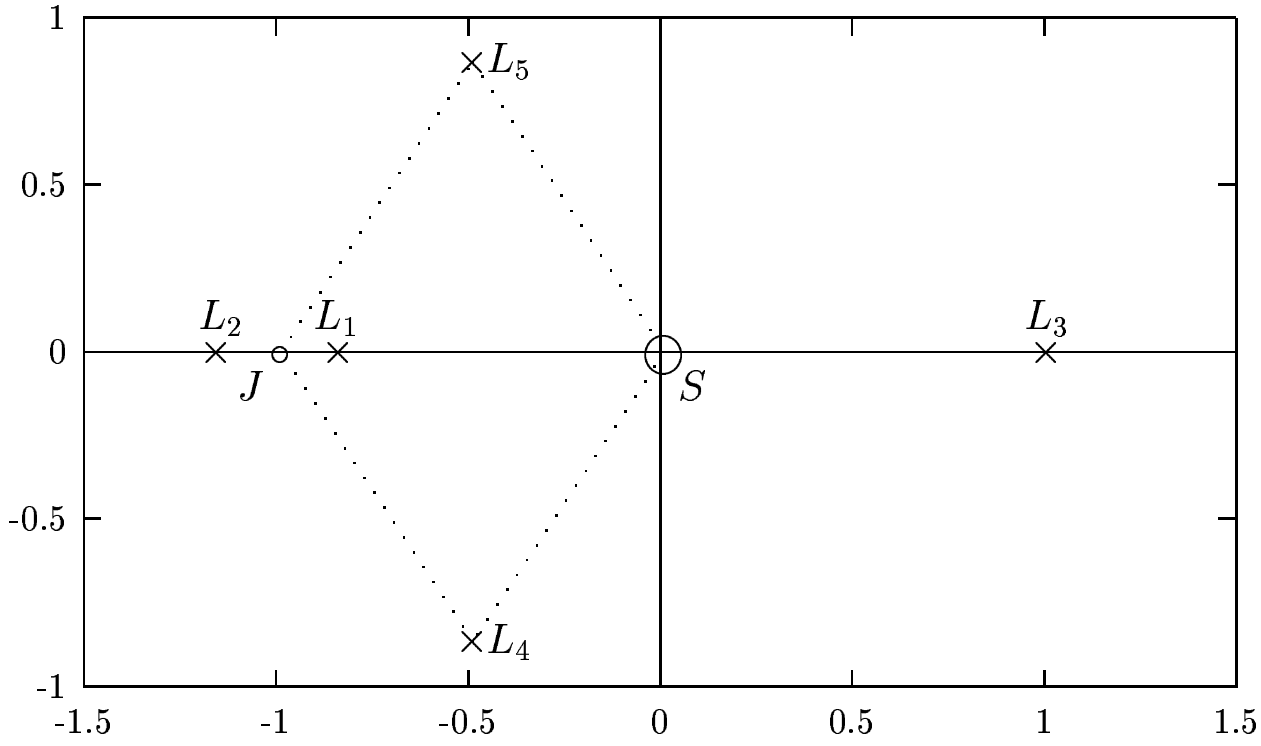
Dynamics near the collinear points of the RTBP

- À. Jorba. A methodology for the numerical computation of normal forms, centre manifolds and first integrals of Hamiltonian systems. *Exp. Math.*, 8(2):155–195, 1999.
- À. Jorba and J. Masdemont. Dynamics in the centre manifold of the collinear points of the Restricted Three Body Problem. *Phys. D*, 132:189–213, 1999.



$$H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + Y P_X - X P_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

We focus on the dynamics around L_2. It is well known that the linear dynamics around this point is of the type saddle×centre×centre.

So the equations of motion can be written as

$$\begin{aligned} \dot{u} &= Au + O_2(u,v), \quad u \in \mathbb{R}^4 \\ \dot{v} &= Bv + O_2(u,v), \quad v \in \mathbb{R}^2 \end{aligned}$$

where the origin is taken at L_2, A is an elliptic matrix and B is an hyperbolic one.

