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BASIC ELEMENTS OF THE THEORY OF NONAUTONOMOUS DYNAMICAL SYSTEMS

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1. Topological dynamics

Let us recall some concepts from topological dynamics.

Definition 1.1. Let X be a metric space. For each $t \in \mathbb{R}$, let $\tau_t : X \to X$ be a homeomorphism. The pair $(X, \{\tau_t\})$ is called a *flow* or *dynamical system* if the following conditions hold:

- $\tau_0(x) = x$ for all $x \in X$;
- $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \in \mathbb{R}$;
- $\tau : X \times \mathbb{R} \to X,\ (x,t) \mapsto \tau_t(x)$ is continuous.

We write $(X, \{\tau_t\})$ or $(X, \mathbb{R})$ to denote a flow.

Sometimes we write $\tau_t(x) = x \cdot t$. The second of the above conditions is called the group property. If $x \in X$, the *orbit* or *trajectory* through x is by definition $\{\tau_t(x) : t \in \mathbb{R}\} = \mathcal{O}(x)$.

The terms *positive semiorbit* and *negative semiorbit* have the natural meanings. A subset $X_1 \subseteq X$ is called *invariant* if, whenever $x \in X_1$ and $t \in \mathbb{R}$, we have $x \cdot t \in X_1$. One defines positive invariance and negative invariance in the natural way.

Definition 1.2. Let $(X, \{\tau_t\})$ be a flow. If $x \in X$, then we define the *ω-limit set* as $\omega(x) = \{y \in X :$ there is a sequence $t_n \to \infty$ such that $y = \lim\limits_{n\to\infty} x \cdot t_n\}$. The *$\alpha$-limit set* $\alpha(x)$ is defined similarly using sequences $t_n \to -\infty$.

Definition 1.3. Let $(X, \{\tau_t\})$ be a flow and let $M \subseteq X$ be a nonempty compact $\{\tau_t\}$-invariant set. The flow $(M, \{\tau_t\})$ is called *minimal* if M contains no nonempty proper closed $\{\tau_t\}$-invariant subset. One abuses language and says that M is a minimal subset of X.
The flow $(X, \mathbb{R})$ is *minimal* if X itself is minimal.

Proposition 1.4. *Let $(X, \{\tau_t\})$ be a flow on a compact metric space. The following statements are equivalent:*

1. *X is a minimal flow;*
2. *all trajectories lying in X are dense in X.*

Proof. Exercise. □

Proposition 1.5. *If X is a compact (nonempty) metric space and $(X, \{\tau_t\})$ is a flow, then X contains a minimal subset.*

Proof. Consider $\mathcal{M} = \{M \subseteq X : M$ is compact, $\{\tau_t\}$-invariant and nonempty$\}$. If $M_1, M_2 \in \mathcal{M}$, then say that $M_1 \leq M_2$ if and only if $M_1 \subseteq M_2$.
Consider a totally ordered subset $\{M_\alpha : \alpha \in A\}$ of $\mathcal{M}$. Then, by the finite intersection property, $\bigcap\limits_{\alpha \in A} M_\alpha$ is nonempty and so belongs to $\mathcal{M}$.

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By Zorn's lemma, there is a minimal element M_* of $\mathcal{M}$. It is easy to check that M_* is indeed a minimal subset of X. $\square$

Exercise 1.6. If X is a metric space, $(X, \{\tau_t\})$ is a flow, and if $x \in X$ is a point whose positive semiorbit is contained in a compact subset of X, then $\omega(x)$ is a nonempty compact $\{\tau_t\}$-invariant connected subset of X.

Definition 1.7. Let X and Y be a metric space, and let $(X, \mathbb{R})$ and $(Y, \mathbb{R})$ be topological flows. A continuous map $\pi : X \to Y$ is called a *flow homomorphism* if $\pi(x \cdot t) = \pi(x) \cdot t$ for all $x \in X, t \in \mathbb{R}$. A flow homomorphism $\pi : X \to Y$ is called a *flow isomorphism* if it is also a homeomorphism onto Y.

Example 1.8. (a) Let $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the standard n-torus. Let $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ be numbers which are independent over $\mathbb{Q}$ in the sense that, if $\sum_{i=1}^{n} q_i\gamma_i = 0$ for rationals $q_1, \ldots, q_n$, then $q_1 = q_2 = \ldots = q_n = 0$. If $t \in \mathbb{R}$ and $(\psi_1, \ldots, \psi_n) \in \mathbb{T}^n$, define

$$\tau_t(\psi_1, \ldots, \psi_n) = (\psi_1 + \gamma_1 t, \ldots, \psi_n + \gamma_n t)$$

where all coordinates are taken $\mod 1$. Then $(X, \{\tau_t\})$ is a *Kronecker flow*. One can show that a Kronecker flow is minimal, since every orbit must be dense (see [20]); this will be checked in Proposition 2.9. The numbers $\gamma_1, \ldots, \gamma_n$ are called *frequencies*.

(b) Let $\mathbb{T}^n$ be the n-torus, and let $\cdot$ denote a Kronecker flow on $\mathbb{T}^n$. Let $r : \mathbb{T}^n \to \mathbb{R}$ be a continuous function. For each $t \in \mathbb{R}$ define

$$\begin{array}{rccc} \tau_t : & \mathbb{T}^{n+1} = \mathbb{T}^n \times \mathbb{T} & \longrightarrow & \mathbb{T}^{n+1} \\ & (\psi, \varphi) & \mapsto & (\psi \cdot t, \varphi + \int_0^t r(\psi \cdot s)ds) \end{array}$$

where $\psi = (\psi_1, \ldots, \psi_n), r : \mathbb{T}^n \to \mathbb{R}$ and all coordinates are taken $\mod 1$. Then $(\mathbb{T}^{n+1}, \{\tau_t\})$ is a flow called *Furstenberg flow*.

(c) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz continuous function. Suppose that, for each $x_0 \in \mathbb{R}^n$, the solution $\varphi(t, x_0)$ of the Cauchy problem

$$\left\{ \begin{array}{lcl} x' & = & f(x) \\ x(0) & = & x_0 \end{array} \right. \tag{1.1}$$

exists on the interval $-\infty < t < \infty$, i.e. f is *complete*. Define

$$\begin{array}{rccc} \tau_t : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x_0 & \mapsto & \varphi(t, x_0). \end{array}$$

Then $(\mathbb{R}^n, \{\tau_t\})$ is a flow. Let us verify the group property. If $x_0 \in \mathbb{R}^n$, set $\psi(s) = \tau_{t+s}(x_0) = \varphi(t + s, x_0)$ and verify that $\frac{d\psi}{ds} = f(\psi(s))$ and that $\psi(0) = \tau_t(x_0)$. Since each Cauchy problem (1.1) has a unique solution, we must have $\psi(s) = \tau_s(\tau_t(x_0))$.

(d) Let X be a compact C^2-manifold and let $V : X \to X$ be a C^1 vector field. For $x_0 \in X$, let $\varphi(t, x_0)$ be the solution of $x' = V(x)$ satisfying $\varphi(0, x_0) = x_0$. By compactness of X, $\varphi(\cdot, x_0)$ exists on $(-\infty, \infty)$ for all $x_0 \in X$, so setting $\tau_t(x_0) = \varphi(t, x_0)$ and reasoning as above one obtains a flow $(X, \{\tau_t\})$.

(e) Suppose that $X = \mathbb{T}^2$ is the 2-torus. Let $\psi = (\psi_1, \psi_2)$ be angular coordinates $\mod 1$ on $\mathbb{T}^2$. Let $V(\psi) = (V_1(\psi), V_2(\psi))$ be a C^1 vector field on $\mathbb{T}^2$ such that $V_1(\psi) \neq 0$ for all $\psi \in \mathbb{T}^2$. Let $C = \{(\psi_1, \psi_2) : \psi_2 = 0\}$, and let $f : C \to C$ be the first-return map obtained by solving the problem

$$\left\{ \begin{array}{lcl} \psi' & = & V(\psi) \\ \psi(0) & = & (\psi_1, 0) \end{array} \right.$$

for each $\psi_1 \in [0, 1)$. Then the qualitative behavior of the orbits of the flow $\{\tau_t\}$ is described by that of the iterates $\{f^{(n)} : n \in \mathbb{Z}\}$ of f. In turn, the qualitative behavior of $\{f^{(n)} : n \in \mathbb{Z}\}$ can be described using the rotation number

$$\rho = \lim_{n\to\infty} \frac{\widetilde{f}^{(n)}(\psi_1)}{n}$$

where $\psi_1 \in \mathbb{R}$ and $\widetilde{f}$ is a lift of f to $\mathbb{R}$. If ρ is rational, then f has a periodic point and so $\{\tau_t\}$ has periodic orbits. If ρ is irrational and V is of class C^2, then the flow $(\mathbb{T}^2, \{\tau_t\})$ is minimal; this is a consequence of the fact that there is a homeomorphism $h : C \to C$ such that $h \circ f \circ h^{-1}$ equals the rigid rotation $R_\rho : \psi_1 \mapsto \psi_1 + \rho$ on C. If V is only of class C^1, it may happen that no such homeomorphism exists. In this case, the flow $(\mathbb{T}^2, \{\tau_t\})$ admits a unique minimal set which is a "strange attractor". See [4, chapter 17] for further details.

2. Ergodic theory

We recall some facts from ergodic theory. They concern regular Borel measures which are invariant/ergodic with respect to a given flow.

Fix a compact metric space X and a flow $\{\tau_t\}$ on X.

