## Patterns for a nonlinear diffusion equation

- N. Cónsul and À. Jorba. On the existence of patterns for a diffusion equation on a convex domain with nonlinear boundary reaction. Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2005. To appear.

We focus on the existence of nonconstant (w.r.t. "space" variables) equilibrium solutions for the following diffusion equations with nonlinear reactions on the boundary,

$$
\left\{\begin{aligned}
u_{t}-\Delta u & =0, & & \text { in } \Omega, t>0 \\
u_{\nu} & =k f(u), & & \text { on } \partial \Omega, t>0 \\
u(0, x) & =\psi(x) \in H^{1}(\Omega) & &
\end{aligned}\right.
$$

Here $\Omega=[0,1]^{2}, k$ is a real positive parameter, and $u_{\nu}$ denotes the outer normal derivate. The boundary nonlinearity is given by a cubic reaction $f(u)=-u(u+1)(u-1)$.

It can be proved analytically that, for all $k>0$,

- $u=0$ is a constant unstable equilibrium
- $u= \pm 1$ are constant stable equilibria

Question: Is there any pattern in this problem?
A pattern is a stable and nonconstant equilibria.

The variational formulation of is

$$
\left\langle u_{t}, v\right\rangle+a(u, v)=k\langle f(u), v\rangle_{\partial \Omega},
$$

for any $v$ in the space of test functions $V$. Here, $\langle\cdot, \cdot\rangle$ and $\langle\cdot, \cdot\rangle_{\partial \Omega}$ denote the scalar products in $L^{2}(\Omega)$ and $L^{2}(\partial \Omega)$, respectively and

$$
a(u, v)=\int_{\Omega} \nabla u \cdot \nabla v d x
$$

We will use the standard finite element formulation, based on linear triangular elements. Therefore, we have a linear space of piecewise linear polynomials, of dimension $N$.

Using the usual approximation in this space and the Galerkin method, we obtain a set of nonlinear ordinary differential equations for the $N$ coefficients $u_{i}, 1 \leq i \leq N$, of the approximations of $u$. The system can be expressed in the matrix form,

$$
\begin{equation*}
A \dot{u}+B u=k F(u), \tag{1}
\end{equation*}
$$

with $u=\left(u_{1}, \ldots, u_{N}\right)^{T}$ and

$$
\begin{aligned}
A & =\left(\left\langle\varphi_{i}, \varphi_{j}\right\rangle\right)_{i, j}, \\
B & =\left(a\left(\varphi_{i}, \varphi_{j}\right)\right)_{i, j}, \\
F(u) & =\left(\left\langle f\left(\sum_{i=1}^{N} u_{i} \varphi_{i}\right), \varphi_{j}\right\rangle_{\partial \Omega}\right)_{j}
\end{aligned}
$$

where $1 \leq i, j \leq N$, and $\left\{\varphi_{i}\right\}_{1 \leq i \leq N}$ is the usual basis in the space of $N$-piecewise linear polynomials.

In the finite element discretization, we have used three meshes (of 8321,33025 and 131585 nodes) combined with two steps of extrapolation.

This allows a significant increase of accuracy.

The equilibrium solutions of (1) satisfy

$$
\begin{equation*}
B u-k F(u)=0 . \tag{2}
\end{equation*}
$$

We will solve this equation by means of the Newton method. The advantage of this approach is that the equilibria are found regardless of their stability.

The continuation procedure is very standard; it is based on including the parameter $k$ as an ordinary unknown.

Assume that $p_{*}^{(0)}=\left(u_{*}^{(0)}, k_{*}^{(0)}\right)$ is an approximation to an equilibrium.

Then, the Newton method requires to solve a linear system with an extra unknown $(k)$.

This implies that, generically, we have a 1-D affine space of solutions. Among them, we select the one of minimum $L^{2}$-norm; this implies that we are looking for the point on the manifold (in the $(u, k)$ space) of solutions closest to the initial condition.

Once the Newton method has converged to some $p^{(0)}=\left(u^{(0)}, k^{(0)}\right)$, the kernel of the linearization at $p^{(0)}$ of the operator $(u, k) \mapsto B u-k F(u)$ gives the unitary tangent vector $\tau^{(0)}$ to the curve of solutions at this point.
Then, we can predict an approximation $p_{*}^{(1)}=p^{(0)}+h \tau^{(0)}$ to a new point of the curve. If the value of $h$ is too large, the Newton method starting at $p_{*}^{(1)}$ will not converge to a point on the curve, and if it is too small we will need to compute a lot of points to advance a fixed distance on the curve.

