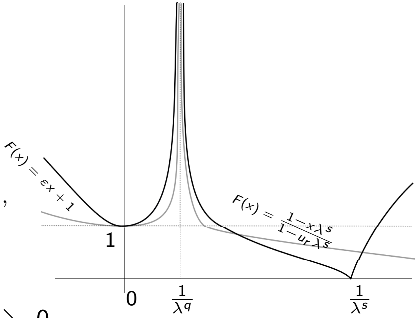

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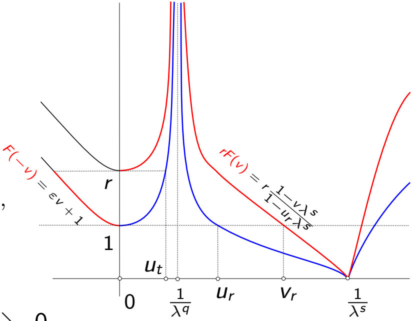

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Canard regime: Canard orbits

- For $c > 0$ consider $u(c, \varepsilon) \in (0, u_t]$ such that

$$F(u(c, \varepsilon)) = 1 + c.$$

- Since $\varepsilon v(u(c, \varepsilon)) + 1 = F(u(c, \varepsilon)) = 1 + c$ then

$$v(u(c, \varepsilon)) = \frac{c}{\varepsilon}$$

- Orbits passing through

$$\mathbf{e}_2^T(\mathbf{p}_L - u(c, \varepsilon)\dot{\mathbf{p}}_L) \approx \varepsilon - u(c, \varepsilon)\varepsilon^{\frac{3}{2}},$$
$$\mathbf{e}_2^T(\mathbf{p}_L + v(u(c, \varepsilon))\dot{\mathbf{p}}_L) \approx \varepsilon + v(u(c, \varepsilon))\varepsilon^{\frac{3}{2}} \approx \varepsilon + c\sqrt{\varepsilon},$$

are under canard regime.

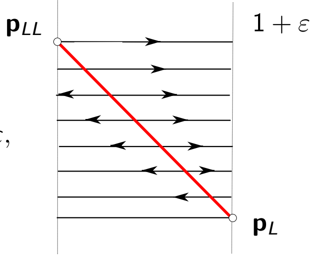
- In particular, since $F(u_t) = \varepsilon^{-\frac{1}{2}}$,

$$\mathbf{e}_2^T(\mathbf{p}_L - u_t\dot{\mathbf{p}}_L) \approx \varepsilon - u_t\varepsilon^{\frac{3}{2}},$$
$$\mathbf{e}_2^T(\mathbf{p}_L + v(u_t)\dot{\mathbf{p}}_L) \approx \varepsilon + \varepsilon^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}} = 1 + \varepsilon,$$

- Taking $c_\alpha = \varepsilon^\alpha$ with $0 < \alpha < 1$,

$$u(c_\alpha, \varepsilon) = \frac{1}{\lambda^q} - \frac{1}{\lambda^q} e^{-\frac{1}{\varepsilon^{1-\alpha}}},$$

define the birth of canard regime.



Canard regime: Canard orbits

- For $0 < c \ll 1$ consider $u(c,\varepsilon) > \frac{1}{\lambda^q}$ such that

$$F(u(c,\varepsilon)) = 1 + c.$$

- Since $1 + c = r\frac{1-v(u(c,\varepsilon))\lambda^s}{1-u_r\lambda^s}$ then

$$v(u(c,\varepsilon)) = \frac{1}{\lambda^s} - \frac{1}{\sqrt{\varepsilon}}(1+c).$$

