Tracing, mixing and entropy V

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Graphs

By a graph we mean a pair $G = (V, E)$ of finite sets, where $E \subset V \times V$ (V – set of vertices, E – set of edges).

The graphs we consider are always edge surjective, i.e. for every $v \in V$ there are $u, w \in V$ such that $(u, v), (v, w) \in E$.

Graph homomorphisms

A map $\phi\colon V_1 \to V_2$ is a graph homomorphism between graphs (V_1, E_1), (V_2, E_2) if for every $(u, v) \in E_1$ we have $(\phi(u), \phi(v)) \in E_2$.

Important property

A homomorphism ϕ is bidirectional if $(u, v), (u, v') \in E_1$ implies $\phi(v) = \phi(v')$ and $(w, u), (w', u) \in E_1$ implies $\phi(w) = \phi(w')$.

bd-covers

If ϕ is a **bidirectional** map between **edge-surjective graps** then we call it bd-cover.

Construction inspired by (Akin, Glasner, Weiss, 2008)

Let $\mathcal{G} = \langle \phi_i \rangle_{i=0}^{\infty}$ be a sequence of bd-covers $\phi_i \colon (V_{i+1}, E_{i+1}) \to (V_i, E_i)$, and let

$$V_{\mathcal{G}} = \varprojlim (V_i, \phi_i) = \{x \in \Pi_{i=0}^{\infty} V_i : \phi_i(x_{i+1}) = x_i \text{ for all } i \geq 0\}$$

be the inverse limit defined by $\mathcal{G}$.
Set

$$E_{\mathcal{G}} = \{e \in V_{\mathcal{G}} \times V_{\mathcal{G}} : e_i \in E_i \text{ for each } i = 1, 2, \ldots\}$$

As usual, V_i is endowed with discrete topology and $\mathbb{X} = \prod_{i=0}^{\infty} V_i$ is endowed with product topology.

Lemma (Shimomura, 2014)

Let $\mathcal{G} = \langle \phi_i \rangle$ be a sequence of bd-covers $\phi_i \colon (V_{i+1}, E_{i+1}) \to (V_i, E_i)$. Then $V_{\mathcal{G}}$ is a zero-dimensional compact metric space and the relation $E_{\mathcal{G}}$ defines a homeomorphism.

Applications

Shimomura's approach provides **very effective** tool for description or construction of Cantor systems. It leads to huge simplifications in proofs and arguments.

Example: odometer

- G_0 has a cycle c_0.
- $\varphi_n : G_{n+1} \to G_n$ is defined by,

$$\phi_n(c_{n+1}) = a_n \cdot c_n.$$

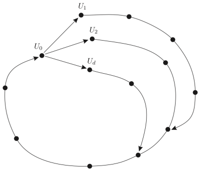
Example: transitive non-minimal system

- G_0 has two cycles $c_{0,1}, c_{0,2}$.
- $\varphi_n : G_{n+1} \to G_n$ is defined, for $i = 1, 2$,

$$\phi_n(c_{n+1,1}) = 3c_{n,1}, \quad \phi_n(c_{n+1,2}) = 2 \cdot c_{n,1} + 2 \cdot c_{n,2} + c_{n,1}.$$

Minimal systems - Gambaudo-Martens approach

1. Ecah graph G_i has special vertex $v_{i,0}$.
2. Each vertex in G_i has at least one outgoing edge but ony $v_{i,0}$ can have ore than one
3. Eeach G_i is strongly connected (there is a path between two vertices)
4. $\varphi_i(v_{i,0}) = v_{i-1,0}$ for every $i \geq 1$,



6. the cycle $c_{i,j}$ in G_i can be written as $v_{i,0} = v_{i,j,0}, v_{i,j,1}, v_{i,j,2}, \ldots, v_{i,j,l(i,j)} = v_{i,0}$ with the length $l(i,j) \geq 1$,
7. $\varphi_i(v_{i,j,1}) = v_{i-1,1,1}$ for $i \geq 1$ and $j = 1, 2, \ldots, r_i$.

Simple Gambaudo-Martens coverings

A GM-covering is called simple if, for $i \geq 1$, there exists $m > i$ such that

$$E(\varphi_{m,i}(c_{m,j})) = E(G_i),$$

for each cycle $c_{m,j}$ in G_m.