The adjustement of the value of $h$ is done automatically: if the Newton method needs more than 3 iterates to converge, $h$ is halved; if it needs only 1 , it is doubled.

The stability is found by rewritting (1) as

$$
\dot{u}=A^{-1}(-B u+k F(u)) .
$$

If $u_{0}$ is a solution, the linearization around $u_{0}$ is given by the operator $L: H^{1} \rightarrow H^{1}$,

$$
L v=A^{-1}\left(-B+k D F\left(u_{0}\right)\right) v .
$$

It can be shown that the spectrum of $L$ only consists of real eigenvalues. Therefore we look for couples $(\lambda, v) \in \mathbb{R} \times H^{1}$ such that

$$
\left(-B+k D F\left(u_{0}\right)\right) v=\lambda A v .
$$

The equilibrium $u_{0}$ is asymptotically stable iff the first eigenvalue is strictly positive. The dominant part of the spectra (including the eigenfunctions) has been obtained by means of an inverse power method with a suitable shift.

We will look for families branching off from the constant equilibria.
We know that the constant solutions $u=-1$ and $u=1$ are both asymptotically stable for all $k$ and, therefore, nothing can bifurcate from them.

It is known that, for $k$ sufficiently small, all the equilibrium solutions are constant and, therefore, they must coincide with the zeroes of $f$.

We will look for nonconstant equilibrium solutions as bifurcations of the unstable solution $u=0$. Hence, the first step must be to compute the spectrum of the linearization of the equation at $u=0$ to detect eigenvalues that cross 0 .


Continuation of the first three eigenvalues of the linearization around $u=0$. The value of $k$ is shown in the horizontal axis.

The critical values of $k$ are $k_{1} \approx 1.37650548469797$ and $k_{2} \approx 2.00000000015288$.

These values can be obtained analytically, and are $k_{1}=1.376505484672535 \ldots$ and $k_{2}=2$.

The bifurcation of $u=0$ for $k=k_{1}$ corresponds to a double eigenvalue, for which the corresponding eigenspace is also of dimension 2. To study the neigbourhood of this singular point, we have computed two linearly independent eigenfunctions $v_{1}$ and $v_{2}$.


Eigenfunctions for $k=k_{1}$. Horizontal plane: $(x, y)$ coordinates. Vertical axis: value of $u$. The colour goes from blue for $u=-1$ to red for $u=1$. The intersection between the plane $u=0$ and the box containing the figures is marked with a square.

We have considered the "circle" of values $v_{1} \cos \theta+v_{2} \sin \theta$ for $\theta \in \mathbb{S}^{1}$ as initial conditions for a Newton method (we recall that $k$ is also an unknown)

We have used these initial conditions for a mesh of 1000 equispaced values of $\theta$, and we have only found 8 branches going out from the singular point (in other words, we have only found 4 curves going through the bifurcation point). If we take into account the symmetries of the problem, the 8 branches can be reduced to 2 .


Eigenfunction for $k=k_{2}=2$.

The bifurcation of $u=0$ for $k=k_{2}=2$ corresponds to a single null eigenvalue, whose eigenspace is also of dimension 1.

Therefore, in both cases we have obtained the corresponding eigenfunction $v$ and we have used the values $u \pm h v$, for $h$ small, as initial values for the Newton method.

As before, the initial value for $k$ is the value at the bifurcation point, but the Newton method handles $k$ as a variable.


Schematic representation of the branches that bifurcate from the origin. The value of $k$ is shown in the horizontal axis.

A computation of the eigenvalues of the solutions along the branches shows that the first branch ((1a) in the previous figure) is unstable and changes its stability at $k=k_{1}^{s} \approx 2.84083164$ becoming stable.

The second branch ((1b) in the previous figure) is always unstable.


Change of stability on the branch (1a). Left: Critical equilibrium at $k=k_{1}^{s}$. Right: Kernel of the linearization around this equilibrium; $u=0$ corresponds to the lower side of the box. The colour goes from blue for $u=-1$ to red for $u=1$.