We will kill the coupling between the two equations (by means of changes of variables) up to some order, to obtain

$$\begin{aligned} \dot{u} &= Au + f_n(u,v) + O_{n+1}(u,v), \quad u \in \mathbb{R}^4, \\ \dot{v} &= Bv + g_n(u,v) + O_{n+1}(u,v), \quad v \in \mathbb{R}^2, \end{aligned}$$

such that $g_n(u,0) \equiv 0$. Then, neglecting the remainder $O_{n+1}(u,v)$, we have that $v \equiv 0$ is a solution of the second equation that can be inserted in the first one to obtain

$$\dot{u} = Au + f_n(u,0) + O_{n+1}(u,0), \quad u \in \mathbb{R}^4.$$

Note that we have reduced the system to its (approximate) central manifold.

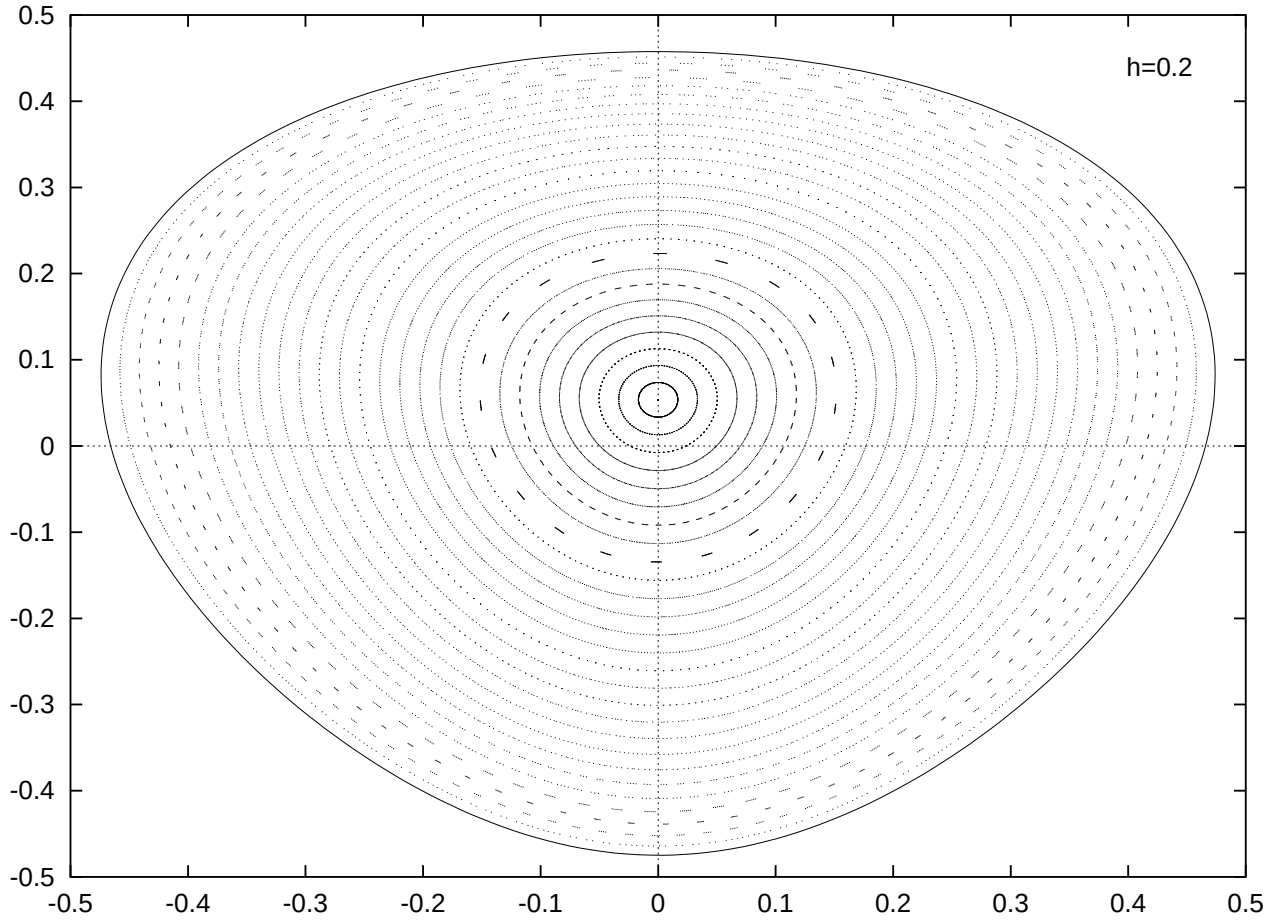
To reduce the computational effort, this procedure is usually applied to the Hamiltonian instead of the differential equations.