Definition 2.1. Let μ be a regular Borel probability measure on X. Say that μ is *invariant* if $\mu(\tau_t(B)) = \mu(B)$ for each Borel set $B \subseteq X$ and each $t \in \mathbb{R}$. Suppose in addition that the measure is indecomposable in the sense that, if $B \subseteq X$ is a Borel set such that $\mu(\tau_t(B)\triangle B) = 0$ for all $t \in \mathbb{R}$, then $\mu(B) = 0$ or $\mu(B) = 1$ ($\triangle$ denotes the symmetric difference: $A\triangle B = (A \cup B) \setminus (A \cap B)$). Then μ is said to be *ergodic*.

Theorem 2.2. *There exists a measure μ on X which is $\{\tau_t\}$-ergodic.*

Proof. We give a proof which has an old-fashioned appearance but which illustrates a remarkable, useful and flexible technique of Krylov and Bogoliubov. This technique is presented in [17], and it is based on the Riesz representation theorem (see [19]).
Let Λ be a bounded linear functional on $C(X)$ such that $\|\Lambda\| = 1$ and $\Lambda(f) \geq 0$ whenever $f \geq 0$. Set $\tau_t(f)(x) = f(\tau_{-t}(x))$ for $t \in \mathbb{R}, x \in X, f \in C(X)$. Say that Λ is $\{\tau_t\}$-invariant if $\Lambda(\tau_t(f)) = \Lambda(f)$ for all $t \in \mathbb{R}, f \in C(X)$. It can be checked that, if Λ is $\{\tau_t\}$-invariant in this sense, then the corresponding regular Borel measure μ has the property that $\mu(\tau_t(B)) = \mu(B)$ for each $t \in \mathbb{R}$ and each Borel set $B \subseteq X$. So such a functional Λ corresponds to an $\{\tau_t\}$-invariant measure μ on X.

Let us construct a bounded linear functional Λ on $C(X)$ such that $\Lambda(1) = 1, \Lambda(f) \geq 0$ whenever $f \geq 0$, and Λ is $\{\tau_t\}$-invariant in the above sense. Let $V = \{f_1, f_2, \ldots, f_n, \ldots\}$ be a countable dense subset of $C(X)$ which is a vector space over $\mathbb{Q}$. Fix $x \in X$. There is a sequence $(t_n^{(1)}) \to \infty$ such that

$$\lim_{n\to\infty} \frac{1}{t_n^{(1)}} \int_0^{t_n^{(1)}} f_1(\tau_s(x))ds \text{ exists;}$$

call the limit $\Lambda(f_1)$. There is a subsequence $(t_n^{(2)}) \to \infty$ of $(t_n^{(1)})$ such that

$$\lim_{n\to\infty} \frac{1}{t_n^{(2)}} \int_0^{t_n^{(2)}} f_2(\tau_s(x))ds \text{ exists;}$$

call the limit $\Lambda(f_2)$. Continuing in this way, we obtain, for each $k \in \mathbb{N}$, a sequence $(t_n^{(k)}) \to \infty$ such that

$$\lim_{n\to\infty} \frac{1}{t_n^{(k)}} \int_0^{t_n^{(k)}} f_k(\tau_s(x))ds = \Lambda(f_k) \text{ exists.}$$

Let $(t_n^{(n)})$ be the diagonal sequence, and note that, for each $f \in V$, one has

$$\Lambda(f) = \lim_{n\to\infty} \frac{1}{t_n^{(n)}} \int_0^{t_n^{(n)}} f(\tau_s(x))ds$$

It is easy to see that Λ is $\mathbb{Q}$-linear, that $\Lambda(f) \geq 0$ whenever $f \geq 0$ and that $\Lambda(1) = 1$. Therefore Λ extends uniquely to a bounded linear functional (denoted by Λ) on $C(X)$. One checks that this functional Λ is $\{\tau_t\}$-invariant, and hence so is μ.

Krylov and Bogoliubov proved the existence of a $\{\tau_t\}$-ergodic measure on X by analyzing so-called regular points; see [17]. This time we will not follow their arguments but will appeal to an abstract result. Let $C^*(X)$ be the dual space of $C(X)$ with the weak-$*$ topology. Then C^* is a locally convex topological vector space. Let $I = \{\Lambda \in C^*(X) : \Lambda(1) = 1, \Lambda \geq 0$ whenever $f \geq 0$, and λ is $\{\tau_t\}$-invariant $\}$. Then I is compact and convex and, by the Krein-Milman theorem, I is the compact convex hull of its extreme points. As an exercise, one can prove that an extreme point of I corresponds via the the Riesz theorem to a $\{\tau_t\}$-ergodic measure μ on X. This completes the proof. □

Next we state without proof the Birkhoff ergodic theorem (in the special case when X is a compact metric space and $\{\tau_t\}$ defines a continuous flow on X).

Theorem 2.3. *Let μ be a $\{\tau_t\}$-ergodic measure on X and let $f \in L^1(X, \mu)$. There is a set $X_f \subseteq X$ satisfying $\mu(X_f) = 1$ such that, if $x \in X_f$, then*

$$\lim_{|t|\to\infty} \frac{1}{t} \int_0^t f(\tau_s(x))ds = \int_X f d\mu.$$

For a proof see, e.g., [17]. The conclusion of the Birkhoff theorem can be strengthened in the case when the flow $(X, \{\tau_t\})$ admits a *unique* $\{\tau_t\}$-invariant measure μ (which must then be $\{\tau_t\}$-ergodic).

Proposition 2.4. *Suppose that $(X, \mathbb{R})$ is uniquely ergodic; that is, there is just one $\{\tau_t\}$-invariant measure μ. Let $f \in C(X)$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(x \cdot s)ds = \int_X f d\mu,$$

where the limit exists and equality holds for all $x \in X$. Moreover the convergence is uniform in $x \in X$.

This can be proved using a variant of the Krylov-Bogoliubov argument.

Let us illustrate the concept of ergodic measure in the context of almost periodic flows and some related Furstenberg flows. First we open a parenthesis and discuss almost periodic flows.

Definition 2.5. Let X be a compact metric space, and let $\{\tau_t\}$ be a flow on X. Say that the flow $(X, \mathbb{R})$ is *(Bohr) almost periodic* if it is minimal and if there is a metric $\tilde{d}$ on X (compatible with the topology on X) such that

$$\tilde{d}(x \cdot t, y \cdot t) = \tilde{d}(x, y) \text{ for all } x, y \in X, t \in \mathbb{R}.$$

One can prove that $(X, \mathbb{R})$ is Bohr almost periodic if and only if it is minimal and, to every $\varepsilon > 0$, there corresponds $T > 0$ such that each interval $[a, a + T] \subseteq \mathbb{R}$ contains s such that

$$d(x \cdot s, x) \leq \varepsilon \text{ for all } x \in X,$$

where d is any metric on X.

Its is well known that the phase space X of an almost periodic flow $(X, \mathbb{R})$ can be given the structure of a compact abelian topological group which admits $\mathbb{R}$ as a dense subgroup. Let us review how such a group structure can be defined. Let $e \in X$. Since $(X, \mathbb{R})$ is minimal, the orbit $\{e \cdot t : t \in \mathbb{R}\}$ is dense in X (why?). If $x = \lim_{k\to\infty} e \cdot t_k$ and $y = \lim_{k\to\infty} e \cdot s_k$ are points in X, set

$$x * y = \lim_{k\to\infty} e \cdot (t_k + s_k).$$

Exercise 2.6. Show that $*$ is a well-defined continuous group operation on X with identity e. If $x = \lim_{k\to\infty} e \cdot t_k$, then $x^{-1} = \lim_{k\to\infty} e \cdot (-t_k)$.

It is clear that the additive group $(\mathbb{R}, +)$ embeds densely in the compact abelian topological group $(X, *)$ via the group homomorphism $i : \mathbb{R} \to X,\ t \mapsto e \cdot t$.

Next, let $\widehat{X}$ be the character group of X:

$$\widehat{X} = \{\chi : X \to \mathbb{T} = \mathbb{R}/\mathbb{Z}\ :\ \chi \text{ is a continuous homomorphism of groups }\}.$$

Then $\widehat{X}$ is indeed an abelian group with respect to pointwise multiplication. It happens to be discrete in the compact-open topology (Pontryagin). Each character $\chi \in \widehat{X}$ is uniquely determined by its restriction to the dense subgroup $i(\mathbb{R}) \subseteq X$. Now

$$\chi(e \cdot t) = e^{2\pi i\lambda t},\ t \in \mathbb{R},$$

for a uniquely determined number $\lambda \in \mathbb{R}$.

Definition 2.7. The *frequency module* $\mathfrak{M}_X$ of the almost periodic flow $(X, \mathbb{R})$ is

$$\mathfrak{M}_X = \{\lambda \in \mathbb{R} : \lambda \text{ is determined as above by a character } \chi \in \widehat{X}\}.$$

The frequency module $\mathfrak{M}_X$ is a countable subgroup of $\mathbb{R}$.

