Numerical methods for invariant curves

- E. Castellà and À. Jorba. On the vertical families of two-dimensional tori near the triangular points of the Bicircular problem. *Celestial Mech.*, 76(1):35–54, 2000.
- À. Jorba. Numerical computation of the normal behaviour of invariant curves of n-dimensional maps. *Nonlinearity*, 14(5):943–976, 2001.

We will focus now on the computation of lower dimensional tori and related issues. We will discuss first numerical methods and then methods based on normalizing transformations. In both cases we will see some applications to Celestial Mechanics.

There are several reasons to justify the need for the computation of invariant tori.

For instance, think of a non-automomous system that depends on time in a periodic or quasi-periodic way.

Another situation is the globalization of a center manifold.

Let f be a diffeomorphism of an open domain of $\mathbb{R}^n$ into itself, and consider the dynamical system

$$\bar{x} = f(x).$$

Assume that this system has an invariant curve with rotation number ω, this is, we assume that there exists a (continuous) map $x : \mathbb{T}^1 \to \mathbb{R}^n$ such that

$$f(x(\theta)) = x(\theta + \omega), \qquad \text{for all } \theta \in \mathbb{T}^1.$$

We are not assuming that f is neither close to integrable, nor it preserves any structure (measure, symplectic form, etc.).

We can also consider the non-autonomous case

$$\left.\begin{array}{rcl}\bar{x} & = & f(x,\theta) \\ \bar{\theta} & = & \theta+\omega\end{array}\right\},$$

where $\theta \in \mathbb{T}^r$.

Both cases are similar, so for the moment I'll focus on the autonomous one.

Suppose that this map has an invariant curve with rotation number ω. The curve is given (in parametric form) by a map $x : \mathbb{T}^1 \to \mathbb{R}^n$. The invariance condition is then,

$$F(\theta) \equiv f(x(\theta)) - x(\theta + \omega) = 0, \quad \forall\, \theta \in \mathbb{T}^1.$$

To start the discussion, assume that we know the rotation number of the curve we are looking for. So we only want to determine the function $x(\theta)$. Let us write $x(\theta)$ as a real Fourier series,

$$x(\theta) = a_0 + \sum_{k>0} a_k \cos(k\theta) + b_k \sin(k\theta),$$

where $a_k, b_k \in \mathbb{R}^n$, $k \in \mathbb{N}$. As it is usual in numerical methods, we will look for a truncation of this series. So, let us fix in advance a truncation value N (the selection of N will be discussed later on), and let us try to determine an approximation to the $2N+1$ unknown coefficients a_0, a_k and b_k, $0 < k \leq N$.

The main idea is to apply a Newton method to find $x(\theta)$ such that $F(x(\theta)) \equiv 0$. We note that F acts on a space of periodic functions.

First, let us define a mesh of $2N+1$ points on $\mathbb{T}^1$,

$$\theta_j = \frac{2\pi j}{2N+1}, \qquad 0 \le j \le 2N.$$

Then, it is not difficult to compute $x(\theta_j)$, $f(\theta_j)$, $f(\theta_j + \omega)$ and, hence, $F(x(\theta_j))$. From these values, we can easily derive the Fourier coefficents of $F(x(\theta))$.

Therefore, we have a procedure to compute the map F.

As this procedure can be easily differentiated, we can also obtain DF.

Then, a Newton method can be applied.

A natural question is about the size of the error of the obtained curve.

To measure such error we use

$$E(x,\omega) = \max_{\theta \in \mathbb{T}^1} |f(x(\theta)) - x(\theta+\omega)|.$$

We estimate $E(x,\omega)$ using a much finer mesh than the one used in the previous computations.

If this error is too big (for instance, bigger than 10^{-12}), we increase N and we apply the Newton process again.

Linear normal behaviour

Let f be a diffeomorphism of an open domain of $\mathbb{R}^n$ into itself, and consider the dynamical system

$$\bar{x} = f(x).$$

Assume that this system has an invariant curve with rotation number ω, this is, we assume that there exists a (continuous) map $x : \mathbb{T}^1 \to \mathbb{R}^n$ such that

$$f(x(\theta)) = x(\theta + \omega), \qquad \text{for all } \theta \in \mathbb{T}^1.$$

We are not assuming that f is neither close to integrable, nor it preserves any structure (measure, symplectic form, etc.).

Let h represent a small displacement with respect to an arbitrary point $x(\theta)$ on the invariant curve. Then,

$$f(x(\theta)+h) = f(x(\theta)) + D_x f(x(\theta))h + O(\|h\|^2).$$

As $f(x(\theta)) = x(\theta+\omega)$, it follows that the linear normal behaviour is described by the following dynamical system,

$$\left.\begin{array}{rcl} \bar{x} & = & A(\theta)x \\ \bar{\theta} & = & \theta+\omega \end{array}\right\}, \qquad (1)$$

where $A(\theta) = D_x f(x(\theta))$. This kind of system is sometimes known as *linear quasi-periodic skew-product.*

Definition 1 *The system (1) is called reducible iff there exists a (may be complex) change of variables* $x = C(\theta)y$ *such that (1) becomes*

$$\left.\begin{array}{rcl} \bar{y} & = & By \\ \bar{\theta} & = & \theta + \omega \end{array}\right\}, \tag{2}$$

where the matrix $B \equiv C^{-1}(\theta+\omega)A(\theta)C(\theta)$ *does not depend on* θ.

