Distortion in Lattice Actions on Surfaces

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A Motivating Conjecture

Conjecture. *[R. Zimmer] Any C^∞ volume preserving action of $SL(n, \mathrm{Z})$ on a compact manifold with dimension less than n, factors through an action of a finite group.*

We are really interested in results valid for all finite index subgroups of $SL(n, \mathrm{Z})$.

Theorem. *[D. Witte] Let $\mathcal{G}$ be a finite index subgroup of $SL(n, \mathrm{Z})$ with $n \geq 3$. Any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Homeo}(S^1)$$

has a finite image.

Example. *The group $SL(3, \mathrm{Z})$ acts analytically on S^2 by projectivizing the standard action on R^3.*

S^2 is the set of unit vectors in R^3. If $x \in S^2$ and $g \in SL(3, \mathrm{Z})$, we can define $\phi(g) : S^2 \to S^2$ by

$$\phi(g)(x) = \frac{gx}{|gx|}.$$

Question. *Let $\mathcal{G}$ be a finite index subgroup of $SL(4, \mathrm{Z})$. Does every homomorphism from $\mathcal{G}$ to $\mathrm{Diff}(S^2)$ or $\mathrm{Homeo}(S^2)$ have a finite image? What about other surfaces?*

Distortion in Groups

Definition. *[Gromov] An element g in a finitely generated group G is called a* distorted *if it has infinite order and*

$$\liminf_{n\to\infty} \frac{|g^n|}{n} = 0,$$

where $|g|$ denotes the minmal word length of g in some set of generators. If $\mathcal{G}$ is not finitely generated then g is distorted if it is distorted in some finitely generated subgroup.

Example. *The subgroup G of $SL(2, \mathrm{R})$ generated by*

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfies

$$A^{-1}BA = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = B^4 \text{ and } A^{-n}BA^n = B^{4^n}$$

so B is distorted.

Example. *The group of integer matrices of the form*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is called the Heisenberg group.

If

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then their *commutator* $f = [g, h] := g^{-1}h^{-1}gh$ is

$$f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } f \text{ commutes with } g \text{ and } h.$$

This implies

$$[g^n, h^n] = f^{n^2}$$

so f is distorted.

Example. [G. Mess] *Consider the subgroup of* $\mathrm{Diff}_\omega(\mathrm{T}^2)$ *generated by the automorphism given by*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and a translation $T(x) = x + w$ *where* $w \neq 0$ *is parallel to the unstable manifold of* A*. The element* T *is distorted.*

Proof:
Let λ be the expanding eigenvalue of A. The element $h_n = A^n T A^{-n}$ satisfies $h_n(x) = x + \lambda^n w$ and $g_n = A^{-n} T A^n$ satisfies $g_n(x) = x + \lambda^{-n} w$. Hence $g_n h_n(x) = x + (\lambda^n + \lambda^{-n})w$. Since $trA^n = \lambda^n + \lambda^{-n}$ is an integer we conclude $T^{trA^n} = g_n h_n$, so $|T^{trA^n}| \leq 4n + 2$. Thus

$$\lim_{n\to\infty} \frac{|T^{trA^n}|}{trA^n} = 0,$$

so T is distorted.

Question. *Is an irrational rotation of S^1 distorted in $\mathrm{Diff}(S^1)$ or $\mathrm{Homeo}(S^1)$? Is an irrational rotation of S^2 distorted in $\mathrm{Diff}(S^2)$ or in area preserving diffeomorphisms of S^2?*

Example. [D. Calegari] *An irrational rotation of S^2 is distorted in $\mathrm{Homeo}(S^2)$. In fact there is a C^0 action of the Heisenberg group on S^2 whose center generated by an irrational rotation.*

Remark [F. Le Roux] There is no action of the Heisenberg group $\mathcal{H}$ on S^2 which is C^1 and has a central element which is an irrational rotatation. (The example of Calegari cannot be made C^1.)

