Tracing, mixing and entropy IV

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Expansive dynamics - decomposition theorems

Theorem (Topological decomposition theorem)

Let (X, T) be a dynamical system with T surjective. Assume additionally that (X, T) is expansive with the shadowing property. Then the following assertions hold.

1. *(decomposition due to Smale) There are finitely many closed, T-invariant and pairwise disjoint sets $B_1, \dots, B_l \subset \Omega(T)$ such that:*
   1. $\Omega(T) = \sum_{i=1}^{l} B_i$,
   2. *Each dynamical system (B_i, T) is topologically transitive.*

   Sets B_i are called basic sets.
2. *(decomposition due to Bowen) For each basic set B there is k and a finite sequence of pairwise disjoint closed sets $C_0, \dots, C_{k-1}$ such that:*
   1. $T(C_i) = C_{i+1}$ *for* $i = 0, \dots, k-1$, *where for technical reasons* $C_k = C_0$,
   2. $B = \sum_{i=0}^{k-1} C_i$,
   3. (C_i, T^k) *is topologically mixing for each* i.

   Sets C_i are called elementary sets.

A result by Richeson and Wiseman

Remark (Expansive dynamics is very special)

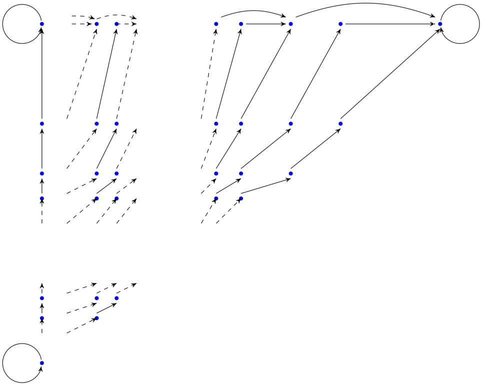
- It is well known that any equicontinuous system on the Cantor set has the shadowing property.
- Map with shadowing can have infinitely many chain-recurrent classes (e.g. identity on Cantor set).

Theorem (Richeson and Wiseman)

Suppose that (X, T) is chain transitive. Then one of the following assertions hold:

1. *there exists $n > 0$ such that (X, T) permutes cyclically n closed and open chain-recurrent classes of (X, T^n) and (X, T^n) is chain mixing on each of these classes;*
2. (X, T) *factors onto odometer.*

1. Let $T\colon X \to X$ be a homeomorphism.
2. $\Gamma_\varepsilon(x) = \{y : d(T^n(x), T^n(y)) < \varepsilon \text{ for all } n \in \mathbb{Z}\}$
3. Clearly, T is expansive iff there is $\lambda > 0$ such that $\Gamma_\lambda(x) = \{x\}$ for every x.
4. T is N-expansive if for some λ all $\Gamma_\lambda(x)$ have at most N-elements.



Shadowing and odometers

1. x is regularly recurrent if for every open neighborhood U of x there exists $k \in \mathbb{N}$ such that $T^{kn}(x) \in U$ for all $n \in \mathbb{N}_0$
2. The following conditions are equivalent:
    1. (X, T) is an odometer
    2. (X, T) is minimal, equicontinuous and every point is regularly recurrent.
3. If (X, T) is minimal with a regularly recurrent point then it is almost 1-1 extension of odometer (extension with residual set of singleton fibres)

Theorem

If (X, T) has shadowing and x is a recurrent point, then for every $\varepsilon > 0$ there is an odometer (Λ, T) such that $d_H(\omega_T(x), \Lambda) < \varepsilon$.

Specification property - another tracing property

Map f has the **(periodic) specification property** if, for any $\delta > 0$, there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \ldots, y_s\}$ of s points of X, and a sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ of $2s$ integers with $j_{m+1} - k_m \geq N_\delta$ for $m = 1, \ldots, s-1$, there is a point $x \in X$ such that, for each positive integer $m \leq s$ and all integers i with $j_m \leq i \leq k_m$, two following conditions hold:

1. $d(f^i(x), f^i(y_m)) < \delta$,
2. $f^n(x) = x$, where $n = N_\delta + k_s$ (periodicity condition).

We say about **weak specification property** in case that both conditions hold only for $s = 2$.

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Specification properties and mixing

1. Every mixing interval map has periodic specification property
2. Every mixing map with shadowing has specification property (not necessary periodic points)
   - if additionally expansive then it is periodic specification property
3. In subshifts, weak specification implies specification property
   1. Equivalently, there is N such that for any admissible words u, v there is w with $|w| = N$ such that uwv is admissible.

Transitivity and shadowing - a step further...

Theorem

If (X, T) is a dynamical system with the shadowing property, then the following conditions are equivalent:

1. *(X, T) is totally transitive,*
2. *(X, T) is weakly mixing,*
3. *(X, T) is mixing,*
4. *T is surjective and (X, T) has the specification property,*

If any of the above conditions is satisfied and (X, T) is expansive then it has the periodic specification property.