We define the operator

$$T_\omega : \psi(\theta) \in C(\mathbb{T}^1, \mathbb{C}^n) \mapsto \psi(\theta+\omega) \in C(\mathbb{T}^1, \mathbb{C}^n),$$

and let us consider now the following generalized eigenvalue problem: to look for couples $(\lambda, \psi) \in \mathbb{C} \times C(\mathbb{T}^1, \mathbb{C}^n)$ such that

$$A(\theta)\psi(\theta) = \lambda T_\omega \psi(\theta). \tag{3}$$

In what follows we will assume, without explicit mention, that $\omega \notin 2\pi\mathbb{Q}$ (the case $\omega \in 2\pi\mathbb{Q}$ can be reduced to constant coefficients by iterating the system a suitable number of times).

Proposition 1 *Consider the generalized eigenvalue problem (3) for a given invariant curve $x(\theta)$. Then, if f does not depend on θ, 1 is an eigenvalue of (3). The corresponding eigenfunction is $x'(\theta)$.*

Proposition 2 *Let* λ *be an eigenvalue of (3). Then, for any* $k \in \mathbb{Z}$, $\lambda \exp(\mathrm{i}\, k\omega)$ *is also an eigenvalue of (3).*

Proof: We denote by $\psi \in C(\mathbb{T}^1, \mathbb{R}^n)$ the eigenfunction corresponding to the eigenvalue λ. Then, check that $\hat{\psi}(\theta) = \exp(-\mathrm{i}\, k\theta)\psi(\theta)$ is an eigenfunction of eigenvalue $\lambda \exp(\mathrm{i}\, k\omega)$.

Remark 1 *This shows that the closure of the set of eigenvalues of (3) can be written as a union of circles with centre at the origin. If the system is autonomous, we have shown that the closure of the eigenvalues must contain the unit circle.*

Proposition 3 *Let us assume that the initial system can be reduced to constant coefficients, by means of a transformation $x = C(\theta)y$. Let B be the reduced matrix. In this situation, one has*

1. *If λ is an eigenvalue of B, then λ is an eigenvalue of (3).*
2. *If λ is an eigenvalue of (3), then there exists $k \in \mathbb{Z}$ such that $\lambda \exp(\mathrm{i}\, k\omega)$ is an eigenvalue of B.*

In particular we have shown that, in the reducible case, the set of eigenvalues is not empty.

Definition 2 *Two eigenvalues* λ_1 *and* λ_2 *are said to be unrelated iff* $\lambda_1 \neq \exp(\mathrm{i}\, k\omega)\lambda_2$, $\forall k \in \mathbb{Z}$. *Otherwise, we will refer to them as related.*

Proposition 4 *Assume that there exist* n *unrelated eigenvalues* $\lambda_1, \ldots, \lambda_n$ *for the eigenproblem (3). Then, (1) can be reduced to (2), where* $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Corollary 1 *The generalized eigenvalue problem (3) cannot have more than* n *unrelated eigenvalues.*

NUMERICAL APPROXIMATION

We consider the (standard) eigenvalue problem for the operator

$$\psi(\theta) \mapsto \left(T_{-\omega} \circ A(\theta)\right) \psi(\theta),$$

in the space $C(\mathbb{T}^1, \mathbb{R}^n)$.

To discretize it, we truncate the Fourier series of the elements of $C(\mathbb{T}^1, \mathbb{R}^n)$ for a given value.

Once the operator is written in matrix form, we apply a standard numerical method to look for the eigenvalues and eigenvectors.

ACCURACY

To simplify the discussion, let us assume that the system is reducible, and let μ_0 be one of the eigenvalues of the reduced matrix B. Then, the operator $T_{-\omega} \circ A(\theta)$ must have all the values $\mu_k \equiv \mu_0 \exp(\mathrm{i}\, k\omega)$ $(k \in \mathbb{Z})$ as eigenvalues. Of course, the discretized version of the operator only contains a finite number of those values and, as is usual in these situations, their accuracy depends on the size of $|k|$.

Can we detect the most accurate eigenvalues?

Idea: Consider the "norm"

$$\|\psi\|^{(p)} = \sum_{j\in\mathbb{Z}} |\psi_j||j|^p.$$

If it is well-defined, the truncation error is

$$TE(\psi, N) = \sum_{|j|>N} |\psi_j||j|^p.$$

If we consider an eigenfunction like $\exp(-\mathrm{i}\, k\theta)\psi(\theta)$, the previous expression can only be small for a reduced set of values of $|k|$.

Key idea : $TE(\psi, N)$ is small when $\|\psi\|^{(p)}$ is small.

Example: The quasi-periodic Hill equation

$$\left.\begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -(a^2 + bp(\theta_1, \theta_2))x \\ \dot{\theta}_1 & = & \omega_1 \\ \dot{\theta}_2 & = & \omega_2 \end{array}\right\}$$

where $p(\theta_1, \theta_2) = \cos(\theta_1) + \cos(\theta_2)$, $(x, y) \in \mathbb{R}^2$, $(\theta_1, \theta_2) \in \mathbb{T}^2$ and $\omega_{1,2}$ are the forcing frequencies.