Many Lattices have Distortion

Theorem. *[Lubotzky-Mozes-Ragnunathan] Suppose* Γ *is a non-uniform irreducible lattice in a semi-simple Lie group* $\mathcal{G}$ *with* $\mathrm{R}-$*rank* ≥ 2*. Suppose further that* $\mathcal{G}$ *is connected, with finite center and no nontrivial compact factors. Then* Γ *has distortion elements, in fact, elements whose word length growth is at most logarithmic.*

Margulis' normal subgroup theorem

Definition. *A group is called* almost simple *if every normal subgroup is finite or has finite index.*

Theorem. *[Margulis] Assume Γ is an irreducible lattice in a semi-simple Lie group with $\mathrm{R}-$rank ≥ 2, e.g. any finite index subgroup of $SL(n, \mathrm{Z})$ with $n \geq 3$. Then Γ is almost simple.*

Proposition. *If $\mathcal{G}$ is a finitely generated almost simple group which contains a distortion element and $\mathcal{H} \subset \mathcal{G}$ is a normal subgroup, then the only homomorphism from $\mathcal{H}$ to R is the trivial one.*

Thurston's stability theorem

Theorem. *[Thurston] Suppose $\mathcal{G}$ is a finitely generated group,*

$$\phi : G \to \mathrm{Diff}^1(M^n)$$

is a homomorphism and there is $x_0 \in M$ such that for all $g \in \mathcal{G}$

$$\phi(g)(x_0) = x_0 \text{ and } D\phi(g)(x_0) = I.$$

Then either ϕ is trivial or there is a non-trivial homomorphism from $\mathcal{G}$ to R.

N.B.: For definitive results on C^1 actions on S^1 see E. Ghys, *"Actions de réseaux sur le cercle."*

Theorem. [Toy Theorem] *Suppose $\mathcal{G}$ is a finitely generated almost simple group and has a distortion element and suppose μ is a finite probability measure on S^1. If*

$$\phi : \mathcal{G} \to Diff_\mu(S^1)$$

is a homomorphism then $\phi(\mathcal{G})$ is finite.

Proof:

- The rotation number $\rho : Diff_\mu(S^1) \to \mathrm{R}/\mathrm{Z}$ is a homomorphism.

- If f is distorted $\rho(f^n) = 0$ for some $n > 0$ so $\mathrm{Fix}(f^n)$ is non-empty.

- $supp(\mu) \subset \mathrm{Fix}(f^n)$

- $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \phi(g)$ pointwise fixes $supp(\mu)\}$ is infinite and normal, and hence finite index.

- $\phi(\mathcal{G}_0)$ is trivial by Thurston stability.

Theorem. *[F-Handel] Suppose S is a closed oriented surface of genus at least one and μ is a Borel probabilty measure on S with infinite support. Suppose $\mathcal{G}$ is finitely generated, almost simple and has a distortion element. Then any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Diff}_\mu(S)$$

has finite image.

This result was previously known in the special case of symplectic diffeomorphisms by a result of L. Polterovich.

The result above also holds with $supp(\mu)$ finite if $\mathcal{G}$ is a Kazhdan group (aka $\mathcal{G}$ has property T).

Distortion and Measure

Theorem. *[F-Handel] Suppose that S is a closed oriented surface, that f is a distortion element in $\mathrm{Diff}(S)_0$ and that μ is an f-invariant Borel probability measure.*

1. *If S has genus at least two then $\mathrm{Per}(f) = \mathrm{Fix}(f)$ and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$.*

2. *If $S = T^2$ then for some $\mathrm{Per}(f) \neq \emptyset$. All points of $\mathrm{Per}(f)$ have the same period, say n, and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f^n)$*

3. *If $S = S^2$ and if f^n has at least three fixed points for some smallest $n > 0$, then $\mathrm{Per}(f) = \mathrm{Fix}(f^n)$ and $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f^n)$.*

Heisenberg again

Theorem. *[F-Handel] Suppose S is a closed oriented surface with Borel probabilty measure μ and $\mathcal{G}$ is a finitely generated, almost simple group with a subgroup isomorphic to the Heisenberg group. Then any homomorphism*

$$\phi : \mathcal{G} \to \mathrm{Diff}_{\mu}(S)$$

has finite image.