Theorem (Mai & Ye)

- *The only minimal (X, T) with the shadowing property are odometers.*
- *Transitive + shadowing but not minimal implies PTE.*

Shadowing in tent maps (Coven & Yorke)

$$T_s(x) = \begin{cases} sx & , 0 \leq x < 1 \\ s(2-x) & , 1 \leq x \leq 2. \end{cases}$$

1. T_s has shadowing for almost every $s \in (\sqrt{2}, 2]$.
2. Shadowing fails on uncountable, dense set of parameters.
3. Each T_s is mixing on the core $[2s - s^2, s]$

Sufficient condition for shadowing in dimension one

Definition

Let $f\colon X \to X$ be continuous and let $\varepsilon > 0$. A point $x \in X$ is ε-linked to a point $y \in X$ by f if there exists an integer $m \geq 1$ and a point z such that $f^m(z) = y$ and $d(f^j(x), f^j(z)) \leq \varepsilon$ for $j = 0, \ldots, m$.
We say $x \in X$ is linked to $y \in X$ by f if x is ε-linked to y by f for every $\varepsilon > 0$. A set $A \subset X$ is linked by f if every $x \in A$ is linked to some $y \in A$ by f.

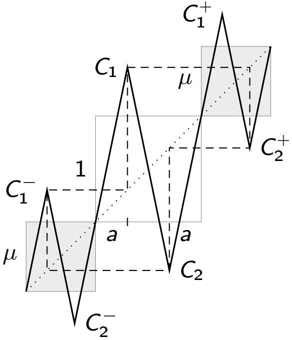
- The following is generalization of result by Coven, Kan and Yorke for tent maps (note that $C(f)$ contains endpoints).

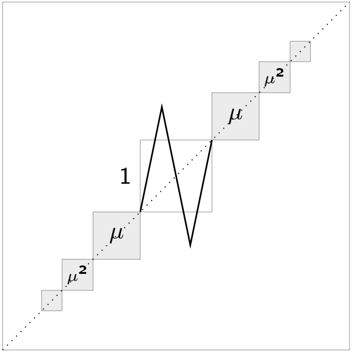
Theorem (Chen)

Suppose $f\colon [0,1] \to [0,1]$ is a map that is conjugate to a continuous piecewise linear map with a constant slope $\mathbf{s > 1}$. Then f has the shadowing property if and only if set of local extrema $C(f)$ is linked by f.

Finitely many pieces of monotonicity are essential

- Example from joint work with C. Good and M. Puljiz





- Use to construct transitive not mixing example with linking.

Prohorov metric - a recall

1. Prohorov metric D is defined by

$$D(\mu,\nu)=\inf\left\{\varepsilon\colon \quad \begin{array}{l}\mu(A)\le\nu(A^\varepsilon)+\varepsilon \text{ and } \nu(A)\le\mu(A^\varepsilon)+\varepsilon \\ \text{for any Borel subset } A\subset X\end{array}\right\}$$

for $\mu,\nu\in M(X)$.

2. The topology induced by D coincides with the *weak*-topology*
3. In practice, it is enough to use for any $\mu,\nu\in M(X)$,

$$D(\mu,\nu)=\inf\{\varepsilon\colon \mu(A)\le\nu(A^\varepsilon)+\varepsilon \text{ for all Borel subsets } A\subset X\}.$$

Approximation of measures

1. Assume that $\operatorname{diam} X = 1$ (for simplicity of calculations)
2. Denote $\mathbf{m}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i(x)}$.
3. If $0 \le k < n \le m$ then $D(\mathbf{m}_m(x), \mathbf{m}_{n-k}(T^k(x)) \le (m-n+k)/n$
4. If $d_n(x,y) < \varepsilon$ and $n \le k \le (1+\varepsilon)n$ then $D(\mathbf{m}_n(x), \mathbf{m}_k(y)) < \varepsilon$.
5. $D((1-\alpha)\mu + \alpha\nu, \mu) \le \min\{D(\mu,\nu), \alpha\}$.

6. for every ergodic μ and $\varepsilon > 0$ there is x and n such that $D(\mathbf{m}_s(x), \mu) < \varepsilon$ for every $s \ge n$.
7. for every $\mu \in M_T(X)$ and $\varepsilon > 0$ there are ergodic $\mu_1, \dots, \mu_k$ and $\alpha_1, \dots, \alpha_k$ such that

$$D(\mu, \sum_{i=1}^{k} \alpha_i \mu_i) < \varepsilon$$

An old result of Sigmund

1. Siegmund was first to develop approximation technique for density of some measures
2. Generic points with tracing can be used to approximate measures.

