Experimental Mathematics in dynamical Systems

Roberto Barrio[1]

[1] Computational Dynamics group, Dept. Applied Mathematics – University of Zaragoza, SPAIN
rbarrio@unizar.es, http://cody.unizar.es/

In collaboration with:
David H. Bailey (Lawrence Berkeley National Laboratory, USA),
Jonathan M. Borwein (University of Newcastle, Australia),
M. Lefranc (University of Lille, France),
A. Shilnikov, T. Xing (Georgia State University, Atlanta),
D. Wilczak (Jagiellonian University, Krakow, Poland),
J. T. Lázaro, J.R. Pacha (Universidad Politécnica de Cataluña, Spain),
Sergio Serrano, Marcos Rodríguez, M. A. Martínez, Angeles Dena (Universidad de Zaragoza, Spain)

DDAYS'14, Badajoz
12-14 November, 2014

Outline

- Our point of view and a fast review of our current research

---

1. Numerical techniques to explore complex behaviors
2. Experimental mathematics: high-precision and more ...
3. Need to mixing techniques (so collaborations): “bailar juntos”
4. Need to prove some results: CAP
5. Need to use “problem”-specific techniques: kneading plots

Outline

1. Numerical techniques to explore complex behaviors
2. Experimental mathematics: high-precision and more ...
3. Need to mixing techniques (so collaborations): “bailar juntos”
4. Need to prove some results: CAP
5. Need to use “problem”-specific techniques: kneading plots

Numerical tools (the need of a lot of techniques ...)

- General chaos indicators:
  MLE, Lyapunov exponents, MEGNO, FLI, OFLI2, FMA ...
- Computation of invariants
  - Continuation of periodic orbits ...: MATCONT, AUTO, CONTIDES (Abad, B., Dena and Rodríguez, 2012[illegible]3) ...
  - Bifurcation analysis: MATCONT, AUTO, ...
- Particular chaos and dynamics indicators
  - Neurocomputing:
    Spike-counting diag. (Storace, 2009; B. and Shilnikov, 2011),
    Duty-cycle diagrams (B. and Shilnikov, 201[illegible]), ...
  - Specific phenomena: Shrimps and topological changes (B., Blesa, Serrano and Shilnikov, 20[illegible]), T-points (B. and Shilnikov, 2011), ...
- Detailed studies
  - Arbitrary precision:
    NUMERICAL MICROSCOPE (Abad, B. and Dena, 2012)
  - Rigorous studies CAP: Rigorous DATA BASE of periodic orbits (CAPD) (B., Dena, Tucker, 2014), Hyperchaos (2014), ...

INSIGHT, NOT JUST NUMBERS!

What is TIDES?

TIDES: a **T**aylor series **I**ntegrator for **D**ifferential **E**quation**S**

- Taylor series method using Variable-Stepsize Variable-Order formulation and extended formulas for the variational equations.

- Free numerical software based on extended Taylor series method: **TIDES**[*].
  - ***Extremely easy*** to use via a MATHEMATICA preprocessor.

NOW INCLUDED IN **SAGE** (2014)!!!

- ***Automatic*** construction of Fortran or C codes for solving ODEs

- ***Automatic*** construction of C codes for solving solutions of ODEs and ***variational equations up to any order*** (and sensitivities with respect to any parameter up to any order)

- ***Easy to use arbitrary precision*** (do you need 500 digits?, 1000?)

Where?: `http://cody.unizar.es/software`
or email: `tides.taylor@gmail.com`, `rbarrio@unizar.es`

---

[*] A. Abad, R. Barrio, F. Blesa and M. Rodriguez, "Algorithm 924: TIDES, a Taylor series Integrator for Differential EquationS," *ACM Trans. Math. Software*, Volume 39(1), 2012

Outline

1. Numerical techniques to explore complex behaviors
2. Experimental mathematics: high-precision and more ...
3. Need to mixing techniques (so collaborations): “bailar juntos”
4. Need to prove some results: CAP
5. Need to use “problem”-specific techniques: kneading plots

Experimental mathematics in Dynamical Systems

Origin of the discipline in Spain

⟶ Prof. Carles Simó (Universidad de Barcelona)

Experimental mathematics (?)[†]

“Experimental mathematics” has emerged in the past 25 years or so to become a competitive paradigm for research in the mathematical sciences.

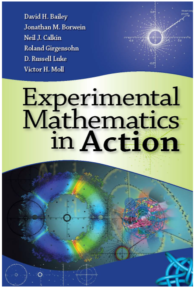
So what exactly is “experimental mathematics”? While several definitions have been offered, perhaps the most succinct definition is given in the Borwein-Devlin book *The Computer as Crucible*:

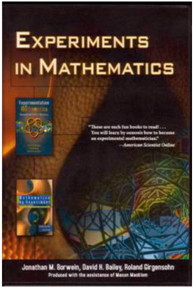
> Experimental mathematics is the use of a computer to run computations — sometimes no more than trial-and-error tests — to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search.

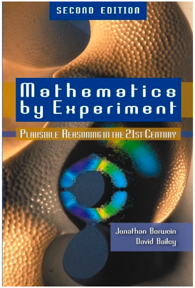
Here we should distinguish “experimental mathematics” from “computational mathematics” and “numerical mathematics.” While there is no clear delineation, the latter two terms generally encompass computational methods for concrete applied mathematics and engineering applications.