Exercise 2.8. Let $X = \mathbb{T}^n$ and let $(X, \mathbb{R})$ be a Kronecker flow with frequencies $\gamma_1, \ldots, \gamma_n$. Then

$$\mathfrak{M} = \left\{ \sum_{i=1}^{n} z_i\gamma_i : z_i \in \mathbb{Z}, 1 \le i \le n \right\}.$$

Proposition 2.9. *A Kronecker flow is minimal.*

Proof. Let us prove this result for $n \ge 2$. Let $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$; let $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ be $\mathbb{Q}$-independent. Define

$$\tau_t(\psi_1, \ldots, \psi_n) = (\psi_1 + \gamma_1 t, \ldots, \psi_n + \gamma_n t),\ (\psi_1, \ldots, \psi_n) \in \mathbb{T}^n,\ t \in \mathbb{R}.$$

Let us check that the orbit $\{\tau_t(0, \ldots, 0) = (\gamma_1 t, \ldots, \gamma_n t) : t \in \mathbb{R}\}$ is dense in $\mathbb{T}^n$. Let $Y = \mathrm{cls}\{\tau_t(0, \ldots, 0) : t \in \mathbb{R}\} \subseteq \mathbb{T}^n$; note that Y contains a dense subgroup which is isomorphic to $\mathbb{R}$. Then Y is a closed connected subgroup of $\mathbb{T}^n$; therefore, it is a Lie group, so it is either discrete or isomorphic to $\mathbb{T}^l$ for some $l \in \mathbb{N}$ (and it is not discrete since it is connected). But we know that the frequency module of $(Y, \mathbb{R})$ is

$$\left\{ \sum_{i=1}^{n} z_i\gamma_i : z_i \in \mathbb{Z}, 1 \le i \le n \right\}.$$

Consequently, Y has rank n and so $l = n$. □

Again, let $(X, \{\tau_t\})$ be an almost periodic (minimal) flow. If $B \subseteq X$ is a Borel set and $x \in X$, write $B*x = \{b*x : b \in B\}$. It is known that there is a unique Borel regular probability measure ν on X such that

$$\nu(B * x) = \nu(B)$$

for all Borel sets $B \subseteq X$ and all $x \in X$. This measure ν is called the (normalized) *Haar measure* on X. It is clear that ν is $\{\tau_t\}$-invariant: just restrict x to $\{e \cdot t : t \in \mathbb{R}\}$.

Exercise 2.10. Let μ be a $\{\tau_t\}$-invariant measure on X. Let $B \subseteq X$ be a Borel set and let $x \in X$. Then $\mu(B * x) = \mu(B)$.

It follows from this exercise that the $\{\tau_t\}$-invariant measure μ coincides with the normalized Haar measure on X. This implies that ν is the unique $\{\tau_t\}$-invariant measure on X, so ν is actually $\{\tau_t\}$-ergodic. We thus have a class of examples of uniquely ergodic flows, namely the almost periodic flows.

Next let $X = \mathbb{T}^n$, and let $(X, \mathbb{R})$ be a Kronecker flow with frequencies $\gamma_1, \ldots, \gamma_n$. Let ν be the normalized Haar measure on X. Let $r : X \to \mathbb{R}$ be a continuous function, and let

$$r_0 = \int_X r d\nu = \lim_{|t| \to \infty} \frac{1}{t} \int_0^t r(x \cdot s) ds$$

where the limit is uniform in $x \in X$. Set

$$\tau_t(x, \varphi) = \left(x \cdot t, \varphi + \int_0^t r(x \cdot s) ds \right)$$

where $x \in X, \varphi \in \mathbb{R}$, and $t \in \mathbb{R}$. Then $\{\tau_t\}$ is a Furstenberg flow on $X \times \mathbb{T} \cong \mathbb{T}^{n+1}$. Let m be the normalized Lebesgue measure on the circle $\mathbb{T}$ and let $\mu = \nu \times m$.

Let us study the Furstenberg flow $(X \times \mathbb{T}, \{\tau_t\})$. We do this via some exercises, the solutions of which may require some hints from [12].

Exercise 2.11. The measure μ is $\{\tau_t\}$-invariant.

It is convenient to divide the class of $\mathcal{C}$ of continuous real-valued functions on X into two subclasses, namely

$$\mathcal{C}_1 = \{r \in \mathcal{C} : \int_0^t [r(x \cdot s) - r_0]ds \text{ is bounded on } \mathbb{R} \text{ for some } x \in X\},$$

and $\mathcal{C}_2 = \mathcal{C} \backslash \mathcal{C}_1$ (complement).

Exercise 2.12. (a) If $r \in \mathcal{C}_1$, then there is a continuous function $R : X \to \mathbb{R}$ such that

$$R(x \cdot t) - R(x) = \int_0^t [r(x \cdot s) - r_0]ds$$

for all $x \in X$ and all $t \in \mathbb{R}$.

(b) If $r \in \mathcal{C}_1$ and if r_0 is rationally independent of $\gamma_1, \dots, \gamma_n$, then the flow $(X \times \mathbb{T}, \{\tau_t\})$ is isomorphic to the Kronecker flow on $\mathbb{T}^{n+1} \cong X \times \mathbb{T}$ with frequencies $\gamma_1, \dots, \gamma_n, r_0$. In this case, μ is the unique $\{\tau_t\}$-ergodic measure on $X \times \mathbb{T}$.

(c) Let $r \in \mathcal{C}_1$. If $\mathfrak{M}_X$ is the frequency module of $(X, \mathbb{R})$ and if $kr_0 \in \mathfrak{M}_X$ for some integer $k \geq 1$, then $X \times \mathbb{T}$ laminates into minimal subsets, each of which is almost periodic and each of which is a k-cover of X under the natural projection $\pi : X \times \mathbb{T} \to X$. (It is assumed that k is the minimal integer ≥ 1 such that $kr_0 \in \mathfrak{M}_X$.)

In fact, one can prove the following result.

Proposition 2.13. *Let $(X, \mathbb{R})$ be a minimal flow and let $r : X \to \mathbb{R}$ be a continuous function. Suppose that there exists $\bar{x} \in X$ such that $\left|\int_0^t r(\bar{x} \cdot s)ds\right| \leq C$ for some $C > 0$. Then there exists a continuous function $R : X \to \mathbb{R}$ such that*

$$R(x \cdot t) - R(x) = \int_0^t r(x \cdot s)ds, \ x \in X.$$

Proof. Introduce a flow $\{\hat{\tau}_t\}$ on $X \times \mathbb{R}$ as follows:

$$\hat{\tau}_t(x, u) = \left(\tau_t(x), u + \int_0^t r(x \cdot s)ds\right).$$

Consider the point $(\bar{x}, 0) \in X \times \mathbb{R}$; it has a bounded orbit in $X \times \mathbb{R}$. The closure $\text{cls}\{\hat{\tau}_t(\bar{x}, 0) : t \in \mathbb{R}\} \subseteq X \times \mathbb{R}$ is compact and $\{\tau_t\}$-invariant, so it contains a minimal set M.
Note that the projection $\pi : M \to X$ is a flow homomorphism which is surjective; this is because $\pi(M) \subseteq X$ is compact, $\{\tau_t\}$-invariant and nonempty and because $(X, \mathbb{R})$ is minimal.
It is yet to be proved that $\pi^{-1}(x) \cap M$ contains just one point for all $x \in X$. Suppose that there exists $\hat{x} \in X$ such that $\pi^{-1}(\hat{x}) \cap M$ contains two points, say $(\hat{x}, u_1)$ and $(\hat{x}, u_2)$. Let $\delta = u_2 - u_1$ and let $T_\delta : X \times \mathbb{R} \to X \times \mathbb{R}$, $(x, u) \mapsto (x, u + \delta)$ be the vertical translation by δ units.
Then $T_\delta(\hat{\tau}(\cdot)) = \hat{\tau}(T_\delta(\cdot))$; this means that $T_\delta(M)$ is a minimal subset of $X \times \mathbb{R}$ and, therefore, $T_\delta(M) = M$. It follows that $T_{k\delta}(M) = M$, $k \in \mathbb{Z}$, which means that M is not bounded, a contradiction.
Thus $\pi^{-1}(x) \cap M = \{(x, R(x))\}$ for all $x \in X$ and some $R(x) \in \mathbb{R}$. By compactness of M, R must be continuous.
Finally, we have

$$\begin{aligned} \hat{\tau}_t(x, R(x)) &= (x \cdot t, R(x \cdot t)), \\ \hat{\tau}_t(x, R(x)) &= \left(x \cdot t, R(x) + \int_0^t r(x \cdot s)ds\right), \end{aligned}$$

so that $R(x \cdot t) - R(x) = \int_0^t r(x \cdot s)ds, \ x \in X.$ □

The case when $\int_0^t [r(x \cdot s) - r_0]ds$ is unbounded for some (hence all) $x \in X$ is more complex.

Exercise 2.14. If $r \in \mathcal{C}_2$, then $(X \times \mathbb{T}, \{\tau_t\})$ is minimal.

Let us divide $\mathcal{C}_2$ into two subclasses. Namely set

$$\mathcal{C}_2' = \{r \in \mathcal{C}_2 : \text{ there exists a } \nu\text{-measurable function } R : X \to \mathbb{R} \text{ such that}$$
$$R(x \cdot t) - R(x) = \int_0^t [r(x \cdot s) - r_0] ds \text{ for all } x \in X, t \in \mathbb{R}.$$

Then set $\mathcal{C}_2'' = \mathcal{C}_2 - \mathcal{C}_2'$.

Exercise 2.15. (a) If $r \in \mathcal{C}_2''$ then $(X \times \mathbb{T}, \{\tau_t\})$ is a uniquely ergodic flow.
(b) If $r \in \mathcal{C}_2'$ and if r_0 is rationally independent of $\gamma_1, \dots, \gamma_n$, then $(X \times \mathbb{T}, \{\tau_t\})$ is again a uniquely ergodic flow.

Exercise 2.16. If $r \in \mathcal{C}_2'$ and if $kr_0 \in \mathfrak{M}_X$ for some integer $k \geq 1$, then there are uncountably many $\{\tau_t\}$-ergodic measures μ on $X \times \mathbb{T}$. Describe these ergodic measures.