We use the Poincaré section $\theta_1 = 0$. We denote by $\Phi_{\theta_2}(t)$ the 2×2 fundamental matrix for the (x, y) variables, obtained by taking $\Phi_{\theta_2}(0) = Id$, $\theta_1 = 0$ and $\theta_2 \in \mathbb{T}^1$. Let $A(\theta_2) = \Phi_{\theta_2}(2\pi)$ (note that for $t = \frac{2\pi}{\omega_1}$, θ_1 is again on the Poincaré section). Then, the dynamics can also be described by the following linear quasi-periodic skew-flow:

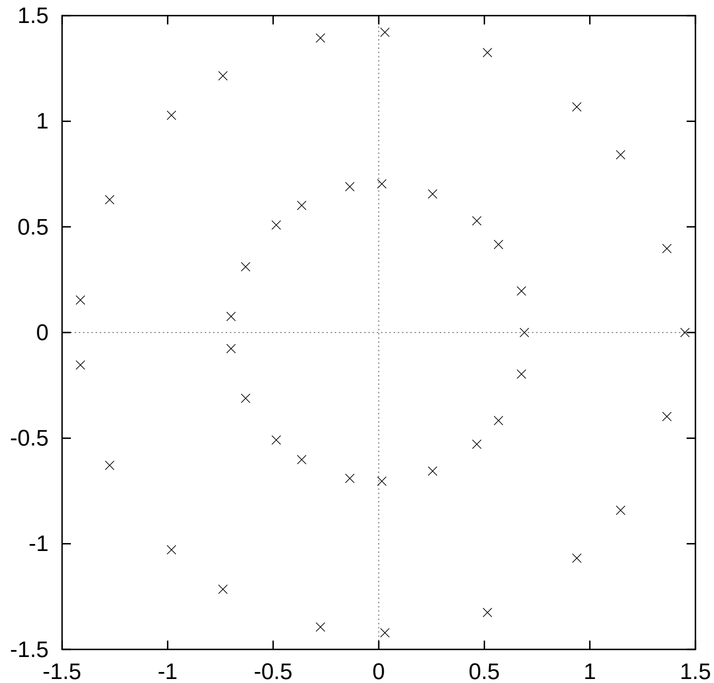
$$\left.\begin{array}{rcl} \bar{z} & = & A(\theta_2) z \\ \bar{\theta}_2 & = & \theta_2 + \dfrac{2\pi}{\omega_1}\omega_2 \end{array}\right\},$$

where $z \in \mathbb{R}^2$ and $\theta_2 \in \mathbb{T}^1$.

We take the values $\omega_1 = 1$ and $\omega_2 \equiv \gamma = \frac{1}{2}(1+\sqrt{5})$. So, the frequency for the cocycle is $\omega = \frac{4\pi}{1+\sqrt{5}} \approx 3.8832220774509332$.

We select the values $b = 0.2$ and $a^2 = 0.7$. The program starts with a value $N = 8$, to find that the estimation of the error for the eigenvectors is 5.6×10^{-12}. As this is greater than the prescribed accuracy (10^{-12}), the program repeats again the calculation with $N = 10$, to reach an estimated accuracy of 1.3×10^{-14}.