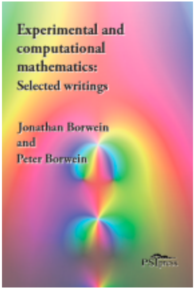
[†] D.H. Bailey, R. Barrio, J.M. Borwein, High-precision computation: Mathematical physics and dynamics, Appl. Math. Comput. 218(20), pp 10106-10121 (2012)

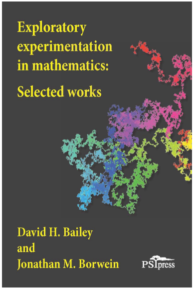
Experimental mathematics: books

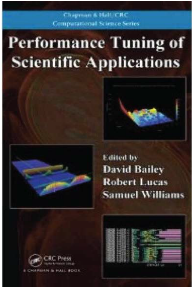




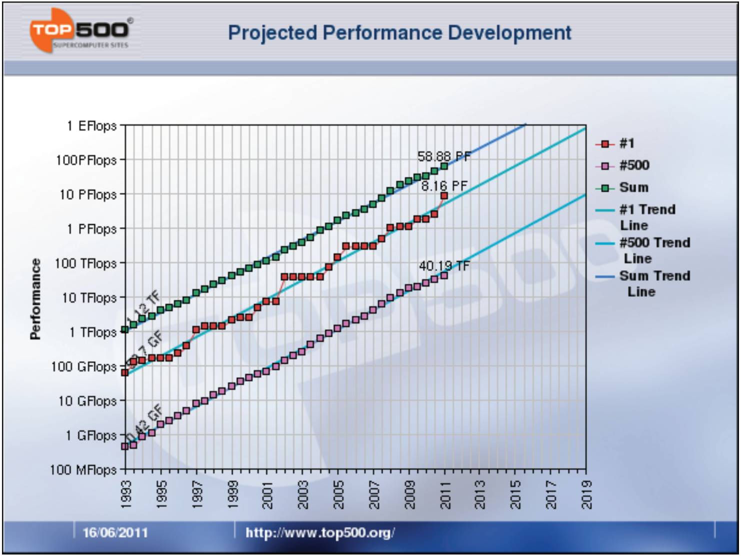








Progress of Scientific Supercomputers: Data from the Top500 List

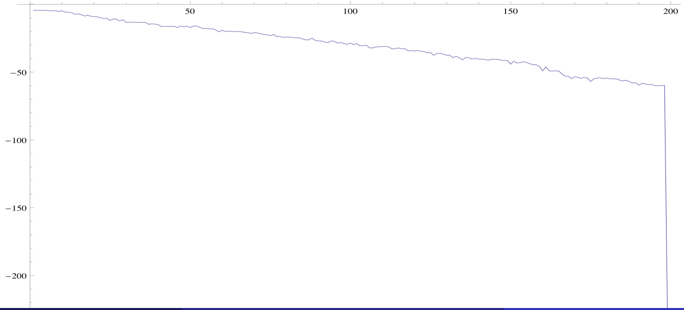


Experimental math: Discovering new mathematical results by computer

- Compute various mathematical entities (limits, infinite series sums, definite integrals) to high precision, typically 100-1000 digits.
- Use algorithms such as PSLQ to recognize these entities in terms of well-known mathematical constants.
- When results are found experimentally, seek to find formal mathematical proofs of the discovered relations.

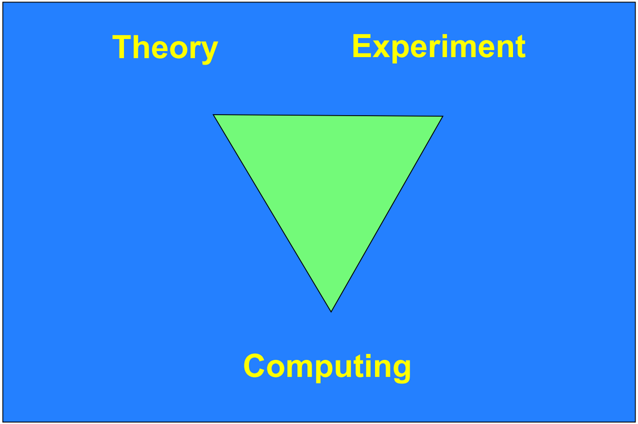
Many results have recently been found using this methodology, both in pure mathematics and in mathematical physics.

Decrease of $\log 10(\min|y_i|)$ in multipair PSLQ run



Computing:
The Third Mode of Scientific Discovery

Mathematics → Computer science → Scientific computing → Mathematics



Theory

Experiment

Computing

Outline

1. Numerical techniques to explore complex behaviors
2. Experimental mathematics: high-precision and more ...
3. Need to mixing techniques (so collaborations): “bailar juntos”
4. Need to prove some results: CAP
5. Need to use “problem”-specific techniques: kneading plots

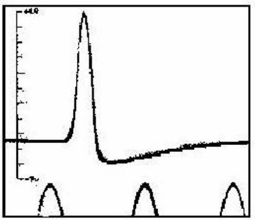
References

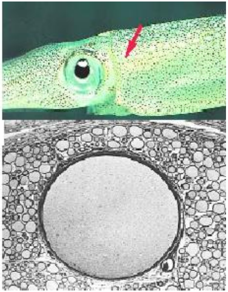
- R. Barrio, M. Lefranc, M.A. Martinez, S. Serrano, *Symbolic dynamical unfolding of spike-adding bifurcations in chaotic neuron models*, Europhysics Letter (EPL), submitted, 2014.
- R. Barrio, M. Lefranc, M.A. Martinez, S. Serrano, in preparation, 2014.
- R. Barrio, M.A. Martinez, S. Serrano, A. Shilnikov, *Macro-and micro-chaotic structures in the Hindmarsh-Rose model of bursting neurons*, Chaos: An Interdisciplinary Journal of Nonlinear Science 24 (2), 023128, 2014.
- R. Barrio, A. Shilnikov, *Parameter-sweeping techniques for temporal dynamics of neuronal systems: case study of Hindmarsh-Rose model*, The Journal of Mathematical Neuroscience (JMN) 1 (1), 1-22, 2011.