- Orbits passing through

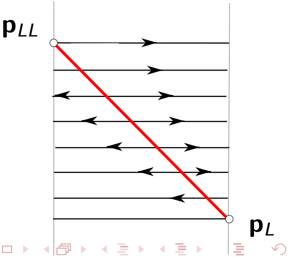
$$\mathbf{e}_2^T(\mathbf{p}_L - u(c,\varepsilon)\dot{\mathbf{p}}_L) \approx \varepsilon - u(c,\varepsilon)\varepsilon^{\frac{3}{2}},$$
$$\mathbf{e}_2^T(\mathbf{p}_{LL} - v(u(c,\varepsilon))\dot{\mathbf{p}}_{LL}) \approx \varepsilon + v(u(c,\varepsilon))\varepsilon^{\frac{3}{2}} \approx c\sqrt{\varepsilon} + (4+c)\varepsilon,$$

  are under canard regime.
- Therefore u_r and $v_r = v(u_r)$ given by

  $F(u_r) = 1, \quad v_r = \frac{1}{\lambda^s} - \frac{1}{\sqrt{\varepsilon}},$

  define the maturation of the canard regime.
- Note that

  $\mathbf{e}_2^T(\mathbf{p}_{LL} - v_r\mathbf{p}_{LL}) \approx 4\varepsilon, \quad \mathbf{e}_2^T\mathbf{p}_L < 2\varepsilon.$



Canard regime: Canard cycles

- From previous analysis canard regime is restricted to (x_r, x_α), where

$$x_r = -\frac{1}{\sqrt{\varepsilon}} + O(\varepsilon^0), \quad x_\alpha = -\sqrt{\varepsilon} - \varepsilon^\alpha(\sqrt{\varepsilon} + a),$$

and $0 < \alpha < 1$.

Theorem

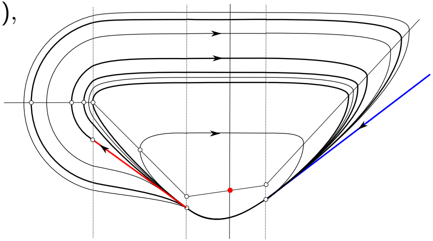
Set ε_0 sufficiently small. There exists a function $a = \tilde{a}(\varepsilon)$, analytic as a function of $\sqrt{\varepsilon}$, defined in the open set $U = (0, \varepsilon_0)$ and such that, for $\varepsilon \in U$, both branches of slow manifold connect if and only if $a = \tilde{a}(\varepsilon)$. The time of flight of the transition is $\tilde{\tau}_C(\varepsilon) > 0$.

$$\tilde{a}(\varepsilon) = \frac{e^{\frac{\pi}{\sqrt{3}}} - 1}{e^{\frac{\pi}{\sqrt{3}}} + 1}\sqrt{\varepsilon} - \frac{4e^{\frac{\pi}{\sqrt{3}}}}{3\left(e^{\frac{\pi}{\sqrt{3}}} + 1\right)^2}\varepsilon^2 + O(\varepsilon^{\frac{5}{2}}),$$

$$\tilde{\tau}_C(\varepsilon) = \frac{2\pi}{\sqrt{3}\sqrt{\varepsilon}} - 2 - 2\varepsilon + O(\varepsilon^{\frac{3}{2}}),$$

Proof: Apply IFT to

$$E_q(t, a, \varepsilon) = \varphi(t; \mathbf{q}_0(a, \varepsilon)) - \mathbf{q}_1(a, \varepsilon) = \mathbf{0}$$



Canard regime: Canard cycles

Theorem

Set ε_0 sufficiently small. There exists $\hat{a}(\varepsilon, x)$, a C^∞ function of $(\sqrt{\varepsilon}, x)$, defined in the open set $U = (0, \varepsilon_0) \times (-x_r, x_\alpha)$, such that, for $(\varepsilon, x_0) \in U$ and $a = \hat{a}(\varepsilon, x_0)$ the system possesses a stable limit cycle, Γ_{x_0}, passing through $(x_0, f(x_0, a, 0))$. Moreover, $a = \hat{a}(\varepsilon, x_0)$ has the same Taylor series expansion in ε as $\tilde{a}(\varepsilon)$ and therefore, Γ_{x_0} is a canard cycle. Moreover, if $x_0 \in (-1, x_s)$, then Γ_{x_0} is a headless canard; and if $x_0 \in (x_r, -1)$, then Γ_{x_0} is a canard with head.