Parallels between $\mathrm{Diff}(S^1)_0$ and $\mathrm{Diff}_\mu(S)_0$

In general there seem to be strong parallels between results about $\mathrm{Diff}(S^1)_0$ and $\mathrm{Diff}_\mu(S)_0$. For example, Witte's theorem and our results above. Also we have

Theorem. *[Hölder] Suppose $\mathcal{G}$ is a subgroup of $\mathrm{Diff}(S^1)_0$ which acts freely (no non-trivial element has a fixed point). Then $\mathcal{G}$ is Abelian.*

Theorem. *[Arnold Conjecture: Conley-Zehnder] Suppose*

$$f \in \mathrm{Diff}_\omega(\mathrm{T}^2)_0$$

is a commutator (ω is Lebesgue measure). Then f has (at least three) fixed points.

Corollary. *Suppose $\mathcal{G}$ is a subgroup of $\mathrm{Diff}_\omega(\mathrm{T}^2)_0$ which acts freely. Then $\mathcal{G}$ is Abelian.*

Nilpotent Groups

Definition. *A group* $\mathcal{N}$ *is called* nilpotent *provided when we define*

$$\mathcal{N}_0 = \mathcal{N},\ \mathcal{N}_i = [\mathcal{N}, \mathcal{N}_{i-1}],$$

there is an $n \geq 1$ *such that* $\mathcal{N}_n = \{e\}$*. Note if* $n = 1$ *it is Abelian.*

Theorem. *[Plante - Thurston] Let* N *be a nilpotent subgroup of* $\mathrm{Diff}^2(S^1)_0$*. Then* N *must be Abelian.*

Theorem. *[Farb - F] Every finitely-generated, torsion-free nilpotent group is isomorphic to a subgroup of* $\mathrm{Diff}^1(S^1)_0$*.*

An Analogue of the Plante - Thurston Theorem

Theorem. *[F - Handel] Let $\mathcal{N}$ be a nilpotent subgroup of $\mathrm{Diff}^1_\mu(S)_0$ with μ a probability measure with $supp(\mu) = S$. If $S \neq S^2$ then $\mathcal{N}$ is Abelian, if $S = S^2$ then $\mathcal{N}$ is Abelian or has an index 2 Abelian subgroup.*

Proof: (For the case $genus(S) > 1$) Suppose

$$\mathcal{N} = \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_m \supset \{1\}$$

is the lower central series of $\mathcal{N}$. then $\mathcal{N}_m$ is in the center of $\mathcal{N}$. If $m > 1$ there is a non-trivial $f \in \mathcal{N}_m$ and elements g, h with $f = [g, h]$. No non-trivial element of $\mathrm{Diff}^1(S)_0$ has finite order since S has genus > 1. So g, h generate a Heisenberg group and f is distorted. Our theorem says $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$, but $\mathrm{supp}(\mu) = S$ so $f = id$. This is a contradition unless $m = 1$ and $\mathcal{N}$ is abelian.

Detecting Non-Distortion

Properties which imply non-distortion:

- exponential growth of length of a curve
- linear displacement in the universal cover
- positive *spread*

Exponential Growth

Definition. *If the surface S is provided with a Riemannian metric a smooth closed curve $\tau \subset S$ has a well defined length $l_S(\tau)$. Define the* exponential growth rate *by*

$$\operatorname{egr}(f, \tau) = \liminf_{n \to \infty} \frac{\log(l_S(f^n(\tau))}{n}.$$

This is easily seen to be independent of the choice of metric.

Exponential Growth

Proposition. *If G is a finitely generated subgroup of $\mathrm{Diff}(S)_0$ and $f \in G$ is distorted in G then $\mathrm{egr}(f, \tau) = 0$ for all closed curves τ.*

Proof: Choose generators $g_1, \ldots, g_j$ of G. There exists $C > 0$ such that $||Dg_i|| < C$ for all i. Thus $l_S(g_i(\tau)) \leq C l_S(\tau)$ for all τ and all i. It follows that

$$\liminf_{n \to \infty} \frac{\log(l_S(f^n(\tau))}{n} \leq \liminf_{n \to \infty} \frac{\log(l_S(\tau)) + \log(C)|f^n|}{n}$$
$$= 0.$$