Mathematical neuron models

The Hodgkin-Huxley model is a mathematical model that describes how action potentials in neurons are initiated and propagated.

Alan Lloyd Hodgkin and Andrew Huxley described the model in 1952 to explain the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. They received the 1963 Nobel Prize in Medicine for this work.

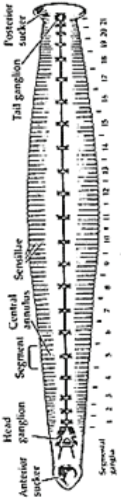


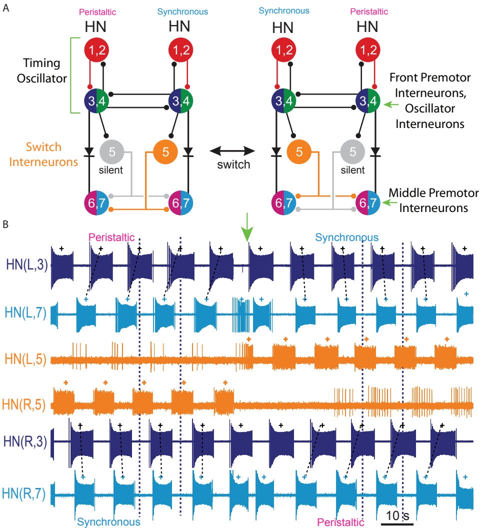


Other mathematical models:

FitzHugh-Nagumo model, Morris-Lecar model, Hindmarsh-Rose model (1984), ...

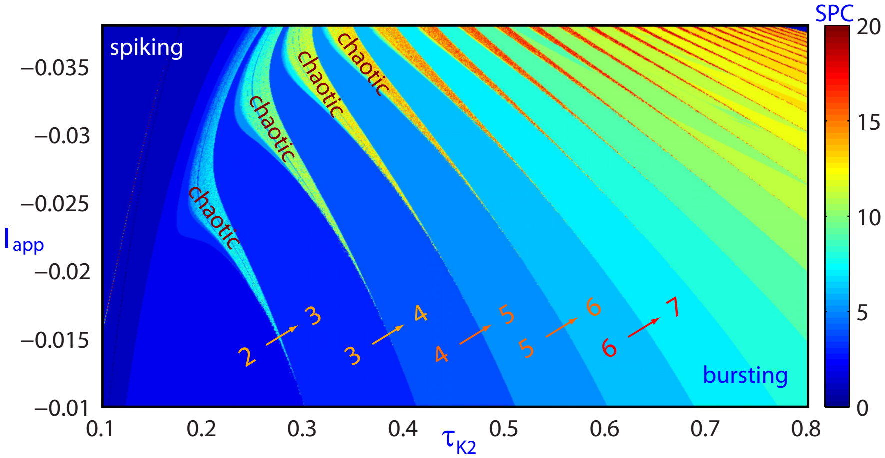
The leech heart interneuron model[‡]





[‡] Weaver, A.L., Roffman, R.C., Norris, B.J., Calabrese, R.L. A role for compromise: synaptic inhibition and electrical coupling interact to control phasing in the leech heartbeat CpG. Frontiers in behavioral neuroscience. 2010.

The leech heart interneuron model



- Onion-like structures.
- Spike-adding bifurcations $\longrightarrow$ UPOs foliated in the attractor $\longrightarrow$??

The Hindmarsh and Rose model

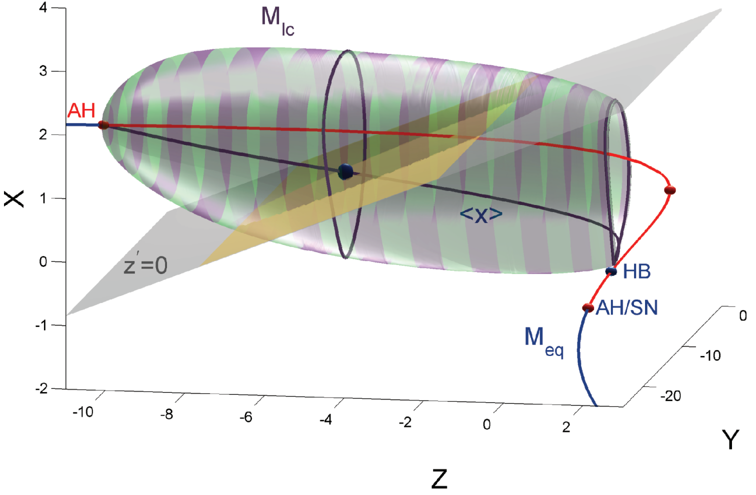
The Hindmarsh and Rose model (1984)

A phenomenological system of ODEs for modeling bursting and spiking oscillatory activities in isolated neurons:

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I \\ \dot{y} &= c - dx^2 - y \\ \dot{z} &= \varepsilon(s(x - x_0) - z) \end{aligned} \qquad (1)$$

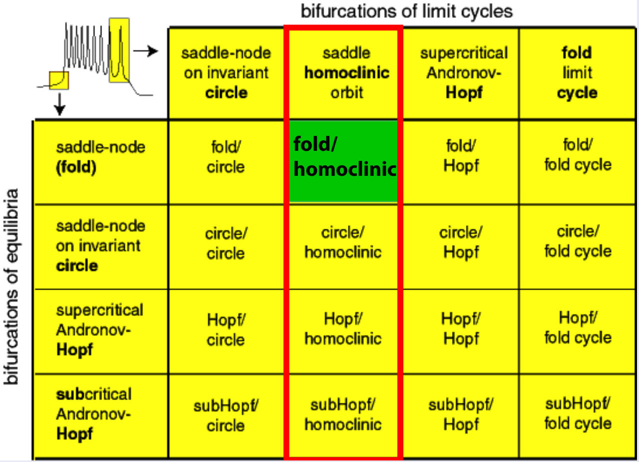
- x is treated as the membrane potential, while y and z describe some fast and slow gating variables for ionic currents.
- Slow “activation" of z is due to the small parameter $0 < \varepsilon \ll 1$.
- The parameters are typically set as $a = 1$, $c = 1$, $d = 5$, $s = 4$, $x_0 = -1.6$ and $\varepsilon = 0.01$.
- A key parameter is the external applied current I.