Theorem

If (X, T) has periodic specification property then for any measure $\mu \in M_T(X)$ and any $\varepsilon > 0$ there is a periodic point p and measure $\nu \in M_T(X)$ supported on p such that $D(\mu, \nu) < \varepsilon$. In other words, ergodic measures supported on periodic points are dense in $M_T(X)$.

A step further...

1. Measure $\mu \in M_T(X)$ is entropy approachable by ergodic measures, if for every $h^* < h_\mu(T)$ and every neighborhood U of μ there is an ergodic measure $\nu \in U$ with $h_\nu(T) > h^*$.
2. (X, T) is entropy dense if every invariant measure is entropy approachable.
3. If (X, T) has the specification property then it is entropy-dense [Eizenberg, Kifer, Weiss].
4. If $\mu \mapsto h_\mu(T)$ is upper semicontinuous then a measure of maximal entropy exists.
5. Bowen proved that specification+expansive $\Rightarrow$ unique measure of maximal entropy.

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Katok entropy formula

1. Katok entropy formula states the following:

$$h_\mu(T) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log N_T^\mu(n,\varepsilon,\delta),$$

   where $N_T^\mu(n,\varepsilon,\delta)$ denotes the smallest number of (n,ε)-Bowen balls covering a subset in X of μ-measure at least $(1-\delta)$ for some ergodic measure μ and an arbitrary $\delta > 0$.

2. Ergodic decomposition for entropy provides for every $\mu \in M_T(X)$ and $\varepsilon > 0$ ergodic $\mu_1,\ldots,\mu_k$ (not necessarily distinct) such that (for case $h_\mu(T) < \varepsilon$)
   - $D(\frac{1}{k}\sum_{i=1}^k \mu_i, \mu) < \varepsilon$,
   - $|\frac{1}{k}\sum_{i=1}^k h_{\mu_i}(T) - h_\mu(T)| < \varepsilon$.

3. If (X, T) is transitive with shadowing then we may approximate any $\mu \in M_T(X)$ by ergodic ν with close entropy.

Results of Pfister and Sullivan - a “weak specification”

Definition

We say that a dynamical system (X, T) has the approximate product structure if for any $\varepsilon > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ there exists an integer $N > 0$ such that for any $n \geq N$ and $\{x_i\}_{i=1}^{\infty} \subset X$ there are $\{h_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and $y \in X$ satisfying $h_1 = 0$, $n \leq h_{i+1} - h_i \leq n(1 + \delta_2)$ and

$$\left|\left\{0 \leq j < n \,:\, \rho\left(T^{h_i+j}(y), T^j(x_i)\right) > \varepsilon\right\}\right| \leq \delta_1 n \text{ for all } i \in \mathbb{N}.$$

Theorem (Pfister & Sullivan)

If (X, T) has approximate product property then ergodic measures are entropy dense.

Remark

If transitive (X, T) has shadowing property, then it has approximate product property.

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Entropy density - results with Jian Li

Let μ be any invariant measure for (X, T)

- When (X, T) has approximate product property then:
  1. $\lim_{n\to\infty} \mu_n = \mu$,
  2. $\liminf_{n\to\infty} h_{\mu_n}(T) \geq h_\mu(T)$,
  3. for some ergodic μ_n (supp μ_n is ... ?).
- If additionally $\mu \mapsto h_\mu(T)$ is upper semicontinuous then:
  1. $\limsup_{n\to\infty} h_{\mu_n}(T) \leq h_\mu(T)$, so $\lim_{n\to\infty} h_{\mu_n}(T) = h_\mu(T)$.

Theorem (Li, O.)

Suppose that (X, T) has shadowing property and is transitive. In this case:

1. *invariant measures whose supports are odometers (this includes periodic orbits) are dense in $M_T(X)$.*
2. *there is a sequence of ergodic measures μ_n such that:*
   1. *support of each μ_n is almost 1-1 extension of an odometer,*
   2. *$\lim_{n\to\infty} \mu_n = \mu$,*
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   3. $\lim_{n\to\infty} h_{\mu_n}(T) = h_\mu(T)$.

- For dynamical system (X, T), continuous function $\Phi : X \to \mathbb{R}$ and $n \in \mathbb{N}$ we define the Birkhoff average:

$$\frac{1}{n}\sum_{i=0}^{n-1}\Phi(T^i x).$$

- The set of points for which the above sum converges is called Φ-regular
- The complementary set is called Φ-irregular, denoted $I_\Phi(T)$
- By irregular set we mean the union:

$$I(T) = \bigcup_{\Phi\in\mathcal{C}(X,\mathbb{R})} I_\Phi(T).$$

- clearly $\mu(I(T)) = 0$ for every (ergodic) invariant measure μ.

