Tracing, mixing and entropy III

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Digraphs and topological properties

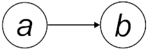
1. A pair $G = (V, E)$ is called a directed graph if
   1. V is finite (the set of vertices of G)
   2. $E \subset V \times V$ (the set of edges of G)
2. Let $Top(X)$ denotes the set of all nonempty open subsets of X and let $G = (V, E)$. A map $f \in C(X)$ has the mapping property (denoted $f \vdash G$) defined by G if
   1. for any map $\psi : V \to Top(X)$ there is $k \leq 1$ such that

$$(u, v) \in E \quad \Longrightarrow f^k(\psi(u)) \cap \psi(v) \neq \emptyset.$$

Simple examples

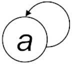
Consider the graph presented below

δ_1 :



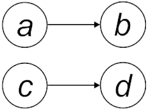
The mapping property defined by this graph is transitivity. The graph

α_1 :

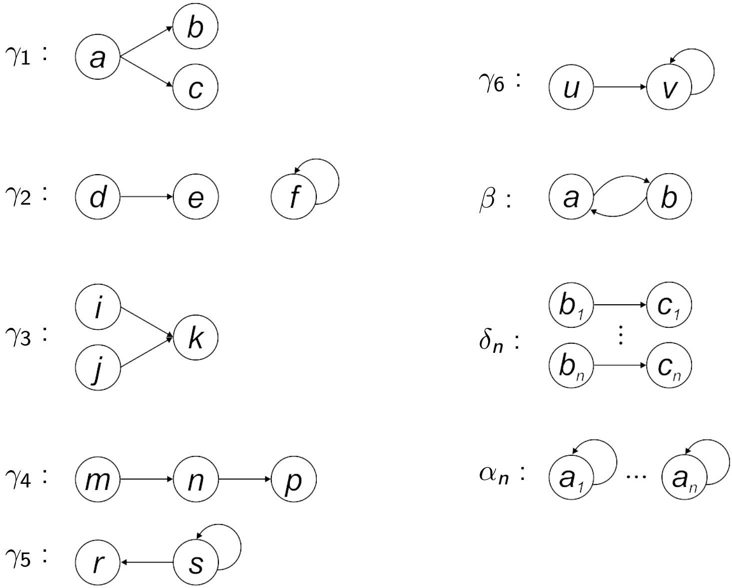


defines nonwandering, and weak mixing is defined by:

δ_2 :



Selected mapping properties



Results of Banks

A map $\pi : K \to H$ between graphs H, G is *pseudo-homomorphisms* if for every edge $(u,v) \in E(H)$ there is $(a,b) \in E(K)$ such that $u = \pi(a)$, $v = \pi(b)$.

Remark

Let H be a pseudo-homomorphic image of a subgraph $K \subset G$.
If $f \vdash G$ then $f \vdash H$.

Theorem

1. A mapping property is equivalent to weak mixing iff it is not given by β, δ_1 or α_n.
2. If f is flip transitive and f^2 is transitive then f is weakly mixing.

1. $\delta_n \implies \gamma_6 \implies \gamma_5 \implies \gamma_2$
2. $\delta_n \implies \gamma_3 \implies \gamma_2$
3. $\delta_n \implies \gamma_4 \implies \gamma_2$
4. $\gamma_2 \implies \gamma_1 \implies \delta_2$

Periodic decomposition

1. A collection of sets $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ is a regular periodic decomposition (of length n) if
   1. $\overline{T(D_i)} \subset D_{i+1 \mod n}$,
   2. $\overline{\operatorname{int} D_i} = D_i$ for $i = 0, \ldots, n-1$,
   3. $\operatorname{int} D_i \cap \operatorname{int} D_j = \emptyset$ for every $i \neq j$.

Theorem

Let $\mathcal{D} = \{D_0, \ldots, D_{n-1}\}$ be a regular periodic decomposition for f. Then the following conditions hold:

1. $\overline{T^k(D_i)} = D_{i+k \mod n}$ for all $i \leq i < n$ and $k \geq 1$.
2. $T^k(D_i) \subset D_i$ iff $k = 0 \mod n$,
3. $T^{-j}(\operatorname{int} D_i) \subset \operatorname{int} D_{i-j \mod n}$ for $i = 0, \ldots, n-1$ and $j \geq 0$,
4. $\bigcup_{i \neq j} D_i \cap D_j$ is invariant and nowhere dense in X.

In particular $\mathrm{Trans}(T) \subset \bigcup_{i=1}^{n-1} \operatorname{int} D_i$.