Fast-slow decomposition



Fast-slow decomposition

Bursting classification (Rinzel-Izhikevich)



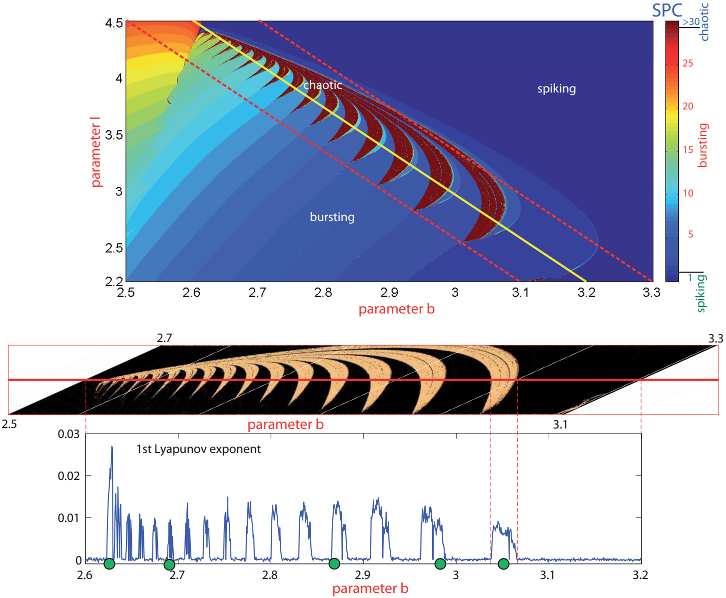
bifurcations of limit cycles

bifurcations of equilibria

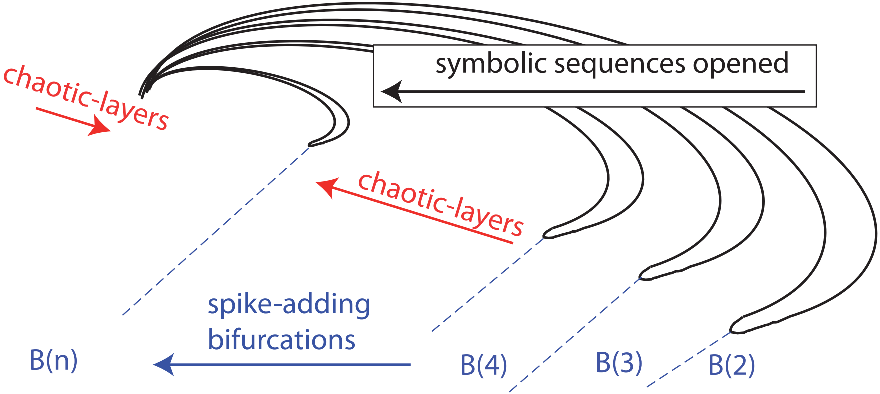
	saddle-node on invariant **circle**	saddle **homoclinic** orbit	supercritical Andronov-**Hopf**	**fold** limit **cycle**
saddle-node **(fold)**	fold/ circle	**fold/ homoclinic**	fold/ Hopf	fold/ fold cycle
saddle-node on invariant **circle**	circle/ circle	circle/ homoclinic	circle/ Hopf	circle/ fold cycle
supercritical Andronov-**Hopf**	Hopf/ circle	Hopf/ homoclinic	Hopf/ Hopf	Hopf/ fold cycle
subcritical Andronov-**Hopf**	subHopf/ circle	subHopf/ homoclinic	subHopf/ Hopf	subHopf/ fold cycle

Square-wave bursting (fold/hom bursting)

But ... why and where is the chaotic behavior?



Scheme of the macroscopic chaotic structures ($\forall \varepsilon$)



symbolic sequences opened

chaotic-layers

chaotic-layers

spike-adding bifurcations

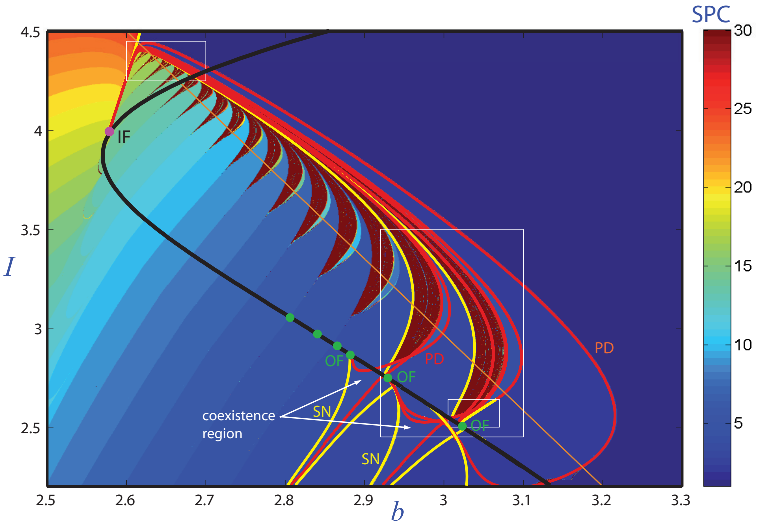
B(n)