Now we are interested in orbits with $v = 0$ (if v is not zero, the solution goes away due to the hyperbolic character of the matrix B), so we can restrict ourselves to the study of the equation for the variable u.

To see the dynamics described by this equation, we observe that

- The phase space is four dimensional.
- We fix the energy level (Hamiltonian). Now the phase space is of dimension three.
- We use $z = 0$ as a surface of section

Then, it is possible to draw the corresponding Poincaré sections.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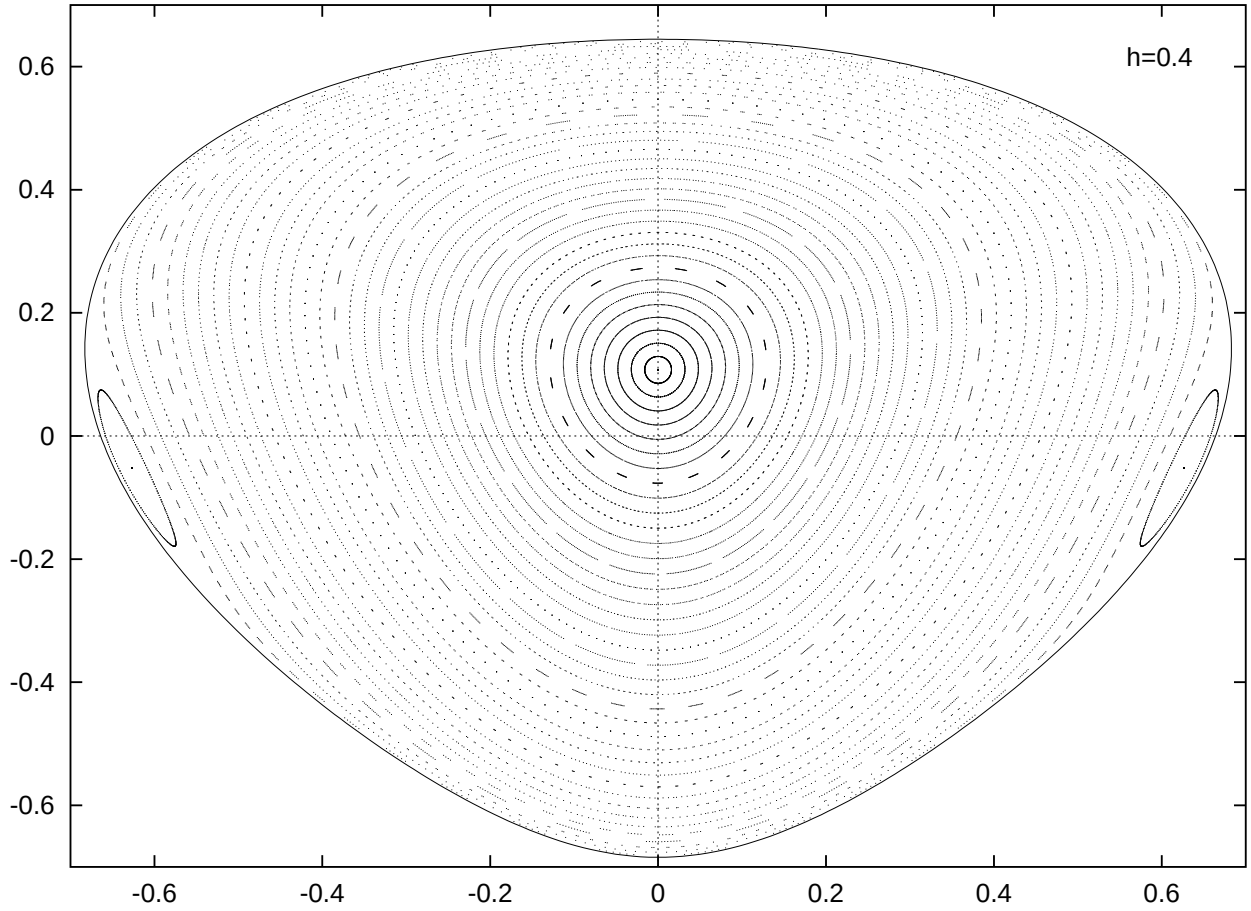
0.3
0.4