It can further be shown that, if $r \in \mathcal{C}_2$ then the flow $(X \times \mathbb{T}, \{\tau_t\})$ is minimal but not almost periodic.

Remark 2.17. For "most" continuous functions r (Baire sense), there is no solution at all of the relation
$$R(\psi \cdot t) - R(\psi) = \int_0^t r(\psi \cdot s) ds.$$

Let us finish this section by introducing the *support* of an invariant measure. Let X be a compact metric space, let $(X, \{\tau_t\})$ be a flow and let μ be a $\{\tau_t\}$-invariant measure. Since μ is Borel regular, there exists an open subset $V \subseteq X$ which can be described as the largest open subset of X having μ-measure zero.

Definition 2.18. The *topological support* of μ is $\mathrm{Supp}(\mu) = X \backslash V$.

Clearly $\mathrm{Supp}(\mu)$ is compact; it is easy to see that it is $\{\tau_t\}$-invariant. If $(X, \{\tau_t\})$ is a minimal flow and if μ is a $\{\tau_t\}$-invariant measure on X, then $\mathrm{Supp}(\mu) = X$ (why?).

Exercise 2.19. Determine a flow $(X, \mathbb{R})$ and an $\{\tau_t\}$-ergodic measure μ on X such that $\mathrm{Supp}(\mu) = X$ but $(X, \mathbb{R})$ is not minimal.

Let $(X, \mathbb{R})$ be a flow and let μ be a $\{\tau_t\}$-ergodic measure on X such that $\mathrm{Supp}(\mu) = X$. Then $(X, \mathbb{R})$ is *topologically transitive*; that is, there exists $x \in X$ such that the orbit $\{x \cdot t : t \in \mathbb{R}\}$ is dense in X. To see this, let $\{V_i : i \in \{1, 2, \dots\}\}$ be a countable basis for the topology of X, and let χ_i be the characteristic function of V_i, $i \in \{1, 2, \dots\}$. By the Birkhoff ergodic theorem,
$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \chi_i(x \cdot s) ds = \mu(V_i) > 0$$
for μ-a.a. $x \in X$, say for $x \in X_i$, $i \in \{1, 2, \dots\}$. Thus if $x \in X_i$, then the positive semiorbit of x enters V_i. Let $X_\infty = \bigcap_{i=1}^{\infty} X_i$; each point $x \in X_\infty$ has a dense positive semiorbit and a fortiori a dense orbit.

Exercise 2.20. Let $(X, \mathbb{R})$ be a flow which is uniquely ergodic with unique $\{\tau_t\}$-invariant measure μ. If $X = \mathrm{Supp}(\mu)$, then $(X, \mathbb{R})$ is minimal.

3. Flows of Bebutov-type and skew-product flows

We state some facts concerning flows of Bebutov-type and associated skew-product flows.

We observed earlier that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz vector field and if all solutions of the autonomous ODE
$$x' = f(x) \tag{3.1}$$
exist on $(-\infty, \infty)$, then those solutions determine a flow on $\mathbb{R}^n$. This simple fact is extremely useful when one studies the qualitative behavior of the solutions of (3.1); i.e., their stability properties, their oscillation properties, their recurrence properties, their asymptotic behavior,

etc. Think of the Poincaré-Bendixson theory as an illustration of this remark, where one uses the concept of ω-limit set.

Now consider a nonautonomous differential equation

$$x' = f(t, x), \; t \in \mathbb{R}, x \in \mathbb{R}^n. \tag{3.2}$$

Suppose that f is continuous in (t, x) and locally Lipschitz continuous in x, so that solutions of (3.2) are locally defined and unique. Suppose that, for each $x_0 \in \mathbb{R}^n$, the solution $\varphi(t, x_0)$ of (3.2) satisfying $\varphi(0, x_0)$ exists on $-\infty < t < \infty$. It is easy to see that, if we set $\tau_t(x_0) = \varphi(t, x_0)$ as before, and if f "really" depends on t, then $\{\tau_t\}$ does *not* define a flow on $\mathbb{R}^n$.

If we augment the system (3.2) as follows:

$$\begin{cases} x' &= f(t, x) \\ t' &= 1 \end{cases} \tag{3.3}$$

and if f satisfies the conditions indicated above, then the solutions of (3.3) define a flow on $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$. However each solution of (3.3) has an empty ω-limit set (α-limit set), so we cannot hope to use these concepts to study the asymptotic properties of solutions of (3.2). Moreover, $f(\cdot, x)$ may have recurrence properties in the t-variable (for example, almost periodicity). One might try to use this fact to study the recurrence properties of solutions of (3.2). However, the recurrence properties of f are washed away when we augment the system.

The Bebutov approach allows one to use the solutions of (3.2) to define a flow which has better properties than that defined by (3.3). Let us begin our discussion of Bebutov flows by considering linear differential systems. Let $\mathbb{M}_n$ be the set of $n \times n$ real matrices and let $|\cdot|$ be the usual norm on $\mathbb{M}_n$.

If $a : \mathbb{R} \to \mathbb{M}_n$ is continuous and bounded, let $\Phi(t)$ be the fundamental matrix solution of $x' = a(t)x$ (that is $\Phi(\cdot) \in \mathbb{M}_n$; $\Phi' = a(x)\Phi$ and $\Phi(0) = I$). Then $\Phi(t)$ is defined on $-\infty < t < \infty$.

Let $\mathfrak{B} = \{b : \mathbb{R} \to \mathbb{M}_n : b(\cdot) \text{ is bounded and continuous}\}$. If $b_1, b_2 \in \mathfrak{B}$, let

$$\rho(b_1, b_2) = \sum_{n=1}^{\infty} \left(\sup_{-n \le t \le n} |b_1(t) - b_2(t)| \right) \cdot 2^{-n}.$$

Then $(\mathfrak{B}, \rho)$ is a metric space.

Definition 3.1. For each $t \in \mathbb{R}$ and $b \in \mathfrak{B}$, set $\tau_t(b)(\cdot) = b(t + \cdot)$. Then $\{\tau_t\}$ defines a flow on $\mathfrak{B}$ called the *Bebutov flow*

Next, let $a \in \mathfrak{B}$ be a *uniformly* continuous function. Let $\Omega_a = \text{cls}\{\tau_t(a) : t \in \mathbb{R}\}$. Then Ω_a is a closed $\{\tau_t\}$-invariant subset of $\mathfrak{B}$.

Lemma 3.2. *Ω_a is compact. One calls Ω_a the* hull *of a.*

Proof. Apply the Arzelà-Ascoli theorem. Consider a sequence $\{a(t + t_k)\}$ where $\{t_k\} \subseteq \mathbb{R}$. Restrict each $a(t + t_k)$ to some fixed compact interval I; call the restriction b_k. Then $\{b_k\}$ is uniformly bounded and equicontinuous.
So, if $a \in \mathfrak{B}$ and a is uniformly continuous, then Ω_a is compact. □

Example 3.3.
(a) Suppose $a(\cdot) = a_0$ is a constant function. Then $a(\cdot)$ is fixed with respect to the Bebutov flow, so $\Omega_a = \{a_0\}$.
(b) Suppose $a \in \mathfrak{B}$ is a 1-periodic function. Then $\tau_t(a)(\cdot) = a(t + \cdot)$ for all $t \in \mathbb{R}$ and, cosequently, $\tau_1(a)(\cdot) = a(\cdot)$. So Ω_a is a topological version of $\mathbb{T}$ and $(\Omega_a, \{\tau_t\})$ is isomorphic to $(\mathbb{T}, \tau_t^R)$, where $\tau_t^R(\psi) = \psi + t$.
(c) Suppose for example that $a(\cdot)$ is Bohr almost periodic. This means that, to each $\varepsilon > 0$, there corresponds $T = T(\varepsilon) > 0$ such that each interval $[\alpha, \alpha+T] \subseteq \mathbb{R}$ contains a number η such that

$$|a(t + \eta) - a(t)| \le \varepsilon \text{ for all } t \in \mathbb{R}.$$

One can show that such a function $a(\cdot)$ is uniformly continuous and hence Ω_a is compact. Actually much more is true. Among other things, one can show that $(\Omega_a, \{\tau_t\})$ is an almost periodic flow. See [11].

Remark 3.4. Where do almost periodic functions come from?
Consider $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and a Kronecker flow with frequencies $(1, \sqrt{2})$ on $\mathbb{T}^2$.
Let (ψ_1, ψ_2) be coordinates on $\mathbb{T}^2$ and set

$$f(\psi_1, \psi_2) = \sin(2\pi\psi_1) + \sin(2\pi\psi_2).$$

Consider the orbit passing through $(0, 0)$:

$$\{(t, \sqrt{2}t) : t \in \mathbb{R}\} \subseteq \mathbb{T}^2.$$

Set $f(t, \sqrt{2}t) = \sin(2\pi t) + \sin(2\pi\sqrt{2}t)$; it turns out that f is Bohr almost periodic.
More generally, any finite sum of periodic functions and even a uniformly convergent series of periodic functions is Bohr almost periodic.
Let $(X, \mathbb{R})$ be an almost periodic flow and let $f : X \to \mathbb{R}$ be continuous. Then, for each $x \in X$, the restriction $t \mapsto f(x \cdot t)$ is Bohr almost periodic.