B(4)

B(3)

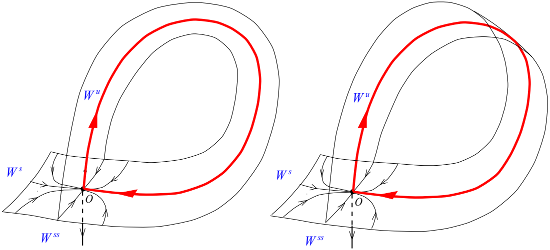
B(2)

The Hindmarsh and Rose model: bifurcation analysis

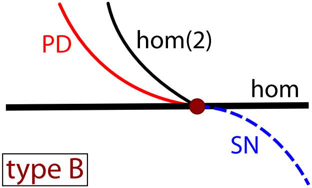


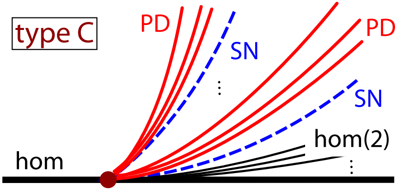
Codimension-two homoclinic bifurcation points

- Inclination flip

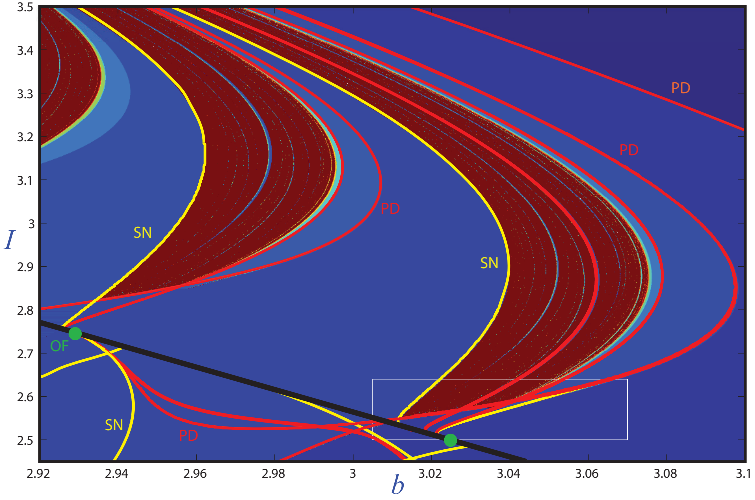


- Inclination flip + orbit flip homoclinic bifurcations

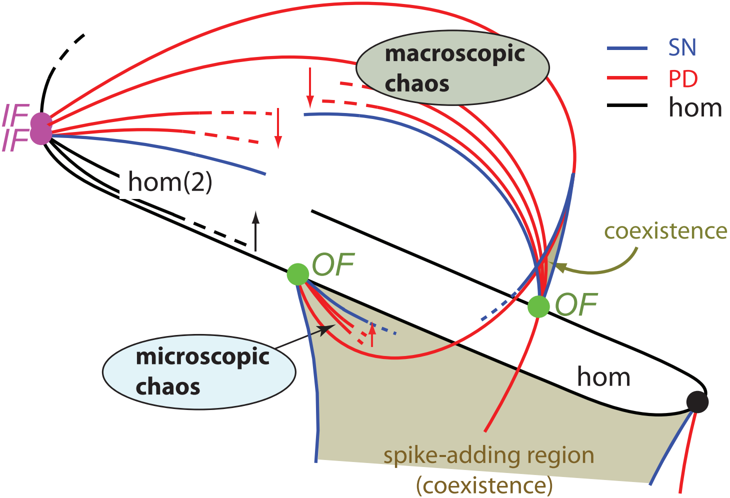


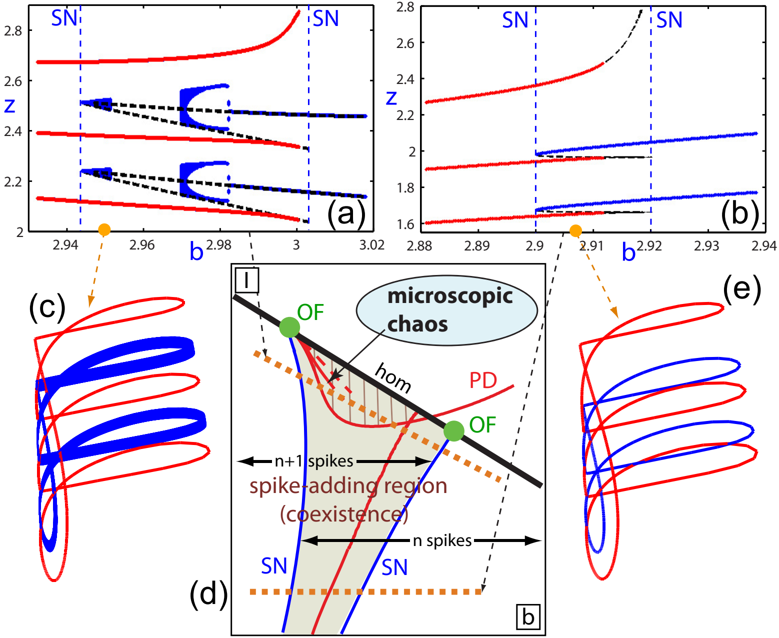


The Hindmarsh and Rose model



Hindmarsh and Rose: bifurcation sketch




SN
SN
2.8
2.6
z
2.4
2.2
2
2.94
2.96
2.98
3
3.02
b
(a)
2.8
2.6
2.4
z
2.2
2
1.8
1.6
2.88
2.89
2.9
2.91
2.92
2.93
2.94
b
(b)
(c)
(e)
l
OF
microscopic chaos
hom
PD
OF
n+1 spikes
spike-adding region (coexistence)
n spikes
SN
SN
(d)
b