Modulus	Argument	Norm	Error
1.421399533740337	1.199981614864327	1.069409e+01	1.334772e-14
0.703531960059501	1.199981614864327	1.069409e+01	1.326585e-14
1.421399533740339	2.683240462586607	2.360600e+01	3.917410e-13
0.703531960059501	2.683240462586607	2.360600e+01	3.928163e-13
0.703531960059501	0.283277232857952	3.291719e+01	5.593418e-12
1.421399533740338	0.283277232857953	3.291719e+01	5.594009e-12
0.689449522457020	0.000000000000000	3.939715e+01	2.489513e+00
1.450432508004731	0.000000000000000	4.203193e+01	2.638401e+00
0.703531960059501	2.116685996870700	4.216431e+01	6.078403e-10
1.421399533740338	2.116685996870700	4.216431e+01	6.078433e-10
0.703531960059500	1.766536080580232	5.141098e+01	1.776935e-08
1.421399533740338	1.766536080580234	5.141098e+01	1.777005e-08
1.421399533740338	0.633427149148420	5.954887e+01	5.827280e-07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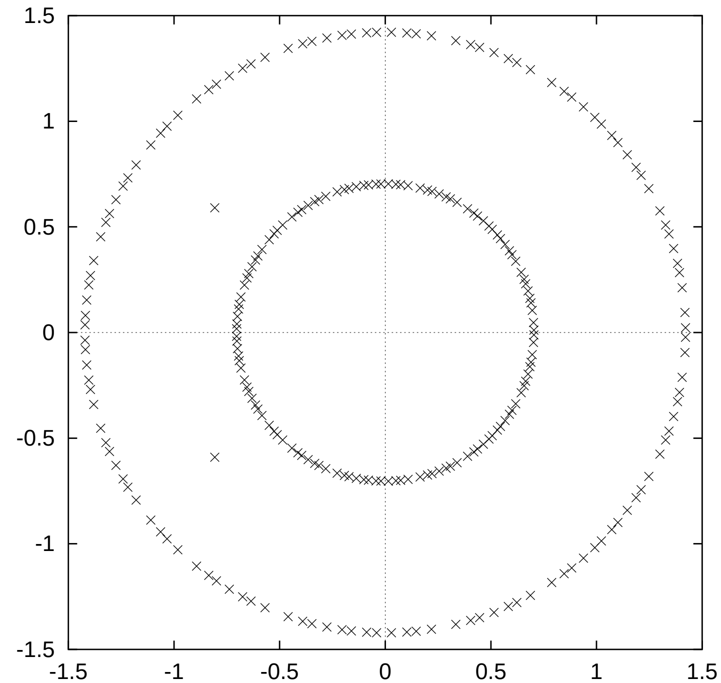
-1.5 -1 -0.5 0 0.5 1 1.5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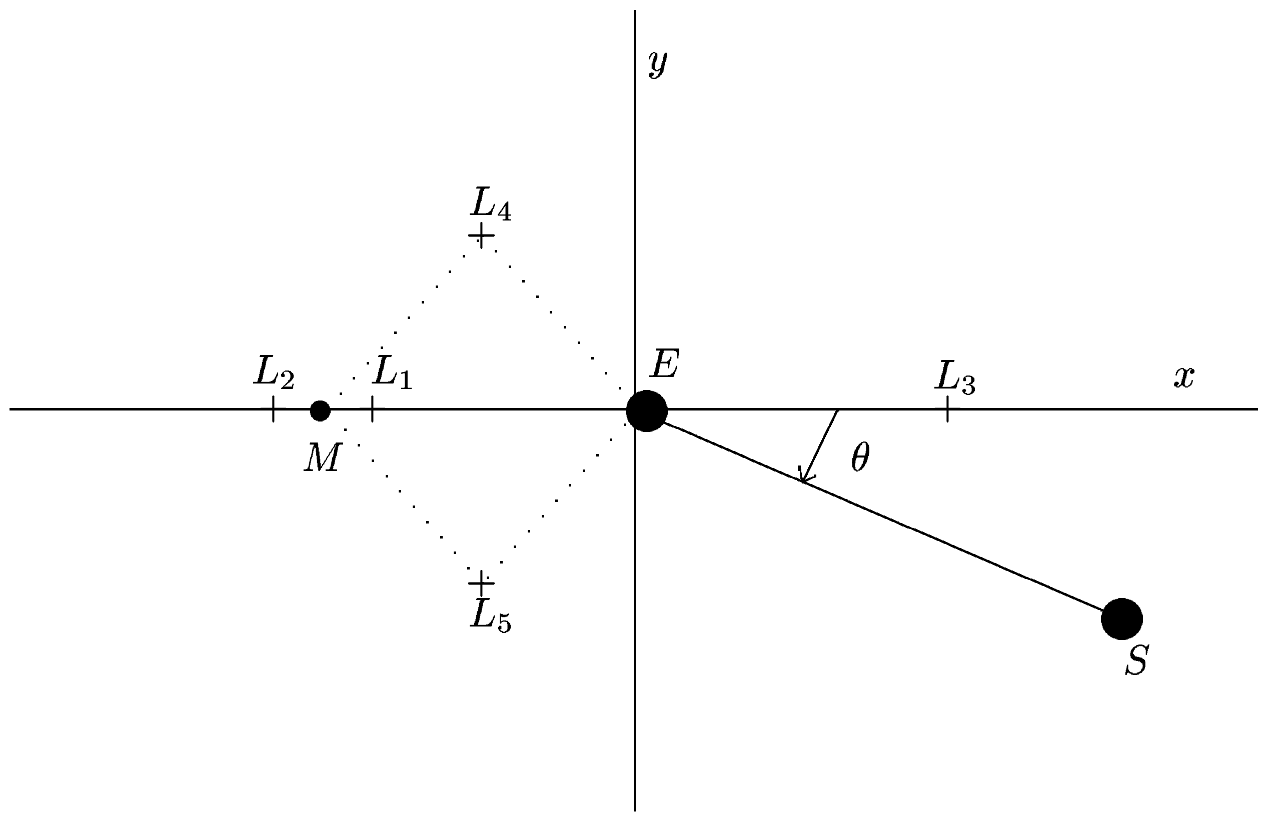
The bicircular problem (BCP)