Let $a : \mathbb{R} \to \mathbb{M}_n$ be uniformly continuous and consider the differential system

$$x' = a(t)x \tag{3.4}$$

Let Ω_a be the hull of a and define $A : \Omega_a \to \mathbb{M}_n,\ \omega \mapsto \omega(0)$.
Consider the family of equations

$$x' = A(\omega \cdot t)x \tag{3.5}$$

where $\omega \cdot t = \tau_t(\omega)$. If $\omega = a$, we obtain the original equation (3.4), which has thus been embedded in a compact, Bebutov-invariant family of linear differential systems.

Proposition 3.5. *Let $\Phi_\omega(t)$ be the fundamental matrix solution of* (3.5)*. If $t \in \mathbb{R}$, define*

$$\begin{array}{rccc} \widehat{\tau}_t : & \Omega_a \times \mathbb{R}^n & \longrightarrow & \Omega_a \times \mathbb{R}^n \\ & (\omega, x_0) & \mapsto & (\tau_t(\omega), \Phi_\omega(t)x_0). \end{array}$$

Then $(\Omega \times \mathbb{R}^n, \{\widehat{\tau}_t\})$ is a flow.

Proof. First use the uniqueness property of the solutions of the equations (3.5) to prove the *cocycle identity*:

$$\Phi_\omega(t + s) = \Phi_{\omega\cdot t}(s)\Phi_\omega(t).$$

The group property of $\{\widehat{\tau}_t\}$ follows from the cocycle identity. The continuity property of $\{\widehat{\tau}_t\}$ follows from the Gronwall lemma. ☐

Example 3.6. (a) Suppose a is a constant function with value A. Then

$$\Omega_a = \{A\},\ \Phi(t) = e^{At} \text{ and } \widehat{\tau}_t(A, x_0) = (A, e^{At}x_0).$$

(b) Suppose a is 1-periodic. If $\omega \in \Omega_a$, then ω is a translate of a. Let $\Phi(t)$ be the fundamental matrix solution of $x' = a(t)x$. One can study the "dynamical" properties of solutions of $x' = a(t)x$ by studying the iterates of the "period matrix" $P = \Phi(1)$.

(c) Consider $x' = a(t)x$ where $a : \mathbb{R} \to \mathbb{M}_n$ is Bohr almost periodic. Is there an analogue of the period matrix? Think about it…

One calls $(\Omega \times \mathbb{R}^n, \{\widehat{\tau}_t\})$ a (linear) skew-product flow. It can be viewed as a nonautonomous analogue of the flow $\{\widehat{\tau}_t(x_0) = e^{At}x_0\}$ which is determined by the linear differential system with constant coefficients $x' = Ax$. It is now natural to look for analogues of eigenvalues and generalized eigenspaces in the context of nonautonomous linear differential systems. We will do so a bit later.

It is worth noting that the flow $(\mathfrak{B}, \{\tau_t\})$ can be substituted by other Bebutov-type flows. For example, let $\mathfrak{B} = L^\infty(\mathbb{R}, \mathbb{M}_n)$ with the weak-$*$ topology. If $k > 0$, let $\mathfrak{B}_k = \{b \in \mathfrak{B} : |b|_\infty \leq k\}$. Then $\mathfrak{B}_k$ is compact. Define

$$\begin{array}{rccc} \tau_t : & \mathfrak{B}_k & \longrightarrow & \mathfrak{B}_k \\ & b & \mapsto & \tau_t(b)(\cdot) = b(t + \cdot) \end{array}$$

for each $t \in \mathbb{R}$. Then $(\mathfrak{B}_k, \{\tau_t\})$ is a flow. The analogue of Proposition 3.5 holds for this flow.

Next we consider nonlinear flows which arise from a Bebutov-type construction. We give just one example. Let $\mathfrak{F}$ be the set of all functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which are jointly continuous and which have the following property: for each compact set $K \subseteq \mathbb{R}^n$ there is a constant l_K (which depends also on f) such that

$$|f(t,x) - f(t,y)| \leq l_K |x-y|$$

for all $t \in \mathbb{R}$ and all $x, y \in K$. Give $\mathfrak{F}$ the compact-open topology, then set

$$\tau_t(f)(\cdot, x) = f(t + \cdot, x)$$

for each $t \in \mathbb{R}, x \in \mathbb{R}^n, f \in \mathfrak{F}$. The pair $(\mathfrak{F}, \{\tau_t\})$ is a Bebutov-type flow.

Next suppose that $f \in \mathfrak{F}$ is uniformly continuous on $\mathbb{R} \times K$ for each compact subset $K \subseteq \mathbb{R}^n$. Let $\Omega_f = \text{cls}\{\tau_t : t \in \mathbb{R}\}$ be the hull of f. Then Ω_f is Bebutov-invariant and compact. In this case, let $\varphi(t, \omega, x_0)$ be the solution of $x' = \omega(t,x)$ satisfying $\varphi(0, \omega, x_0) = x_0$ for each $\omega \in \Omega_f, x_0 \in \mathbb{R}^n$. Suppose that $\varphi(\cdot, \omega, x_0)$ is defined on $(-\infty, \infty)$ for each $(\omega, x_0) \in \Omega_f \times \mathbb{R}^d$.

Exercise 3.7. Set $\widehat{\tau}_t(\omega, x_0) = (\tau_t(\omega), \varphi(t, \omega, x_0))$ for each $t \in \mathbb{R}$ and each $(\omega, x_0) \in \Omega_f \times \mathbb{R}^d$. Then $\{\widehat{\tau}_t\}$ defines a flow on $\Omega_f \times \mathbb{R}^d$.

We have defined a typical (nonlinear) skew-product flow. We can now try to study the qualitative properties of the solutions of the equation $x' = f(t,x)$ and of the equations $x' = \omega(t,x)$ by studying the dynamics of the flow $\{\widehat{\tau}_t\}$. Note that the recurrence properties of f are encoded in the flow $(\Omega_f, \{\tau_t\})$.

We finish this section by noting that it is possible to modify the choice of $\mathfrak{F}$ in such a way as to take account of eventual higher smoothness in x of the vector field f.

4. Exponential dichotomies

Next we discuss *exponential dichotomies.* There are several ways to introduce this basic concept. We do so following a line of thought which goes back (at least) to Lyapunov.

Consider a linear differential equation with constant coefficients

$$x' = Ax, \ x \in \mathbb{R}^n. \tag{4.1}$$

The fundamental matrix solution of this equation is $\Phi(t) = e^{At}$. Making a linear change of variables $x = Cy$, one can put A in Jordan form $J = C^{-1}AC$ and then $\Phi(t) = Ce^{Jt}C^{-1}$, from which one sees that, if $x_0 \in \mathbb{R}^n$, then

$$\Phi(t)x_0 = \sum_{j=1}^{r} e^{\eta_j t} p_j(t)$$

where $\eta_1, \dots, \eta_r$ are the eigenvalues of A and $p_1(t), \dots, p_r(t)$ are polynomials. There is clearly a close relation between the exponential growth/decay of $x(t) = \Phi(t)x_0$ as $t \to \pm\infty$ and the real parts $\text{Re}(\eta_j)$ of the eigenvalues of A. Thus for example

$$\lim_{t \to \infty} \frac{1}{t} \ln |x(t)| = max\{\text{Re}(\eta_j) : p_j \neq 0, 1 \leq j \leq r\}.$$

Furthermore, the imaginary parts $\text{Im}(\eta_j)$ of these eigenvalues are related to the "rotation" of $x(t)$ in some not-very-well-defined sense.

Let us make some observations which involve the real parts $\text{Re}(\eta_j)$ of the eigenvalues $\eta_1, \dots, \eta_r$ of A.

First of all, suppose $\text{Re}(\eta_j) < 0$ for all $j \in \{1, 2, \dots, r\}$. Let g be a sufficiently regular function defined on $[0, \infty) \times W$, where W is a neighborhood of $x = 0$ in $\mathbb{R}^n$. If $g(t,x) = o(|x|)$ uniformly in $t \geq 0$ as $x \to 0$, then a theorem of Perron states that the solution $\psi(t) \equiv 0$ of the nonlinear system

$$x' = Ax + g(t,x) \tag{4.2}$$

is exponentially asymptotically stable as $t \to \infty$. So one has a sufficient condition for asymptotic stability of the zero solution of (4.2) which involves only the linear terms of the right-hand side.

Second, if no eigenvalue $\eta_1, \dots, \eta_r$ of A has real part zero, then $\mathbb{R}^n = W_s \oplus W_u$ where W_s (W_u) is the intersection of $\mathbb{R}^n$ with the direct sum of the generalized eigenspaces of A corresponding to eigenvalues with negative (positive) real parts. Let $P : \mathbb{R}^n \to W_s$ be the projection whose image is W_s and whose kernel is W_u. Let $g : \mathbb{R} \to \mathbb{R}^n$ be a bounded measurable function. Then the nonhomogeneous equation

$$x' = Ax + g(t)$$

admits a unique solution x_g which is bounded on all of $\mathbb{R}$:

$$x_g = \int_{-\infty}^{t} \Phi(t)P\Phi(s)^{-1}g(s)ds - \int_{t}^{\infty} \Phi(t)(I-P)\Phi(s)^{-1}g(s)ds.$$

This simple and explicit formula is very useful.