Topological templates

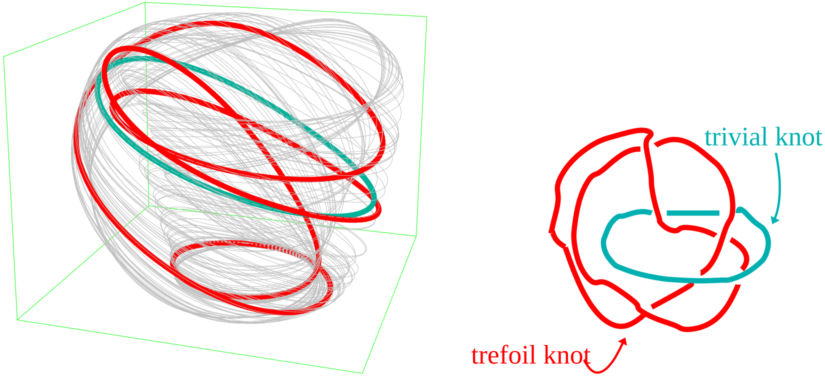
Given a three-dimensional (3D) hyperbolic chaotic flow Φ_t, Birman and Williams define the following equivalence relation which identifies points of the invariant set Λ having the same asymptotic future:

$$\forall x, y \in \Lambda, \quad x \sim y \Leftrightarrow \lim_{t \to \infty} ||\Phi_t(x) - \Phi_t(y)|| = 0$$

The Birman-Williams theorem states:

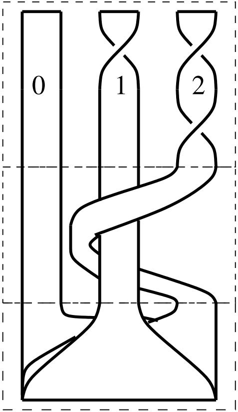
1. In the set of equivalence classes, the hyperbolic flow Φ_t induces a semi-flow $\bar{\Phi}_t$ on a branched manifold $\mathcal{K}$. The pair $(\bar{\Phi}_t, \mathcal{K})$ is called a *template*, or knot-holder
2. Unstable periodic orbits of Φ_t in Λ are in one-to-one correspondence with unstable periodic orbits of ($\bar{\Phi}_t$ in $\mathcal{K}$). Moreover, every link of unstable periodic orbits of (Φ_t, Λ) is isotopic to the corresponding link of $(\bar{\Phi}_t, \mathcal{K})$.

Topological templates



Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Topological templates: subtemplates



torsion

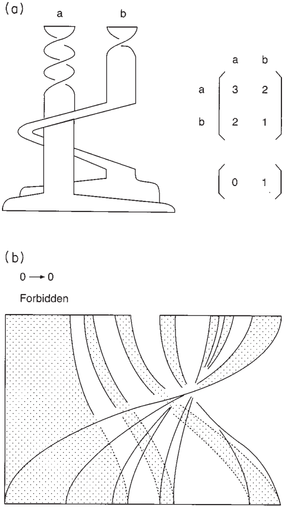
$t_{00} = 0,$
$t_{11} = 1,$
$t_{22} = 2$

linking

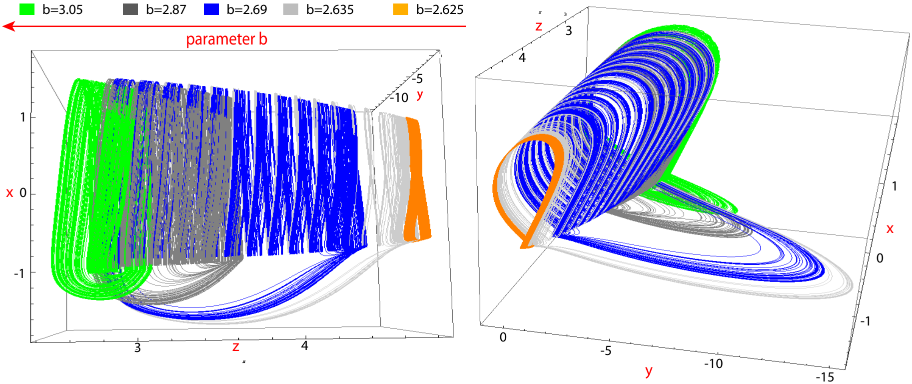
$t_{01} = 0,$
$t_{02} = 0,$
$t_{12} = 2$

layering

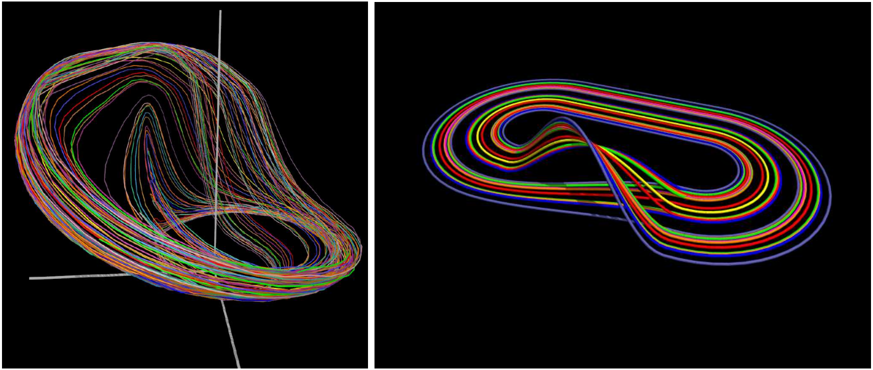
$l_{01} = 1,$
$l_{02} = 1,$
$l_{12} = -1$