Odometers

Let $\mathbf{s} = (s_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of positive integers such that s_n divides s_{n+1}. For each $n \geq 1$ define $\pi_n \colon \mathbb{Z}_{s_{n+1}} \to \mathbb{Z}_{s_n}$ by the natural formula $\pi_n(m) = m \pmod{s_n}$ and let $G_{\mathbf{s}}$ denote the following inverse limit

$$G_{\mathbf{s}} = \varprojlim_n (\mathbb{Z}_{s_n}, \pi_n) = \left\{ x \in \prod_{i=1}^{\infty} \mathbb{Z}_{s_n} : x_n = \pi_n(x_{n+1}) \right\},$$

where each $\mathbb{Z}_{s_n}$ is given the discrete topology, and on $\prod_{i=1}^{\infty} \mathbb{Z}_{s_n}$ we have the Tychonoff product topology. On $G_{\mathbf{s}}$ we define $T_{\mathbf{s}} \colon G_{\mathbf{s}} \to G_{\mathbf{s}}$ by

$$T_{\mathbf{s}}(x)_n = x_n + 1 \pmod{s_n}.$$

Then $G_{\mathbf{s}}$ is a compact metrizable space and $T_{\mathbf{s}}$ is a homeomorphism, therefore $(G_{\mathbf{s}}, T_{\mathbf{s}})$ is a dynamical system (odometer).

Periodic decomposition in interval map sub-dynamics

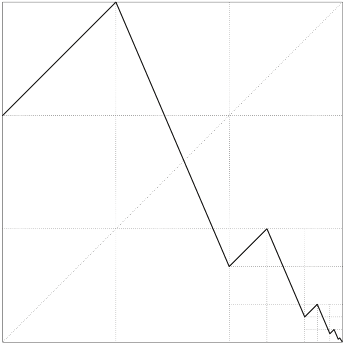


Figure: Classical Delahaye's example

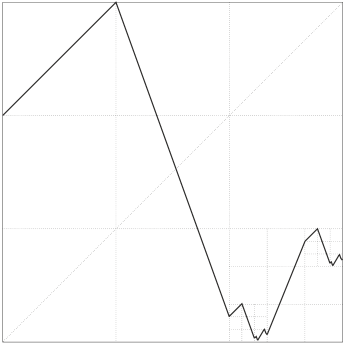


Figure: Modified example with splitting

Periodic decomposition of transitive maps

Corollary

The elements of regular periodic decomposition for a minimal map are pairwise disjoint.

Theorem

Transitive map has at most one regular periodic decomposition of length n.

Theorem

Two decompositions for transitive map have always common refinement.

Theorem

Let $\mathcal{D} = \{D_0, \ldots, D_n\}$ be a regular periodic decomposition for T.
Then T is transitive iff $T^n|_{D_i}$ is transitive for $i = 0, \ldots, n$.

Periodic decomposition of transitive maps

Theorem

let T be transitive with T^p not transitive, where p is a prime number. Then f admits a regular periodic decomposition of length p.

Theorem

Let $\mathcal{D} = \{D_0, \ldots, D_{n-1}\}$ be a periodic decomposition for a transitive (X, T). The following conditions are equivalent:

1. $\mathcal{D}$ is terminal (i.e. does not have sub-decomposition),
2. $T^n|_{D_i}$ is totally transitive for $i = 0, \ldots, n-1$.

Theorem

Let (X, T) be transitive and X locally connected. Then every regular periodic decomposition for T has a connected refinement (i.e. sub-decomposition).

Decomposition - further properties

Theorem

Let (X, T) be transitive. If X has a connected component C with $\mathrm{int}\, C \neq \emptyset$ then

1. X has finitely many connected components
2. and they form a regular periodic decomposition.

Corollary

A terminal decomposition for a transitive map on a locally connected space X consists of connected sets. A minimal map on a connected space X is totally transitive.

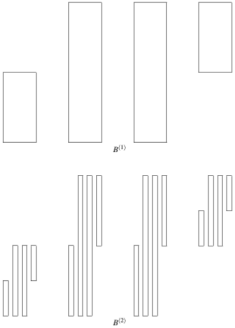


Figure: Auslander-Floyd system

Characterization by B-D-H-S-Š

1. A space X is almost totally disconnected if the set of its degenerate components, considered as a subset of X, is dense in X.
2. A space X is cantoroid if it is almost totally disconnected without isolated points.

Theorem

An almost totally disconnected compact metric space admits a minimal map if and only if it is either a finite set or a cantoroid.

Decompositions in dimension one

Theorem

Let T be transitive and let $\mathcal{C}$ be a refinement of a regular periodic decomposition $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$. Then $C \in \mathcal{C}$ intersects D_j iff the parent of C intersects D_j.

Theorem

Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a regular periodic decomposition for a transitive map f and let p be a periodic point whose period is co-prime with n. Then $Orb^+(p) \subset \bigcap_{i=0}^{n-1} D_i$.

Corollary

1. *Transitive map T on topological graph has always terminal decomposition.*
2. *Transitive map T on $[0,1]$ has always terminal decomposition of length 1 or 2.*

Shadowing property

1. a finite sequence $x_1, \ldots, x_n$ is δ-pseudo orbit if $d(T(x_i), x_{i+1}) < \delta$ for $i = 1, \ldots, n-1$
2. a point z ε-traces δ-pseudo orbit if $d(T^i(z), x_i) < \varepsilon$.
3. (X, T) has shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ε-traced.