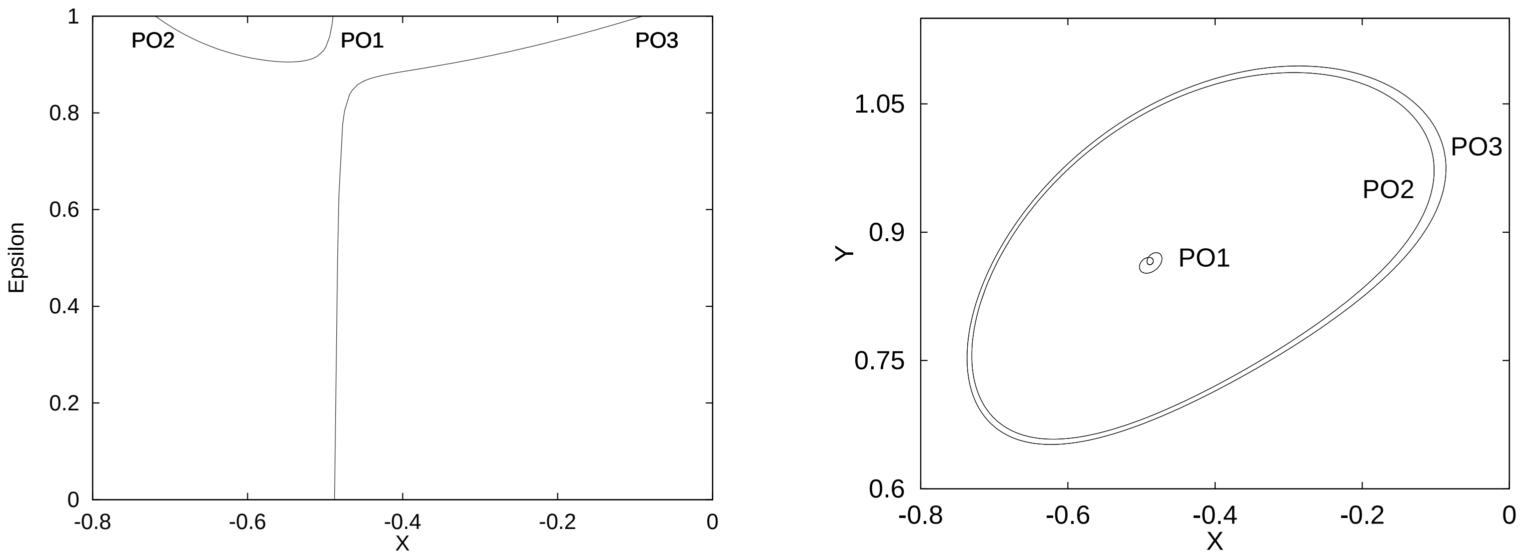
It is a model for the study of the dynamics of a small particle in the Earth-Moon-Sun system.



The BCP can be described by the Hamiltonian system,

$$\begin{aligned} H_{BCP} &= \frac{1}{2}\left(p_x^2+p_y^2+p_z^2\right)+yp_x-xp_y- \\ & \frac{1-\mu}{r_{PE}}-\frac{\mu}{r_{PM}}-\frac{m_S}{r_{PS}}- \\ & \frac{m_S}{a_S^2}\left(y\sin\theta-x\cos\theta\right), \end{aligned}$$

where $r_{PE}^2=(x-\mu)^2+y^2+z^2$, $r_{PM}^2=(x-\mu+1)^2+y^2+z^2$, $r_{PS}^2=(x-x_S)^2+(y-y_S)^2+z^2$, $x_S=a_S\cos\theta$, $y_S=-a_S\sin\theta$, and $\theta=\omega_S t$.