Lemma

A zero-dimensional dynamical system is minimal if and only if it can be represented as the inverse limit of a simple GM-covering.

Minimal systems - old in new language

Over the years several techniques were obtained to construct minimal systems with desired properties.

- There are several general methods, which are modifications of celebrated Jewett-Krieger theorem. They provide many examples with desired dynamical properties (as a consequence of properties of selected invariant measure).
- But techniques that may be directly applied in various context (e.g. to detect dynamical properties in concrete cases) are also of interest.

1. Examples of weakly mixing minimal system
    - If (X, T) is transitive and there is a point x with dense orbit such that
    - for each open $U \ni x$ there is n such that
    - $T^n(U) \cap U \neq \emptyset$ and $T^{n+1}(U) \cap U \neq \emptyset$
    - Then (X, T) is weakly mixing.
2. Examples of minimal systems with positive entropy

Krieger's Marker Lemma (Downarowicz's version)

1. (X, T) - a zero-dimensional system.
2. By an *n-marker* we mean a clopen set $F \subset X$ such that:
   - no orbit visits F twice in n steps (i.e. $F, T^{-1}(F), \ldots, T^{-(n-1)}(F)$ are pairwise disjoint)
   - every orbit visits F at least once (by compactness, this implies that for some $N \in \mathbb{N}$, we have $F \cup T^{-1}(F) \cup \ldots \cup T^{-(N-1)}(F) = X$).

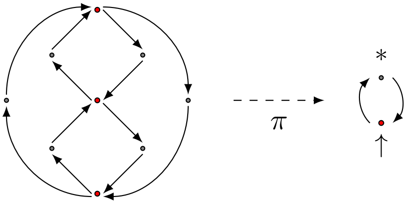
Theorem (Krieger's Marker Lemma, aperiodic case)

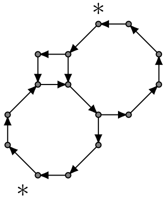
If (X, T) is an aperiodic zero-dimensional system then for every $n \in N$ there exists an n-marker. The parameter N in above can be selected equal to $N = 2n - 1$.

- If we fix any clopen set U, then in Krieger's Marker Lemma we may require that $F \cap U \neq \emptyset$.

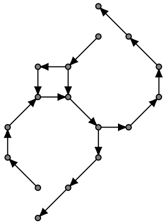
Examples of applications of Marker Lemma

Avoiding stacking of markers





Acyclic graph by breaking $*$



Shadowing in Cantor systems (Good, Meddaugh)

- An inverse system (of spaces or of dynamical systems) satisfies the Mittag-Leffler condition provided that for all $i \geq 0$ there exists $j \geq i$ such that for every $k \geq j$ we have $\varphi_{k,i}(X_k) = \varphi_{j,i}(X_j)$.
- if φ_i are injective, they satisfy Mittag-Leffler condition.

Theorem (Simplified version)

Let X be the Cantor set, or indeed any compact, totally disconnected metric space. The map $T: X \to X$ has shadowing if and only if (T, X) is conjugate to the inverse limit of a sequence satisfying the Mittag-Leffler condition and consisting of shifts of finite type.

Typical homeomorphisms on Cantor set

1. There exists a residual set $\mathcal{R} \subset \mathcal{H}(C)$ such that if $T, S \in \mathcal{R}$ then (C, T), (C, S) are conjugate.
2. First proof by Kechris and Rosendal on existence of $\mathcal{R}$.
3. Akin, Glasner and Weiss - proof with complicated construction of an element
4. Bernardes, Darji - nice characterization via graph covers (for homeomorphisms and continuous surjections)

Theorem (Bernardes, Darji)

The set of all $T \in \mathcal{H}(C)$ with the following property is a residual conjugacy class of $\mathcal{H}(C)$. For every $m \in \mathbb{N}$, there are a partition P of C of mesh $< 1/m$ *and a multiple $q \in \mathbb{N}$ of m such that every component of $G_T(P)$ is a balanced dumbbell with plate weight $q!$ that contains both a left and a right loop of T.*

A representative of conjugacy class

1. Roughly speaking:

   attractor-repellor pair of two odometers $\times$ identity on Cantor set

2. Properties of maps in the class include:
   - Zero (sequence) entropy
   - Shadowing property
   - Lack of periodic points
   - $\mathrm{Rec}(T)$ is a Cantor set with empty interior
   - ...