Entropy of non-compact sets

- Let $E \subset X$,
- $\mathcal{G}_n(E,\varepsilon)$ be the collection of all at most countable covers of E by sets of the form $B_u(x,\varepsilon)$ with $u \geq n$. Put

$$C(E;t,n,\varepsilon,f) := \inf_{\mathcal{C}\in\mathcal{G}_n(E,\varepsilon)} \sum_{B_u(x,\varepsilon)\in\mathcal{C}} e^{-tu}$$

and

$$C(E;t,\varepsilon,f) := \lim_{n\to\infty} C(E;t,n,\varepsilon,f).$$

- Then we define

$$h_{\mathrm{top}}(E;\varepsilon,f) := \inf\{t : C(E;t,\varepsilon,f) = 0\} = \sup\{t : C(E;t,\varepsilon,f) = \infty\}$$

- The Bowen topological entropy of E is

$$h_{\mathrm{top}}(f,E) := \lim_{\varepsilon\to 0^+} h_{\mathrm{top}}(E;\varepsilon,f).$$

Irregular sets, specification and shadowing

Theorem (Chen, Tassilo, Shu; Li, Wu)

If (X, T) has the specification property then:

1. *when nonempty, $h_{top}(I(T)) = h_{top}(T)$.*
2. *$I(T)$ is either empty or residual,*

Theorem (Thompson)

If (X, T) has the (almost) specification property then for any Φ:

1. *$I_\Phi(T) = \emptyset$ or*
2. *$h_{top}(I_\Phi(T)) = h_{top}(T)$.*

Theorem (Dong, O., Tian)

if (X, T) has the shadowing property then one of the conditions hold:

1. *$h_{top}(T) = 0$ and $I(T) = \emptyset$,*
2. *$h_{top}(T) > 0$, $I(T) \neq \emptyset$ and $h_{top}(I(T)) = h_{top}(T)$*

Work with Foryś-Krawiec, Kupka and Tian

Theorem

Assume (X, T) has the shadowing property and Y is chain recurrent class. If $I_\Phi(T) \cap Y \neq \emptyset$ for some $\Phi\colon X \to \mathbb{R}$ then:

$$h_{top}(I_\Phi(T)) \geq h_{top}(Y).$$

As a consequence we obtian

$$h_{top}(I_\Phi(T)) \geq \sup\{h_{top}(Y) : Y \subseteq X, Y \cap I_\Phi(T) \neq \emptyset \text{ and } Y \text{ is a chain recurrent class}\}.$$

Question

Can we replace $\geq$ by $=$?

Partial answer

Theorem

Let (X, T) be a dynamical system with the shadowing property, $\Phi\colon X \to \mathbb{R}$ continuous. Then:

$$h_{top}(T, I_\Phi(T)) = \sup\{h_{top}(T, Y) : I_\Phi(T) \cap Y^\omega \neq \emptyset \text{ and } Y \subseteq X \text{ is chain recurrent class }\}.$$

where

$$Y^\omega = \{x : \omega(x) \subset Y\}.$$

Almost specification (very rough approximation of definition)

Map f has almost specification if for any sequence of integers $n_1, \ldots, n_k$ and numbers $\varepsilon_1, \ldots, \varepsilon_k > 0$:

1. for every sequence of orbits

$$x_1, f(x_1), \ldots, f^{n_1}(x_1), x_2, f(x_2), \ldots, x_k, f(x_k), \ldots, f^{n_k}(x_k)$$

2. there is a tracing point z which is further than ε from appropriate segment of orbit at no more than $g(n_i, \varepsilon_i)$ positions, where $g(\cdot, \cdot)$ is a function (depending only on f) such that $\lim_n g(n, \varepsilon)/n = 0$.

Why almost specification? - β-shifts

- β-transformation ($\beta > 1$):

$$T_\beta : x \mapsto \beta x \pmod 1)$$

- β-shift X_β - shift defined by natural partition for T_β.

Theorem (Pfisfer & Sullivan, 2005; Buzzi, 1997)

Every (X_β, σ) has the almost specification property, but for some β (in fact set of full Lebesgue measure) the specification property is not satisfied.

Theorem (Pfisfer & Sullivan, 2007)

Specification $\Longrightarrow$ Almost specification

- Thompson proved full measure for irregular sets $I_\Phi(T)$ under almost specification.

Measure center

Theorem (Kulczycki, Kwietniak & O.)

If A contains the measure center and $f|_A$ has almost specification then f also has almost specification

Theorem (Wu, O. & Chen)

If f has almost specification then $f|_A$ has almost specification

1. There are proximal systems (X, f) (and with singleton measure center) such that
   1. (X, f) is transitive but not weakly mixing; or
   2. (X, f) is weakly mixing but not mixing; or
   3. (X, f) is mixing (but obviously cannot have specification property);