1
0.8
0.6
0.4
0.2
0
Epsilon
-0.8
-0.6
-0.4
-0.2
0
X
PO2
PO1
PO3
1.05
0.9
0.75
0.6
Y
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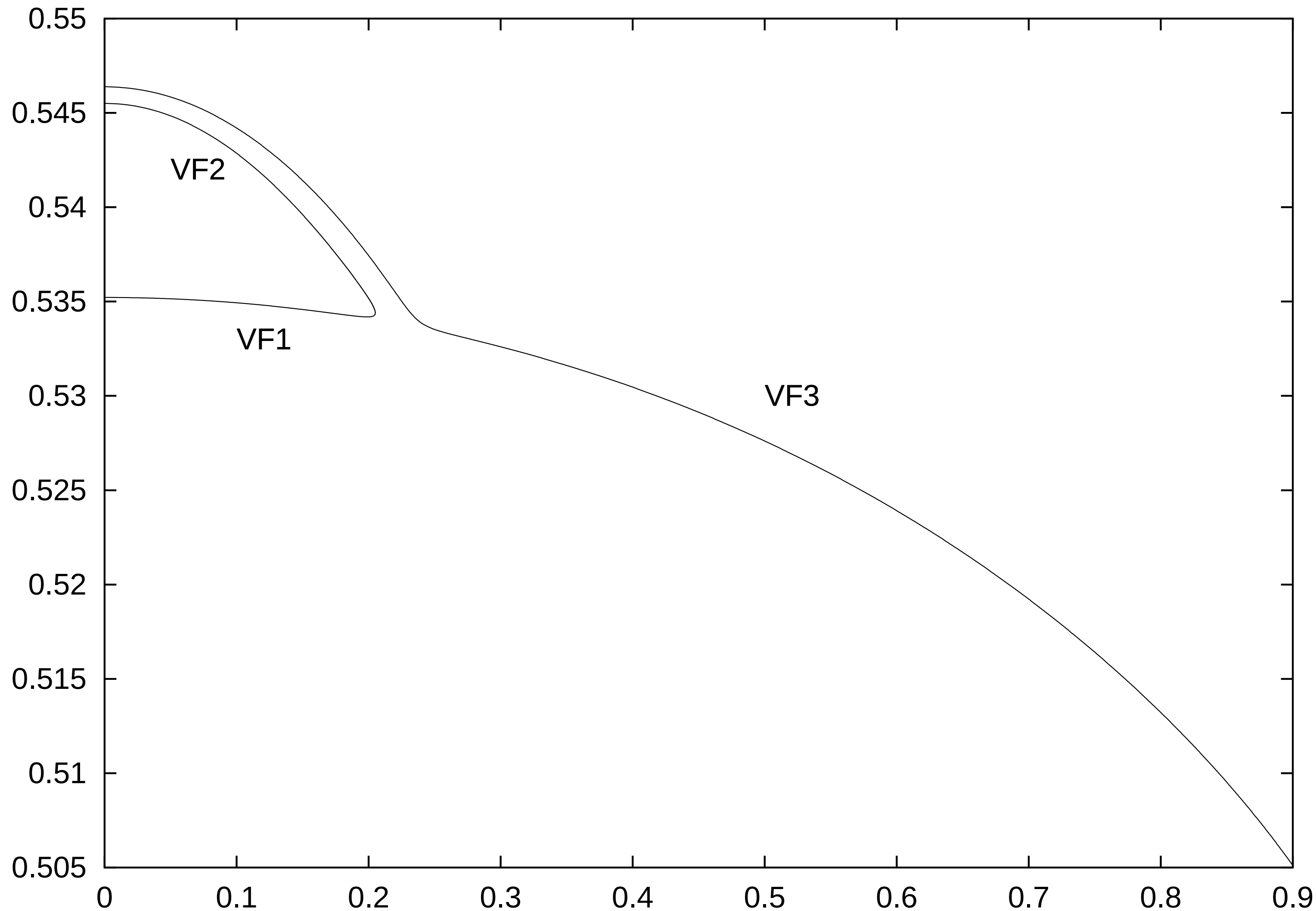
-0.2
0
X
PO3
PO2
PO1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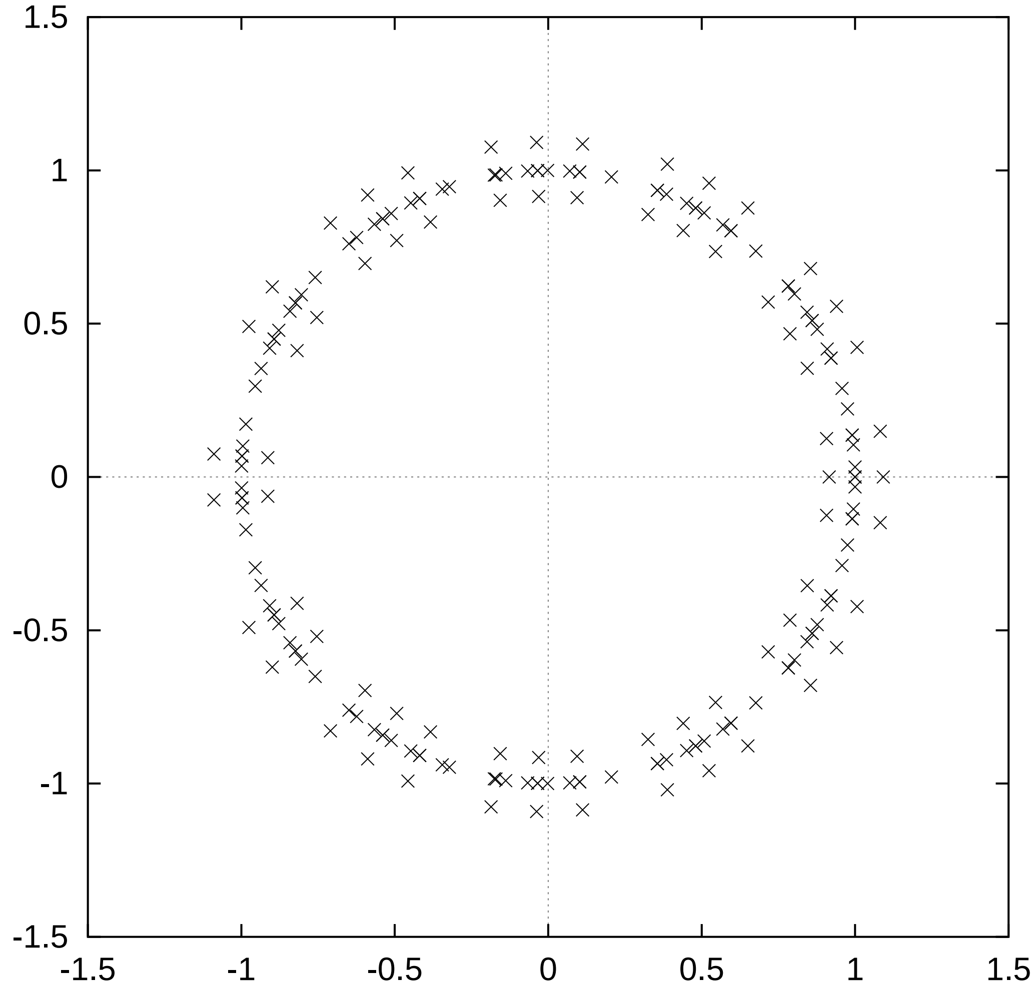
VF2
VF1
VF3