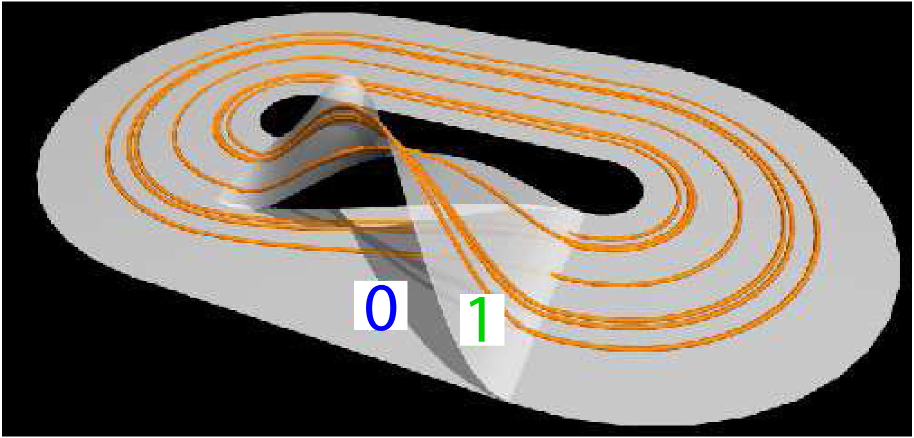
The Hindmarsh and Rose model: chaotic attractors



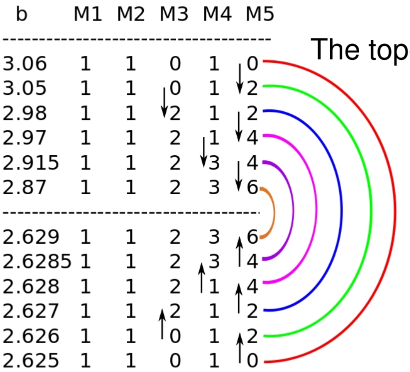
Topological templates: Hindmarsh and Rose model



Topological templates: Hindmarsh and Rose model

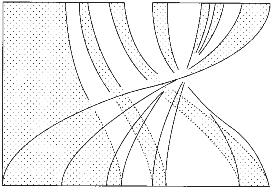


Topological templates: Hindmarsh and Rose model



b	M1	M2	M3	M4	M5
3.06	1	1	0	1	0
3.05	1	1	0	1	2
2.98	1	1	2	1	2
2.97	1	1	2	1	4
2.915	1	1	2	3	4
2.87	1	1	2	3	6
2.629	1	1	2	3	6
2.6285	1	1	2	3	4
2.628	1	1	2	1	4
2.627	1	1	2	1	2
2.626	1	1	0	1	2
2.625	1	1	0	1	0

The topological template for the HR model is the Smale's horseshoe template but with forbidden "lanes".

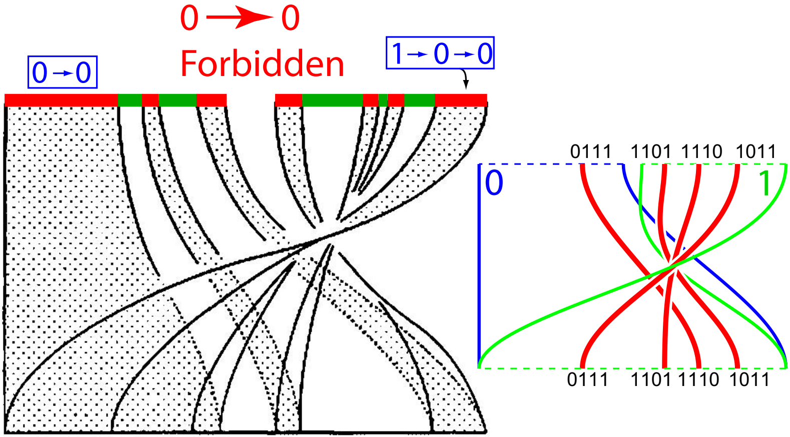


The templates follows the "onion" structure.

♣ Proper *grammar* of the symbolic sequences

⟶ (forbidden symbolic sequences)

Topological templates: successive Cantor structures



- Theoretical framework
  for the “onion-bulb” chaotic structures

Outline

1. Numerical techniques to explore complex behaviors
2. Experimental mathematics: high-precision and more ...
3. Need to mixing techniques (so collaborations): “bailar juntos”
4. Need to prove some results: CAP
5. Need to use “problem”-specific techniques: kneading plots

References

- R. Barrio, A. Martinez, S. Serrano, D. Wilczak, *When chaos meets hyperchaos*, submitted for publication, 2014.
- D. Wilczak, A. Martinez, S. Serrano, R. Barrio, preprint, 2014
- R. Barrio, M. Rodriguez, *Systematic Computer Assisted Proofs of periodic orbits of Hamiltonian systems*, Communications in Nonlinear Science and Numerical Simulation 19 (8), 2660-2675, 2014.
- R. Barrio, A. Dena and W. Tucker, *A data-base of rigorous and high-precision periodic orbits of the Lorenz model*, submitted for publication (2014).
- R. Barrio, M. Rodriguez, F. Blesa, *Computer-assisted proof of skeletons of periodic orbits*, Computer Physics Communications 183 (1), 80-85, 2012.

Playing with hyperchaos

Classical hyperchaos was invented by Sinai (around 1978). He showed that billiards colliding in 3D produce maximal chaos, that is, possess $n-1$ positive Lyapunov characteristic exponents.