Linear Displacement

Definition. *Assume that* $f \in \mathrm{Homeo}(S)_0$ *and that* $S \neq S^2$. *A metric* d *on* S *lifts to an equivariant metric* $\tilde{d}$ *on the universal cover* $\tilde{S}$. *We say that* f *has* $linear$ $displacement$ *if either of the following conditions hold.*

1. $S \neq T^2$, $\tilde{f}$ *is the identity lift and there exists* $\tilde{x} \in \tilde{S} = H$ *such that*

$$\liminf_{n \to \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{x})}{n} > 0.$$

2. $S = T^2$ *and there exist* $\tilde{f}$ *and* $\tilde{x}_1, \tilde{x}_2 \in \tilde{S} = \mathrm{R}^2$ *such that*

$$\liminf_{n \to \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}_1), \tilde{f}^n(\tilde{x}_2))}{n} > 0.$$

Linear Displacement

Proposition. *If G is a finitely generated subgroup of $\mathrm{Homeo}(S)_0$ and $f \in G$ is distorted in G then f does not have linear displacement.*

Proof: We present only the case that S has genus > 1. In this case the identity lifts $\{\tilde{g} : g \in G\}$ form a subgroup $\tilde{G}$ and $\tilde{f}$ is a distortion element in $\tilde{G}$. Let d be the distance function of a Riemannian metric on S and let $\tilde{d}$ be its lift to $\tilde{S}$. For generators $g_1, \dots, g_j$ of G there exists $C > 0$ such that $\tilde{d}(\tilde{g}_i(\tilde{x}), \tilde{x}) < C$ for all $\tilde{x} \in \tilde{S}$ and all i. It follows that

$$\liminf_{n \to \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{x})}{n} \le \liminf_{n \to \infty} C \frac{|f^n|}{n} = 0.$$

Spread

Given curves α, β, γ with β closed then $L_{\beta,\gamma}(\alpha)$ is roughly the number of times α crosses γ while following along β.

Definition. *Define the* spread *of α with respect to f, β and γ to be*

$$\sigma_{f,\beta,\gamma}(\alpha) = \liminf_{n\to\infty} \frac{L_{\beta,\gamma}(f^n(\alpha))}{n}.$$

Proposition. *If G is a finitely generated subgroup of $\mathrm{Diff}(S)_0$ and $f \in G$ is distorted in G then $\sigma_{f,\beta,\gamma}(\alpha) = 0$ for all α, β, γ.*

Lemma. *Suppose that* $g_i \in \mathrm{Diff}(S)_0,\ 1 \leq i \leq k,$ *that* f *is in the group they generate and that* $|f^n|$ *is the word length of* f^n *in the generators* $\{g_i\}$*. Then there is a constant* $C > 0$ *such that*

$$L_{\beta,\gamma}(f^n(\alpha)) \leq L_{\beta,\gamma}(\alpha) + C|f^n|$$

for all α, β, γ *and all* $n > 0$.

Spread

Proposition. *If G is a finitely generated subgroup of $\mathrm{Diff}(S)_0$ and $f \in G$ is distorted in G then $\sigma_{f,\beta,\gamma}(\alpha) = 0$ for all α, β, γ.*

Proof: Since f is distorted in G

$$\liminf_{n\to\infty} \frac{|f^n|}{n} = 0.$$

According to the definition of spread and the lemma we then have

$$\sigma_{f,\beta,\gamma}(\alpha) = \liminf_{n\to\infty} \frac{L_{\beta,\gamma}(f^n(\alpha))}{n}$$

$$\leq \liminf_{n\to\infty} \frac{L_{\beta,\gamma}(\alpha) + C|f^n|}{n} = 0.$$

Proposition. *If S has genus > 1, $f \in \mathrm{Diff}_\mu(S)_0$ has infinite order and $\mu(S \setminus \mathrm{Fix}(f)) > 0$ then one of the following holds:*

1. *There exists a closed curve τ such that*

$$\mathrm{egr}(f, \tau) > 0.$$

2. *f has linear displacement.*

3. *After replacing f with some iterate $g = f^k$ and perhaps passing to a two-fold covering $g : S \to S$ is isotopic to the identity and there exist α, β, γ such that the spread $\sigma_{g,\beta,\gamma}(\alpha) > 0$.*