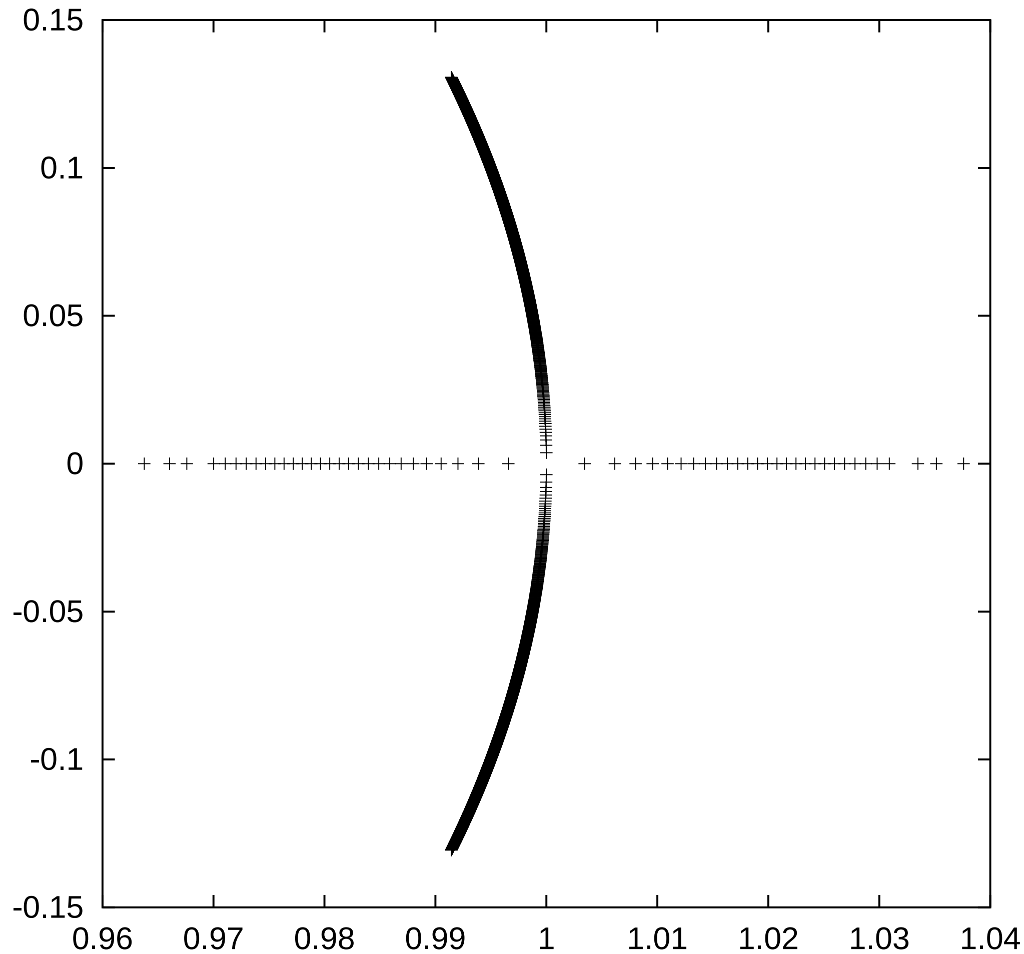


$N = 16$ (total dimension: 198).

	Modulus	Argument
λ_1	1.091942641437887	0.000000000000000
λ_2	0.915799019152856	0.000000000000000
λ_3	0.999999999999985	2.035517841801725
λ_4	0.999999999999985	-2.035517841801725

Normal eigenvalues around an unstable invariant curve of the family VF1. The rotation number is $\omega = 0.535033339385478$, and the value of the $\dot{z}$ coordinate when $z = 0$ is $\dot{z} = 0.080508698608030$.

We can check that $|\lambda_1 \lambda_2 - 1| \approx 4 \times 10^{-15}$.



Motion of one of the couples of eigenvalues in the complex plane, near the change of stability in the families VF1 and VF2.

Growing the stable manifold

We call ψ_j the eigenfunction corresponding to λ_j, $j = 1, \ldots, 4$, and we focus on the couple $(\lambda_1, \psi_1) \in \mathbb{R} \times C(\mathbb{T}^1, \mathbb{R}^n)$.

The linearized unstable manifold is given by $x(\theta) + h\psi_1(\theta)$. To estimate a suitable value for h, we note that