Third and finally, let $\lambda_1, \dots, \lambda_s$ be the distinct values of $\mathrm{Re}(\eta_1), \dots, \mathrm{Re}(\eta_r)$. Let $W_k \subseteq \mathbb{R}^n$ be the intersection of $\mathbb{R}^n$ with the direct sum of the generalized eigenspaces of A corresponding to eigenvalues η_j satisfying $\mathrm{Re}(\eta_j) = \lambda_k$ for $1 \le k \le s$. Then W_k is an A-invariant subspace of $\mathbb{R}^n$, and

$$\mathbb{R}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_s$$

If $0 \neq x_0 \in W_k$, then

$$\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(t)x_0| = \lim_{t\to\infty} \frac{1}{t} \ln |\Phi(t)x_0| = \lambda_k,$$

and solutions $x(t) = \Phi(t)x_0$ such that x_0 does not lie in one of the subspaces W_k have growth properties and "angular" properties which are easily worked out.

Let us now consider a compact Bebutov-invariant set $\Omega \subseteq \mathfrak{B}$. Let $A : \Omega \to \mathbb{M}_n$, $\omega \mapsto \omega(0)$ as before and consider the family of linear differential systems

$$x' = A(\omega \cdot t)x. \tag{4.3}$$

It is natural to look for analogues of the set of the real parts of the eigenvalues and of the corresponding sums of generalized eigenspaces, in the context to the family (4.3). Roughly speaking, two approaches to this question have been developed. One can be viewed as "topologico-dynamical" and is due to Bylov, Sacker, Sell and others. The other is based on ergodic theory and is due to Oseledets (also Millionščikov) with later developments by Pesin, Arnold and others.

Let us consider the topologico-dynamical approach. The point of departure is

Definition 4.1. Say that equations (4.3) admit an *exponential dichotomy* over Ω if there are constants $k > 0, \eta > 0$ and a continuous projection-valued function $P : \Omega \to \mathbb{M}_n$ (thus $P(\omega)^2 = P(\omega)$ for all $\omega \in \Omega$) such that

$$|\Phi_\omega(t)P(\omega)\Phi_\omega(s)^{-1}| \le ke^{-\eta(t-s)} \quad \text{if } t \ge s,$$

$$|\Phi_\omega(t)(I - P(\omega))\Phi_\omega(s)^{-1}| \le ke^{\eta(t-s)} \quad \text{if } t \le s.$$

Using the continuity of P, one can show that the sets

$$W_+ = \bigcup_{\omega\in\Omega} \{(\omega, P(\omega)x) : x \in \mathbb{R}^n\}$$

and

$$W_- = \bigcup_{\omega\in\Omega} \{(\omega, (I - P(\omega))x) : x \in \mathbb{R}^n\}$$

are topological vector subbundles of $\Omega \times \mathbb{R}^n$. They are invariant under the linear skew-product flow $\{\widehat{\tau}_t\}$ on $\Omega \times \mathbb{R}^n$ determined by equations (4.3). Moreover

$$\Omega \times \mathbb{R}^n = W_+ \oplus W_- \text{ (Whitney sum).}$$

Exercise 4.2. The constant-coefficient system $x' = Ax$ admits an exponential dichotomy if and only if no eigenvalue of A has zero real part. Identify the subbundles W_+ and W_- in this case. Hint: what is Ω if A is constant?

Definition 4.3. The *dynamical* or *Sacker-Sell spectrum* Λ of equations (4.3) is by definition

$$\Lambda = \{\lambda \in \mathbb{R} : \text{ the translated equations } x' = [-\lambda I + A(\omega \cdot t)]x \text{ do } \textit{not} \\ \text{admit an exponential dichotomy over } \Omega\}.$$

Exercise 4.4. Identify the dynamical spectrum of the constant-coefficient system $x' = Ax$.

Theorem 4.5 (Sacker-Sell)**.** *Let Ω be connected (this is true if Ω is the hull Ω_a of some uniformly continuous function $a \in \mathfrak{B}$). Then the dynamical spectrum is $\Lambda = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_s, b_s]$ where $-\infty < a_1 \le b_1 < \ldots < a_s \le b_s < \infty$ and $1 \le s \le n$. Moreover, $\Omega \times \mathbb{R}^n = W_1 \oplus \cdots \oplus W_s$ as a Whitney sum, where each W_k is a $\{\widehat{\tau}_t\}$-invariant topological vector subbundle of $\Omega \times \mathbb{R}^n$. One has*

$$W_k = \{(\omega, x) \in \Omega \times \mathbb{R}^n : x = 0 \textit{ or} \\ a_k \le \liminf_{t\to\pm\infty} \frac{1}{t} \ln |\Phi(t)x| \le \limsup_{t\to\pm\infty} \frac{1}{t} \ln |\Phi(t)x| \le b_k\}.$$

Example 4.6. The Sacker-Sell subbundles of the constant-coefficient system $x' = Ax$ are

$$W_k = \mathbb{R}^n \cap \widetilde{W}_k$$

where $\widetilde{W}_k$ is the sum of all generalized eigenspaces corresponding to eigenvalues η_j with $\mathrm{Re}(\eta_j) = \lambda_k$. The Sacker-Sell spectrum is the set of the real parts of eigenvalues of A, so that each interval reduces to a point.

Exercise 4.7. Let $a : \mathbb{R} \to \mathbb{M}_n$ be a continuous, p-periodic matrix function. Consider the linear differential system

$$x' = a(t)x. \tag{4.4}$$

Let $\Phi(t)$ be the fundamental matrix solution of (4.4) and let $M = \Phi(p)$ be the period matrix.

(i) Describe the hull $\Omega_a \subseteq \mathfrak{B}$ of the function a. Describe the Bebutov flow $\{\tau_t\}$ on Ω_a.

Introduce the family of equations (4.3) where $\omega \in \Omega_a$.

(ii) Describe the dynamical spectrum of the family (4.3). Hint: use the eigenvalues of M.

(iii) Describe the Sacker-Sell subbundles of the family (4.3). Hint: use the generalized eigenspaces of M.

Remark 4.8. What happens in the almost periodic case? In the previous cases, the Sacker-Sell spectrum was discrete. Is it true that the dynamical spectrum is discrete in the almost periodic case? The answer is no; there are well-known examples due to Millionščikov and Vinograd for which the Sacker-Sell spectrum is a nontrivial interval.

We close this brief introduction to the theory of exponential dichotomies by stating two basic results. The first can frequently be used to verify that an exponential dichotomy is present. The second says that an exponential dichotomy is insensitive to perturbation of the coefficient matrix.

Let us first recall that, if X is a compact metric space with metric d and if $\{\tau_t\}$ defines a flow on X, then $(X, \{\tau_t\})$ is *chain recurrent* if for each $x \in X, \varepsilon > 0$ and $T > 0$, there are points $x = x_1, x_2, \ldots, x_n, x_{n+1} = x$ and times $t_1 > T, \ldots, t_n > T$ such that

$$d(x_i \cdot t_i, x_{i+1}) \le \varepsilon,\ 1 \le i \le n.$$

Theorem 4.9 (Sacker-Sell-Selgrade)**.** *Suppose that Ω is connected and that $(\Omega, \mathbb{R})$ is chain recurrent. Say that a solution $x(t) = \Phi_\omega(t)x_0$ of equation* (4.3) *is bounded if $\sup_{t\in\mathbb{R}} |x(t)| < \infty$. The family of equations* (4.3) *admits an exponential dichotomy over Ω if and only if, for all $\omega \in \Omega$, the only bounded solution of equation* (4.3) *is the zero solution.*

Note. R. Mañé had a version of this result in the time frame in which Sacker-Sell and Selgrade worked out their proofs of this theorem.

Now we consider the robustness properties of exponential dichotomies. We state a result due to Sacker and Sell. Other perturbation theorems are due to Coppel and to Palmer.

Theorem 4.10. *Let $X \subseteq \mathfrak{B}$ be a compact, Bebutov-invariant set (it is understood that X has the topology induced from $\mathfrak{B}$). Define $A : X \to \mathbb{M}_n,\ x \mapsto x(0)$. Let $\Omega \subseteq X$ be a compact Bebutov-invariant set, and let* (4.3) *denote the corresponding family of differential systems $x' = A(\omega \cdot t)x$. Suppose that the family* (4.3) *admits an exponential dichotomy over Ω. Then there is an open set $U \subseteq X$ containing Ω such that, if $Y \subseteq U$ is a compact Bebutov-invariant set, then the family*

$$x' = A(y \cdot t)x$$

admits an exponential dichotomy over Y.

We conclude this section by noting that the point of view we have taken regarding the theory of exponential dichotomies is not unique. Indeed Massera and Schäffer based their theory of exponential dichotomies on the solvability of the nonhomogeneous linear system $x' = a(t)x + f(t)$ in various function spaces. Palmer makes frequent use of the theory concerning exponential dichotomies on the half-lines $(-\infty, 0)$ and $(0, \infty)$. Latushkin has systematically developed a semigroup approach to the theory of exponential dichotomies, the basic idea of which goes back to papers by Johnson and Chicone-Swanson in 1980.