0.5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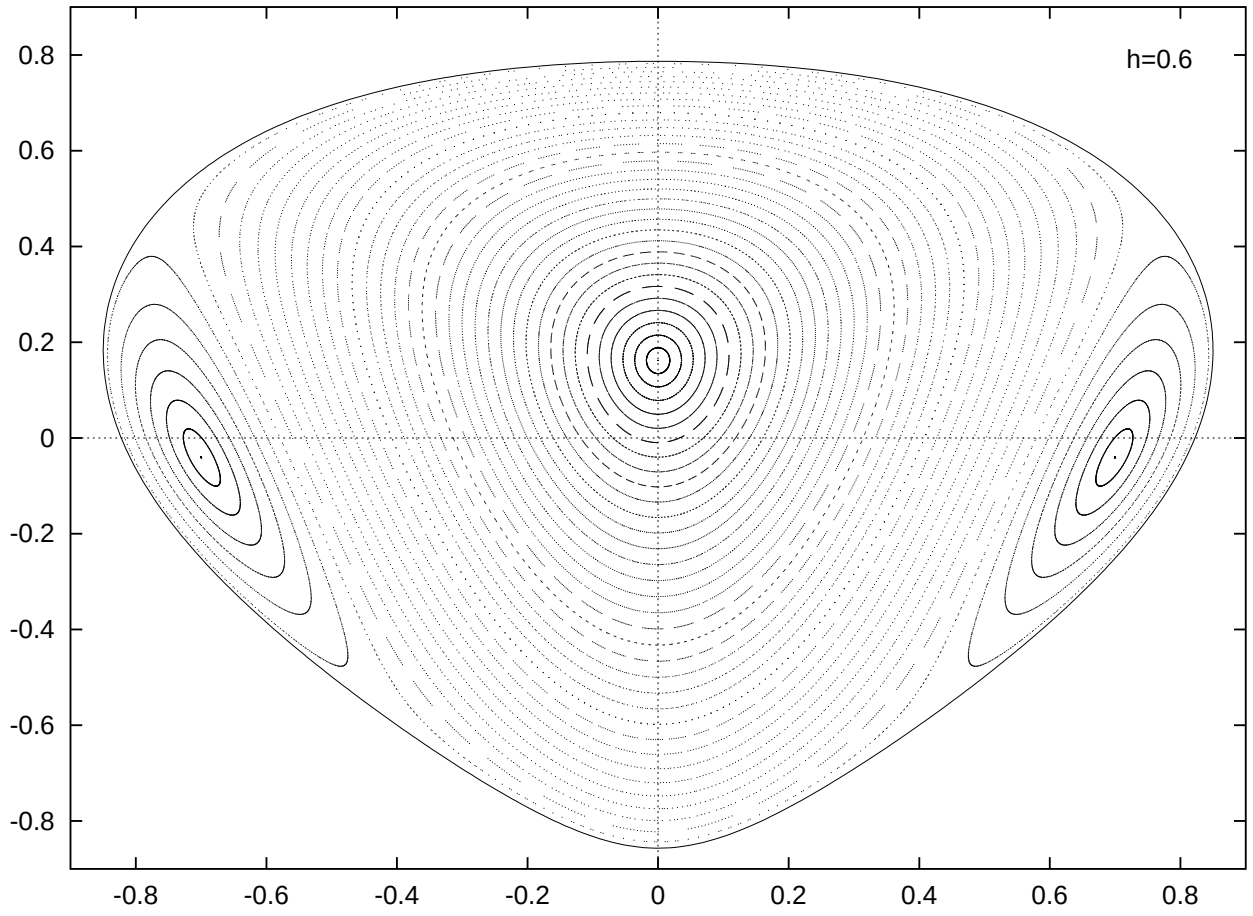
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-0.2
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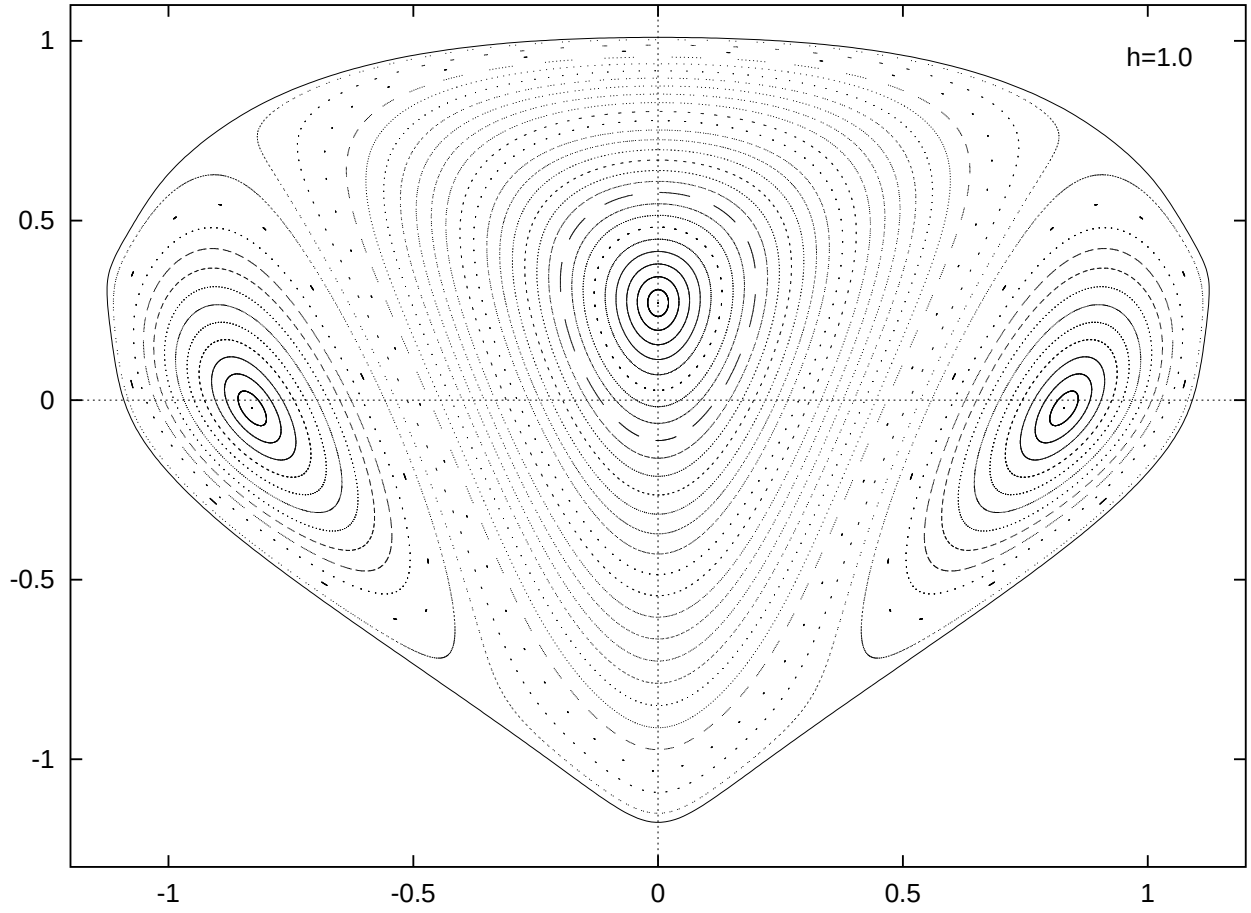
0.4
0.6
0.8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1