Remark

- If (X, T) has shadowing property then elements of periodic decomposition must be pairwise-disjoint.
- So if X is connected, then transitive T is totally transitive, thus weak mixing, thus mixing.

Expansive maps

1. A surjective dynamical system (X,T) is expansive if there is $\varepsilon > 0$ such that for $\{x_i\}_{i\in\mathbb{Z}}$, $\{y_i\}_{i\in\mathbb{Z}}$, $T(x_i) = x_{i+1}$, $T(y_i) = y_{i+1}$ we have
$$\forall i \in \mathbb{Z}\ d(x_i, y_i) < \varepsilon \Longrightarrow x_0 = y_0 \quad (\text{so } x_i = y_i \text{ for all } i \in \mathbb{Z})$$
2. A dynamical system (X,T) is positively expansive if there is $\varepsilon > 0$ such that
$$\forall i \geq 0\ d(T^i(x), T^i(y)) < \varepsilon \Longrightarrow x = y$$
3. definition of expansive was initially defined for homeomorphisms.
4. interval maps are never expansive.
5. classical examples of (positively) expansive dynamical systems are (one sided) subshifts.

A result by Schwartzman

Theorem

Suppose that X is infinite. Then for every $\varepsilon > 0$ there exists a future ε-bounded orbit, i.e.$x \neq y$ such that $d(T^i(x), T^i(y)) < \varepsilon$ for all $i \geq 0$.

Corollary

Suppose T is invertible. The following conditions are equivalent:

1. T is positively expansive,
2. X is finite.

Shifts of finite type

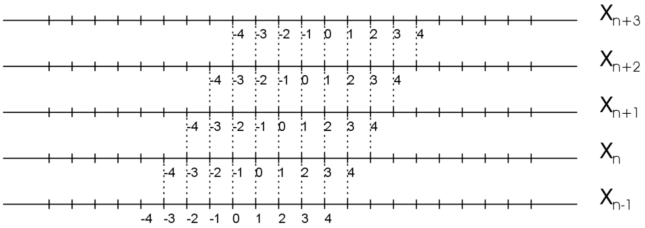
1. If there is $\#\mathcal{F} < \infty$ such that $X = X_{\mathcal{F}}$ then we call X a shift of finite type (SFT).

Theorem (Walters)

A shift $X \subset \Sigma_2$ is SFT iff $\sigma|_X$ has shadowing (same is true for $X \subset \Sigma_2^+$).

Theorem (Parry)

A one-sided shift $X \subset \Sigma_2^+$ is SFT iff $\sigma|_X$ is open.



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Remark

More generally, every open positively expansive map has shadowing.

Chain recurrence

1. a point x is P-related to y (written xPy) if for every $\delta > 0$ there exists a δ-pseudo orbit $z_0, \ldots, z_{n+1}$ such that $x = z_0$ and $y = z_{n+1}$.
2. if xPy and yPx then x is related to y (written $x \sim y$).
3. the set $\mathrm{CR}(T) = \{x \in X \ : \ x \sim x\}$ is called the chain recurrent set of f.
4. $\mathrm{CR}(T)$ is closed, invariant and $\Omega(T) \subset \mathrm{CR}(T)$.
5. If T has shadowing then $\Omega(T) = \mathrm{CR}(T)$.
6. $\mathrm{CR}(T) = \mathrm{CR}(T|_{\mathrm{CR}(T)})$.
7. For every $x \in \mathrm{CR}(T)$ the set $[x]_\sim$ is closed and $T([x]_\sim) = [x]_\sim$.
8. If T has shadowing then $T(\Omega(T)) = \Omega(T)$.
9. If T has shadowing then also $(\Omega(T), T)$ has shadowing.

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Expansive dynamics - decomposition theorems

Theorem (Topological decomposition theorem)

Let (X, T) be a dynamical system with T surjective. Assume additionally that (X, T) is expansive with the shadowing property. Then the following assertions hold.

1. *(decomposition due to Smale) There are finitely many closed, T-invariant and pairwise disjoint sets $B_1, \dots, B_l \subset \Omega(T)$ such that:*
    1. $\Omega(T) = \sum_{i=1}^{l} B_i$,
    2. *Each dynamical system (B_i, T) is topologically transitive.*

    Sets B_i are called basic sets.
2. *(decomposition due to Bowen) For each basic set B there is k and a finite sequence of pairwise disjoint closed sets $C_0, \dots, C_{k-1}$ such that:*
    1. $T(C_i) = C_{i+1}$ *for* $i = 0, \dots, k-1$, *where for technical reasons* $C_k = C_0$,
    2. $B = \sum_{i=0}^{k-1} C_i$,
    3. (C_i, T^k) *is topologically mixing for each* i.

    Sets C_i are called elementary sets.