5. Lyapunov exponents

Now we discuss Lyapunov exponents, beginning with some classical theory. Let $a : \mathbb{R} \to \mathbb{M}_n$ be a bounded measurable function. Consider the system

$$x' = a(t)x \tag{5.1}$$

and let $\Phi(t)$ be its fundamental matrix solution. Lyapunov showed that there are numbers $\lambda_1, \ldots, \lambda_s$, where $1 \leq s \leq n$, with the property that, if $0 \neq x_0 \in \mathbb{R}^n$, then

$$\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t} \ln |\Phi(t)x_0| \in \{\lambda_1, \ldots, \lambda_n\}.$$

Moreover, if we assume that $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ and set

$$V_k = \{x_0 \in \mathbb{R}^n : \lambda(x_0) \leq \lambda_k\},$$

then V_k is a vector subspace of $\mathbb{R}^n$, and $\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_s = \mathbb{R}^n$. Set $d_k = dimV_k - dimV_{k-1}$, $1 \leq k \leq s$. Say that (5.1) is *Lyapunov regular* if

$$\sum_{k=1}^{s} d_k \lambda_k = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} a(s) ds.$$

This concept of regularity is important because it is related to a stability question. Namely, suppose that $\lambda_s < 0$. Then the solution $\psi(t) \equiv 0$ of (5.1) is exponentially asymptotically stable. It is natural to ask if the asymptotic stability of the zero solution is inherited by a small nonlinear perturbation of (5.1). Consider the nonlinear system

$$x' = a(t)x + g(t, x),\ x \in W \subseteq \mathbb{R}^n,\ t \geq 0 \tag{5.2}$$

where W is a neighborhood of the origin; $g : [0, \infty) \times W \to \mathbb{R}^n$ is continuous, and g satisfies

- $|g(t, x)| \leq C_1|x|^q$, $q > 1$;
- $|g(t, x_1) - g(t, x_2)| \leq C_2(\delta)|x_1 - x_2|$ where $\delta = \max(|x_1|, |x_2|)$ and $C_2(\delta) \to 0$ as $\delta \to 0$

for all $t \geq 0$ and $x, x_1, x_2 \in W$. Even if $\lambda_s < 0$, the solution $\psi(t) \equiv 0$ of (5.2) need not be asymptotically stable. However, Lyapunov showed that if in addition, (5.1) is Lyapunov regular, then the solution $\psi(t) \equiv 0$ of (5.2) is asymptotically stable. See [3].

Oseledets found a very useful way to put Lyapunov's theory in an ergodic-theoretic context. We will not give the most general version of the Oseledets theorem (for this see [1]). Instead we formulate a weaker version using the structure we have introduced. Let $\Omega \subseteq \mathfrak{B}$ be a compact Bebutov-invariant set, let $A : \Omega \to \mathbb{M}_n,\ \omega \mapsto \omega(0)$ and consider the corresponding family (4.3) of linear differential systems.

Theorem 5.1 (Oseledets; also Millionščikov)**.** *Let μ be a $\{\tau_t\}$-ergodic measure on Ω. There are a set $\Omega_\mu \subseteq \Omega$ of μ-measure 1 and a set $\{\lambda_1, \ldots, \lambda_s\}$ of real numbers with the following properties.*

(i) *For each $\omega \in \Omega_\mu$, the set*

$$V_k(\omega) = \{0\} \cup \{x_0 \in \mathbb{R}^n \backslash \{0\} : \lim_{t\to\infty} \frac{1}{t} \ln |\Phi_\omega(t)x_0| =$$

$$= \lim_{t\to -\infty} \frac{1}{t} \ln |\Phi_\omega(t)x_0| = \lambda_k\}$$

is a vector subspace of $\mathbb{R}^n$, $1 \le k \le s$.

(ii) *For each $\omega \in \Omega_\mu$, there holds $V_1(\omega) \oplus \ldots \oplus V_s(\omega) = \mathbb{R}^n$.*

(iii) *The sets $V_k = \bigcup_{\omega\in\Omega_\mu} \{(\omega, x_0) : x_0 \in V_k(\omega)\}$ are "measurable subbundles" of $\Omega \times \mathbb{R}^n$ which are invariant under the linear skew-product flow $\{\widehat{\tau}_t\}$ induced by equations* (4.3)*, $1 \le k \le s$.*

(iv) *For each $\omega \in \Omega_\mu$, the equation* (4.3) *is Lyapunov regular.*

The set of numbers $\{\lambda_1, \ldots, \lambda_s\}$ is called the Oseledets spectrum of the family (4.3). The main virtue of the Oseledets theorem is its extreme generality. Note that, μ-a.e., the equation (4.3) is Lyapunov regular and has two-sided Lyapunov exponents.

There are certain relative relations between the Sacker-Sell theory and the Oseledets theory which are discussed in Johnson-Palmer-Sell. We give two of them.

Proposition 5.2. *Suppose that $\Omega \subseteq \mathfrak{B}$ is a compact connected Bebutov-invariant set. Consider the corresponding family of equations* (4.3)*.*

(i) *If μ is a $\{\tau_t\}$-ergodic measure on Ω, then each Oseledets bundle V_k is contained on some Sacker-Sell bundle W_l.*

(ii) *If β is an endpoint of a Sacker-Sell interval, then there is a $\{\tau_t\}$-ergodic measure μ on Ω such that β belongs to the μ-Oseledets spectrum.*

We make a final remark. Suppose that $\Omega \subseteq \mathfrak{B}$ is compact and Bebutov-invariant. A good test case for various hypotheses concerning linear nonautonomous differential systems is that in which $(\Omega, \{\tau_t\})$ is almost periodic and $n = 2$. For example, one might conjeture that, in this case, the dynamical spectrum Λ consists either of one point or of exactly two points. It turns out that this conjecture is false. Examples of Millionščikov (1969) and Vinograd (1974) have the property that Λ is a nondegenerate interval $[a, b]$. This is a remarkable phenomenon which turns out to be important in the spectral theory of the quasi-periodic Schrödinger operator and in other theories which involve almost periodic differential systems.

6. Rotation numbers

We discuss the concept of rotation number. We take as a starting point an observation whose content is admittedly vague and which will turn out to be in some degree misleading. Namely, let A be an $n \times n$ matrix and let η be an eigenvalue of A with nonzero imaginary part. Let $x_0 \in \mathbb{R}^n$ be a vector of the form $x + \overline{x}$ where $Ax = \eta x$. Then $e^{At}x_0$ admits "rotation" with angular velocity $\pm\mathrm{Im}(\eta)$ in the "real part" of $\mathrm{Span}\{x, \overline{x}\}$ in $\mathbb{C}^n$.

We can think of the theories of Sacker-Sell and Oseledets as developments of the theory of Lyapunov exponents, which (roughly speaking) play the role for nonautonomous linear differential systems that the real parts of the eigenvalues of A play for the autonomous linear system $x' = Ax$. We now want to transplant the imaginary parts of the eigenvalues to the nonautonomous setting, i.e., we want to define and discuss rotation of the solutions of a nonautonomous linear system.

Let us begin with the case $n = 2$. As usual, let $\Omega \subseteq \mathfrak{B}$ be a compact, Bebutov-invariant set; let $A : \Omega \to \mathbb{M}_n$, $\omega \mapsto \omega(0)$ and consider the family

$$x' = A(\omega \cdot t)x. \tag{6.1}$$

In this situation, there is a natural way to define rotation. Introduce polar coordinates $(r, \theta) \in \mathbb{R}^2$. Setting

$$A = \begin{pmatrix} a+d & -b+c \\ b+c & -a+d \end{pmatrix},$$

one obtains

$$\frac{r'}{r} = d(\omega \cdot t) + a(\omega \cdot t)\cos(2\theta) + c(\omega \cdot t)\sin(2\theta) \tag{6.2}$$

$$\theta' = b(\omega \cdot t) - a(\omega \cdot t)\sin(2\theta) + c(\omega \cdot t)\cos(2\theta) \tag{6.3}$$

"Define" the *rotation number* α of the family (6.1) to be

$$\alpha = \lim_{t\to\infty} \frac{\theta(t)}{t} \tag{6.4}$$

This is an intriguing idea but not a priori convincing: it is not clear that the limit exists and, even if it does, it is not clear that the resulting quantity has more than superficial interest. We proceed to deal with these objections.

Suppose, to be specific, that $(\Omega, \{\tau_t\})$ is an almost periodic flow. Let ν be the unique $\{\tau_t\}$-invariant measure on Ω. Let $\mathbb{P}$ be the one-dimensional projective space of lines through the origin in $\mathbb{R}^2$. We think of $\mathbb{P}$ as $\mathbb{R}/\pi\mathbb{Z}$, and we think of θ as a π-periodic angular coordinate on $\mathbb{P}$. Let us define a flow $\{\widehat{\tau}_t\}$ on the product space $\Sigma = \Omega \times \mathbb{P}$ by setting

$$\widehat{\tau}_t(\omega, \theta) = (\tau_t(\omega), \theta(t))$$

where $\theta(t)$ is the solution of (6.3) with $\theta(0) = \theta_0$.

Next write

$$\Theta(\omega, \theta) = b(\omega) - a(\omega)\sin(2\theta) + c(\omega)\cos(2\theta)$$

and note that

$$\theta(t) = \theta_0 + \int_0^t \Theta(\widehat{\tau}_s(\omega, \theta_0))ds.$$

We recognize the limit in (6.4) as a time-average of Θ. Using the Birkhoff ergodic theorem, we see that, if μ is a $\{\widehat{\tau}_t\}$-ergodic measure on Σ, then the limit in (6.4) exists μ-a.e., say for $(\omega, \theta) \in \Sigma_\mu$, and is equal there to $\int_\Sigma \Theta d\mu$.

This is certainly a step forward, but more can be said. Note that, if $\pi : \Sigma \to \Omega$ is the projection, then $\Omega_\nu = \pi(\Sigma_\mu)$ has ν-measure 1. If $\omega \in \Omega_\nu$ and $\theta_1, \theta_2 \in \mathbb{R}$,

$$\left| \int_0^t \Theta(\widehat{\tau}_s(\omega, \theta_1))ds - \int_0^t \Theta(\widehat{\tau}_s(\omega, \theta_2))ds \right| \le |\theta_1 - \theta_2| + 2\pi$$

because orbits of the $\{\widehat{\tau}_t\}$-flow are either equal (as sets) or are disjoint. This implies that the limit in (6.4) exists on the set $\Sigma_\nu = \Omega_\nu \times \mathbb{P}$ and does not depend on the choice of $(\omega, \theta) \in \Sigma_\nu$. A further argument of Krylov-Bogoliubov type shows that the limit in (6.4) exists for all $(\omega, \theta) \in \Sigma$ and is uniform on Σ; one uses the unique ergodicity of (Ω, ν). The uniform limit is called the *rotation number* of the family (6.1).