Let us see some details of this computation.

The first part is based on the symbolic manipulation of the power series expansion of the Hamiltonian around the equilibrium point.

The second part is simply the numerical integration of the equations restricted to the central manifold.

We will focus on the first part.

To save computational work, we will perform the changes of variables on the Hamiltonian instead of on the equations of motion.

We will say that a change of variables is canonical when it preserves the Hamiltonian form of the equations.

The method we will use for these changes is the so-called Lie series method. To explain it let us define the Poisson bracket of two functions $f(q,p)$, $g(q,p)$ as

$$\{f,g\} = \nabla f^\top J \nabla g = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

If f and g are homogeneous polynomials, then

$$\deg\{f,g\} = \deg f + \deg g - 2.$$

To construct canonical transformations, we will use the following property:

The time 1 flow of a Hamiltonian $G(q,p)$ is a canonical transformation. The result of applying this transformation to a Hamiltonian $H(q,p)$ can be obtained in the following way:

$$H' = H + \{H,G\} + \frac{1}{2!}\{\{H,G\},G\} + \frac{1}{3!}\{\{\{H,G\},G\},G\} + \cdots,$$

Note that this formula is very suitable for computations.

To apply this formula, it is very useful to have H expanded in power series around the equilibrium point,

$$H(q,p) = H_2(q,p) + H_3(q,p) + H_4(q,p) + H_5(q,p) + \cdots,$$

being $H_j(q,p)$ an homogeneous polynomial of degree j. Moreover, we will ask that $H_2(q,p)$ be in complex diagonal form,

$$H_2 = \lambda q_1 p_1 + \omega_2 q_2 p_2 + \omega_3 q_3 p_3, \qquad \lambda \in \mathbb{R},\ \omega_{2,3} \in \mathbb{C}.$$

Note that, to get this form, one is forced to work with complex coefficients.

To compute the reduction to the central manifold, we will proceed degree by degree. To see how this is done, let us see how to arrange degree 3.

Let $G_3(q,p)$ be an homogeneous polynomial of degree 3. The result of transforming H by the time 1 flow of G_3, H', has been given before. The monomials of degree 3 of H' are given by

$$H'_3 = H_3 + \{H_2, G_3\}\,.$$

If

$$G_3(q,p) = \sum_{|k|=3} g_k q^{k_q} p^{k_p}, \qquad H_3(q,p) = \sum_{|k|=3} h_k q^{k_q} p^{k_p},$$

$$H'_3(q,p) = \sum_{|k|=3} h'_k q^{k_q} p^{k_p},$$

then,

$$\{H_2, G_3\} = \sum_{|k|=3} \langle k^p - k^q, (\lambda, \omega_2, \omega_3)\rangle\, g_k q^{k_q} p^{k_p}\,.$$

To "kill" the monomials of H_3' that are coupling the elliptic and the hyperbolic part, we have to select

$$g_k = \frac{-h_k}{\langle k^p - k^q, (\lambda, \omega_2, \omega_3)\rangle},$$

for the values of k corresponding to this coupling (g_k is taken zero otherwise).

This defines the function G_3, so we can compute the transformed Hamiltonian up to a given order, and we can apply again the same scheme to "arrange" fourth order terms.

This process is continued up to a sufficiently high order.