Proof: Apply IFT to

$E_p(t, a, \varepsilon) = \varphi(t; \mathbf{p}_0(a, \varepsilon)) - \mathbf{p}_1(a, \varepsilon) = \mathbf{0}.$

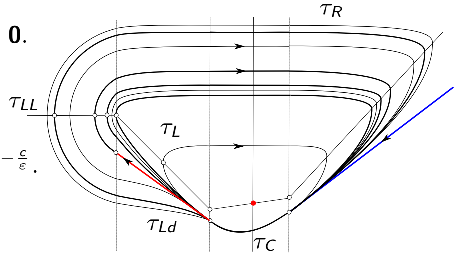
Since $\|\mathbf{p}_k - \mathbf{q}_k\| \leq e^{-\frac{c}{\varepsilon}}$, $k = 1, 2$,

$E_p(t, a, \varepsilon) = E_q(t, a, \varepsilon) + \eta(t, a, \varepsilon)$,

$\|\eta(t, a, \varepsilon)\| \leq e^{-\frac{c}{\varepsilon}}$, $\|D^k \eta(t, a, \varepsilon)\| \leq e^{-\frac{c}{\varepsilon}}$.

Derivative Poincaré map

$e^{\tau_L t_L + \tau_R t_R + \tau_C t_C}, \quad e^{\tau_{Ld} t_{Ld} + \tau_R t_R + \tau_C t_C}.$



Canard regime: Maximal period

Theorem

Set ε_0 sufficiently small. There exists $T : (0, \varepsilon_0) \times (x_r, x_\alpha) \to \mathbb{R}^+$, a $\mathcal{C}^\infty$ function of $(\sqrt{\varepsilon}, x)$ such that $T(\varepsilon, x)$ is the period of the canard cycle Γ_x and satisfies:

a) *there exists $x_P(\varepsilon)$, a $\mathcal{C}^\infty$ function of $\varepsilon^{1/3}$, defined in $(0, \varepsilon_0)$ which provides the maximum of the period T, that is*

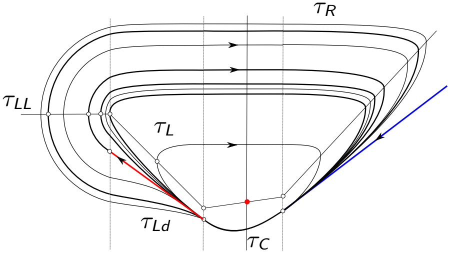
$\frac{\partial T}{\partial x}\Big\|_{(\varepsilon,x)} > 0$	$\frac{\partial T}{\partial x}\Big\|_{(\varepsilon,x_P(\varepsilon))} = 0$	$\frac{\partial T}{\partial x}\Big\|_{(\varepsilon,x)} < 0$
$(x_r, x_P(\varepsilon))$	$x_P(\varepsilon)$	$(x_P(\varepsilon), x_s)$

c) *The maximum satisfies that*

$$x_P(\varepsilon) = -\varepsilon^{-1/6} + O(\varepsilon^{1/3}),$$

$$T(\varepsilon, x_P(\varepsilon)) \approx -\frac{1}{\varepsilon}\ln(\varepsilon).$$

Proof: By computing time of flight

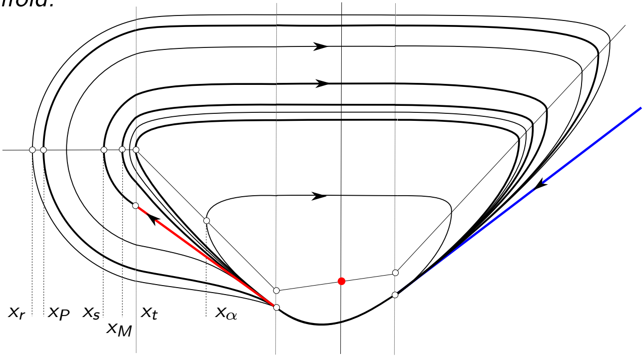


Theorem

Set $\varepsilon_0 > 0$ sufficiently small. Transitory canard $x_t = -1$, maximal canard x_M and the canard with maximal period $x_P = -\varepsilon^{-1/6} + O(\varepsilon^{1/3})$ are different canard cycles and they are ordered as follows

$$x_r < x_P < x_s < x_M < x_t < x_\alpha,$$

where $x_s = -1 - \frac{ep}{2} + O(\varepsilon^{\frac{3}{2}})$ is the width of the canard cycle through the slow manifold.



Birth, transition and maturation of canard cycles in PWL systems

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