We summarize:

Theorem 6.1. *If* $(\Omega, \{\tau_t\})$ *is almost periodic, then*

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \int_0^t \Theta(\widehat{\tau}_s(\omega, \theta))ds$$

is defined and constant for all $(\omega, \theta) \in \Sigma$. *The limit is uniform over* Ω.

This reasoning works in part when $(\Omega, \{\tau_t\})$ is not uniquely ergodic. If ν is a fixed $\{\tau_t\}$-ergodic measure on Ω, then one still obtains a set $\Sigma_\nu = \Omega_\nu \times \mathbb{P}$ where $\nu(\Omega_\nu) = 1$ and the time averages of Θ converge on Σ_ν to a fixed limit α.

There is a well-known application of the rotation number to the theory of the ergodic Schrödinger operator. Let $q : \mathbb{R} \to \mathbb{R}$ be a bounded uniformly continuous function. Let Ω be its Bebutov hull in $\mathfrak{B} = \{b : \mathbb{R} \to \mathbb{R} : b \text{ is bounded and continuous}\}$, and set $Q(\omega) = \omega(0)$ for $\omega \in \Omega$. The Schrödinger operator

$$-\frac{d^2}{dt^2} + q(t)$$

can be viewed as a self-adjoint operator on $L^2(\mathbb{R})$, as can each of the operators

$$-\frac{d^2}{dt^2} + Q(\omega \cdot t).$$

Let us write $(-d^2/dt^2 + q(\omega \cdot t))\psi = E\psi$, then pass to the phase variables $\begin{pmatrix} \psi \\ \psi' \end{pmatrix}$ and rewrite the operator equation as

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q(\omega \cdot t) & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \tag{6.5}$$

Let ν be a $\{\tau_t\}$-ergodic measure on Ω (hence the term "ergodic" Schrödinger operator). Let $\alpha = \alpha(E)$ denote the ν-rotation number of the family (6.5). Then one has the following.

Theorem 6.2 (Johnson-Moser). *The rotation number $\alpha = \alpha(E)$ is continuous, non-decreasing and increases exactly on the spectrum of $L_\omega = -d^2/dt^2 + Q(\omega \cdot t)$ for ν-almost all $\omega \in \Omega$. If Ω is the topological support of ν, then α is constant in an open interval $I \subseteq \mathbb{R}$ if and only if equation* (6.5) *have an exponential dichotomy over Ω for all $E \in I$.*

Johnson-Moser treated the case when (Ω, ν) is almost periodic; the ergodic case presents no essentially new difficulties. If α is constant on an open interval $I \subseteq \mathbb{R}$, then the *value* of α lies in the image of the so-called Schwartzmann homomorphism. In particular, each such value of α lies in a countable subgroup of $\mathbb{R}$ which is determined by the topology of Ω (more exactly by its first Čech cohomology group) and by the ergodic measure ν. This phenomenon is called *gap labelling*. In the almost periodic case, α/π lies in the frequency module $\mathfrak{M}_\Omega$ of Ω if α is constant on an open interval $I \subseteq \mathbb{R}$.

Let us now consider the concept of rotation in the context of higher-dimensional linear system. At first, it is not clear how to realize this concept and it seems fair to say that it is only in the last 20-25 years that the outlines of a theory of "higher-dimensional rotation" have become visible.

Actually, more than one approach is available. One has been developed by San Martin, Arnold and their co-workers. We will discuss another approach, which was initiated by Johnson (also Ruelle) and which has been developed by Novo, Núñez, Obaya, Colonius, Fabbri and Nerurkar. It is at present specific to the symplectic group and to certain other matrix groups.

Let us explain what we have said in a little more detail. Let

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathbb{M}_{2n}$$

where I is the $n \times n$ identity matrix. Let $S_p(n, \mathbb{R}) = \{\Phi \in \mathbb{M}_{2n} : \Phi^t J \Phi = J\}$ where t denotes the transpose. Let $sp(n, \mathbb{R})$ be the Lie algebra of $S_p(n, \mathbb{R})$: it can be described as $sp(n, \mathbb{R}) = \{JA : A^t = A, A \in \mathbb{M}_{2n}\}$.

Next, let Ω be a compact metric space and let $\{\tau_t\}$ be a flow on Ω. We change point of view somewhat and let $A : \Omega \to \mathbb{M}_{2n}$ be a continuous function with values in the set of $2n \times 2n$ symmetric matrices. Consider the family of *Hamiltonian* linear differential systems:

$$Jz' = A(\omega \cdot t)z,\ z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}. \tag{6.6}$$

For each $\omega \in \Omega$, the fundamental matrix solution $\Phi_\omega(t)$ of (6.6) takes values in $S_p(n, \mathbb{R})$.

Let Λ be the Grassmann-type manifold of Lagrange subspaces of $\mathbb{R}^{2n}$. Thus Λ is the set of these n-dimensional vector subspaces $\lambda \subseteq \mathbb{R}^{2n}$ satisfying the following property: if $z_1, z_2 \in \lambda$, then $\langle z_1, Jz_2 \rangle = 0$. Then Λ is a compact manifold of dimension $\frac{n(n+1)}{2}$. Let us introduce the vertical subspace $\lambda_v = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R}^n \right\}$, which is an element of Λ. The vertical *Maslov cycle* is by definition

$$\mathcal{C}_v = \{\lambda \in \Lambda : dim(\lambda \cap \lambda_v) > 0\}.$$

It can be shown that $\mathcal{C}_v$ is a $\mathbb{Z}_2$-cycle of codimension 1 on Λ. It is *two-sided* in a natural sense (V. Arnold). Its complement can be parametrized by the set of $n \times n$ symmetric matrices: if

$\lambda \in \Lambda \backslash \mathcal{C}_v$, then there exists a symmetric matrix $m \in \mathbb{M}_n$ such that

$$\lambda = \left\{ \begin{pmatrix} x \\ m \cdot x \end{pmatrix} : x \in \mathbb{R}^n \right\}.$$

One can check that, if $\lambda \in \Lambda$, then the image subspace $\Phi_\omega(t) \cdot \lambda$ is an element of Λ, for $\omega \in \Omega, t \in \mathbb{R}$. It is now natural to define

$$\alpha = \lim_{T \to \infty} \frac{1}{T} N_T(\Phi_\omega(t)\lambda_0 \cap \mathcal{C}_v) \tag{6.7}$$

where $\lambda_0 \in \Lambda$ and N_T is the number of oriented intersection points of the curve

$$\begin{array}{ccc} [0,T] & \longrightarrow & \Lambda \\ t & \mapsto & \Phi_\omega(t)\lambda_0 \end{array}$$

with the two-sided cycle $\mathcal{C}_v$.

It turns out that α is well-defined in the following sense.

Proposition 6.3. *Let ν be a $\{\tau_t\}$-ergodic measure on Ω. There is a set $\Omega_\nu \subseteq \Omega$ of ν-measure 1 such that, if $(\omega, \lambda_0) \in \Omega_\nu \times \Lambda$, then the limit in* (6.7) *is well-defined and does not depend on the choice of (ω, λ_0).*

The original proof of this result [13] used methods of V. Arnold. Novo, Núñez and Obaya gave another proof using the argument functions on $S_p(n, \mathbb{R})$ of Yakubovich; he developed his theory using results of Gel'fand and Lidskii.

The rotation number seems at first glance to be a somewhat artificial construct. There is evidence (convincing in the opinion of the present author) that this is not so. We first present an application of the rotation number to Atkinson-type spectral problems. Let $B : \Omega \to \mathbb{M}_{2n}$ be a continuous function with values which are symmetric and positive semi-definite. Consider the Atkinson problem

$$Jz' = (A(\omega \cdot t) + EB(\omega \cdot t))z \tag{6.8}$$

where E is a real or complex parameter. Let us suppose that the following *Atkinson condition* is satisfied. As before, write $\Phi_\omega(t)$ for the fundamental matrix solution of $Jz' = A(\omega \cdot t)z$.

Hypothesis 6.4. *For each $\omega \in \Omega$, there is a constant $\delta > 0$ such that, for each $z_0 \in \mathbb{R}^{2n}$ there holds*

$$\int_{-\infty}^{\infty} |B(\omega \cdot t)\Phi_\omega(t)z_0|^2 dt \geq \delta |z_0|^2.$$

The Atkinson condition has a useful control-theoretic interpretation: it means that the control system

$$z' = -A^t(\omega \cdot t)z + B(\omega \cdot t)u$$

is null controllable for each $\omega \in \Omega$.

Now let ν be an ergodic measure on Ω . Let $\alpha = \alpha(E)$ be the ν-rotation number of equations (6.8). It can be checked that α is continuous and non-increasing.

Theorem 6.5 (Johnson-Nerurkar)**.** *Suppose that Ω is the topological support of ν. Suppose that* Hypothesis 6.4 *is valid. Let $I \subseteq \mathbb{R}$ be an open interval such that α is constant on I. Then equations* (6.8) *admit an exponential dichotomy over Ω for all $E \in I$.*

Note that the perturbation EB of A is quite special in that B is positive semidefinite and is otherwise only subject to Hypothesis 6.4.

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