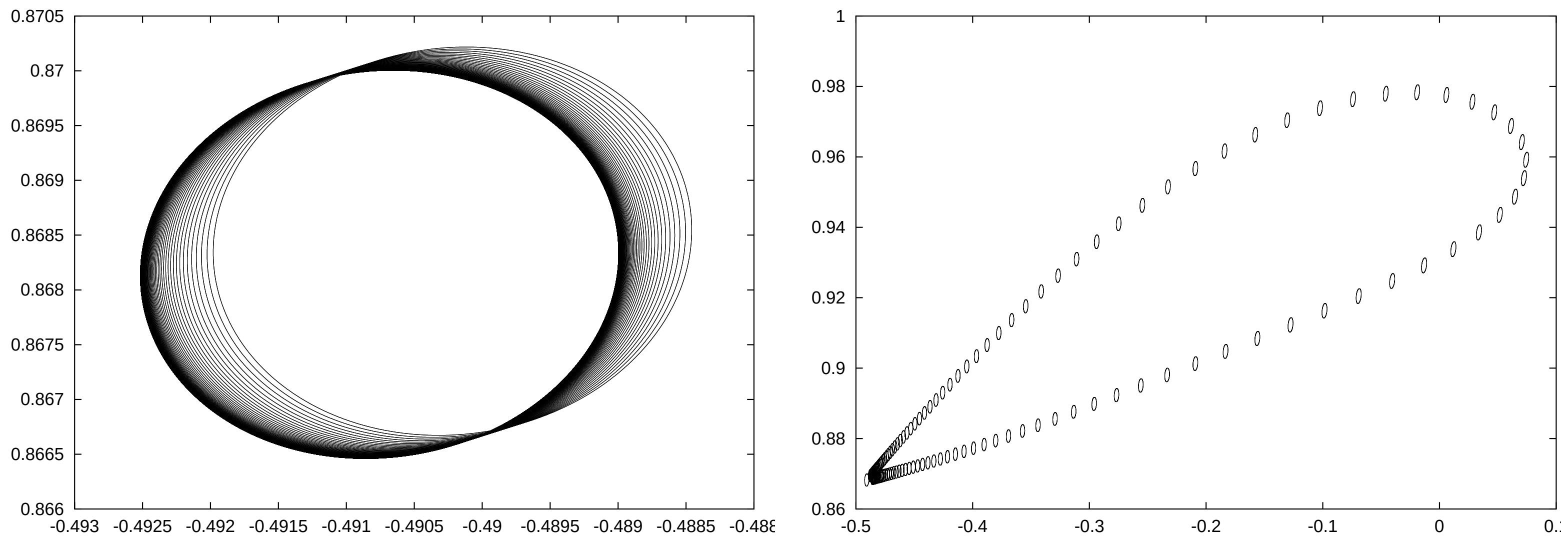
$$\begin{aligned} f(x(\theta) + h\psi_1(\theta)) &= f(x(\theta)) + hD_x f(x(\theta))\psi_1(\theta) + O(h^2) \\ &= x(\theta + \omega) + h\lambda_1\psi_1(\theta + \omega) + O(h^2). \end{aligned}$$

The size of the term $O(h^2)$ can be estimated by

$$E(h) = \max_{\theta \in \mathbb{T}^1} \|f(x(\theta) + h\psi_1(\theta)) - x(\theta + \omega) - h\lambda_1\psi_1(\theta + \omega)\|_2.$$

It follows that $h = 10^{-7}$ is enough to have $E(h) < 10^{-13}$.

We define the curve $C_1 \subset \mathbb{R}^n$ as the image of the map $\theta \mapsto x(\theta) + h\psi_1(\theta)$ and, for $j > 1$, $C_j = f(C_{j-1})$.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1
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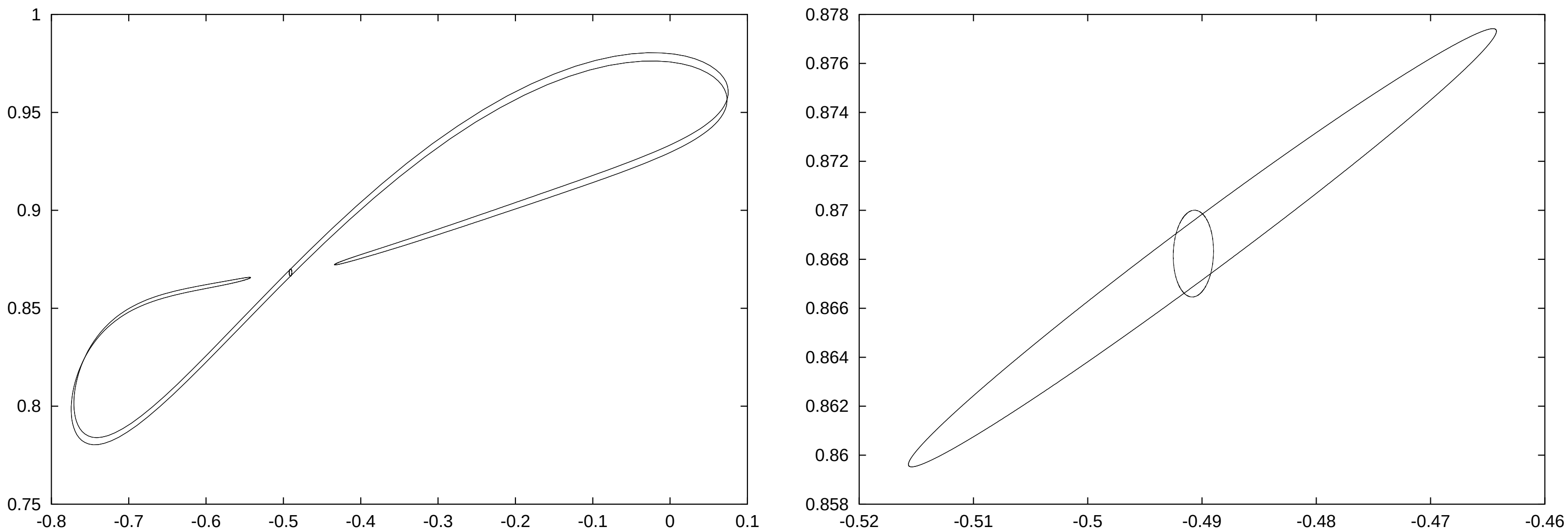
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0.87
0.868
0.866
0.864
0.862
0.86
0.858
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