Ya.G. Sinai, Appendix to the translation of S. Krylov, Works on the Foundations of Statistical Physics, Princeton University Press, Princeton, 1980.

For simple dissipative system with hyperchaos, that is, more than one direction of divergence of trajectories the first system was introduced by O. Rössler (1979).

O. E. Rossler, An equation for hyperchaos, Physics Letters A, 71, 155-157, 1979.

The occurrence of hyperchaotic behavior has been found in an electronic circuit (Matsumoto et al, 1986), NMR laser (Stoop et al, 1988), in a semi-conductor system (Stoop et al, 1989) and in a chemical reaction system (Eiswirth et al, 1992).

M. Eiswirth, Th.-M. Kruel, G. Ertl and F. W. Schneider, Hyperchaos in a chemical reaction, Chemical Physics Letters, 193 (4), 305, 1992.
T. Matsumoto, L. O. Chua and K. Kobayashi, Hyperchaos: laboratory experiment and numerical confirmation, 'IEEE Transactions on Circuits and Systems, CAS-33 (11), 1143-1147, 1986.

R. Stoop, J. Peinke, J. Parisi, B. Rohricht and R. P.Hubener, A p-Ge semiconductor experiment showing chaos and hyperchaos, Physica D, 35, 425-435, 1989.

Experimental Hyperchaos

Electronic circuits

T. Kapitaniak, L. Chua, G. Zhong, Experimental Hyperchaos in Coupled Chua's Circuits, IEEE Transactions on Circuits and Systems, 41 (7), 1994.

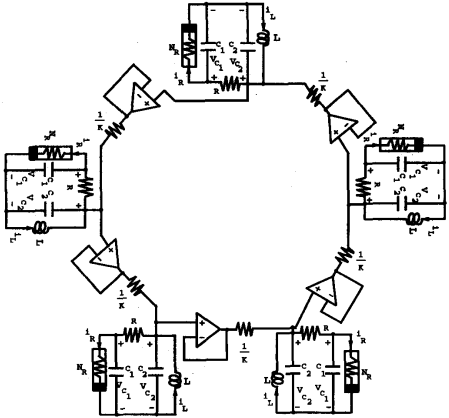
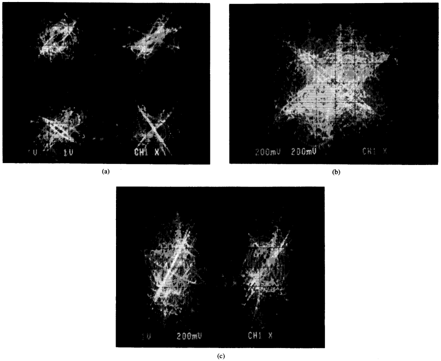


Fig. 1. Five identical coupled Chua's circuits forming a ring.



(a)

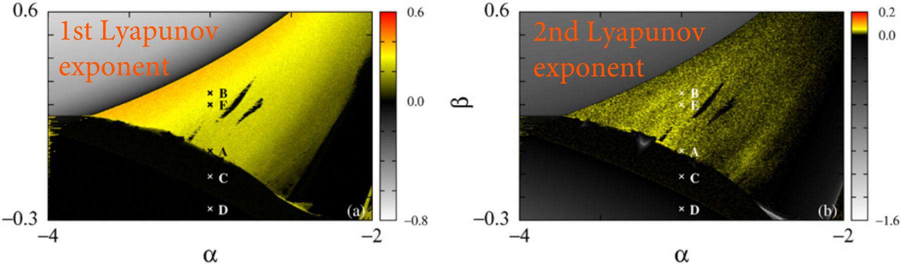
(b)

(c)

Fig. 2. Experimental two-dimensional projections of hyperchaotic attractors: $K_{1-5} = 0.01$; (a) $v_{C1}^{(1)}$ versus $v_{C1}^{(3)}$, Horizontal axis is $v_{C1}^{(1)}$, 1 V/div; Vertical axis is $v_{C1}^{(3)}$, 1 V/div, (b) $v_{C2}^{(1)}$, versus $v_{C2}^{(3)}$, Horizontal axis is $v_{C2}^{(1)}$, 200 mV/div; Vertical axis is $v_{C2}^{(3)}$, 200 mV/div, (c) $v_{C1}^{(1)}$ versus $v_{C2}^{(3)}$, Horizontal axis is $v_{C1}^{(1)}$, 1 V/div; Vertical axis is $v_{C2}^{(3)}$, 200 mV/div.

Playing with hyperchaos

But ... in several simulations [§] ALWAYS (?) quite noisy



- So, chaotic?
- Hyperchaotic?
- WHAT?

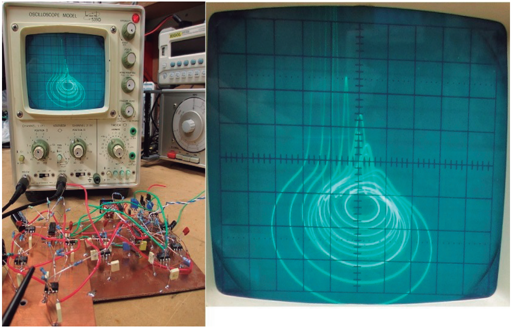
[§] P. Rech, Chaos and hyperchaos in a Hopfield neural network, Neurocomputing, 2011.

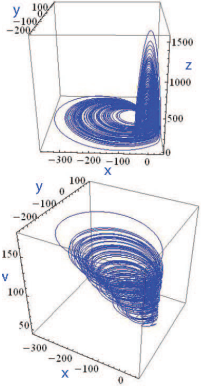
Playing with hyperchaos

Biparametric study of the 4D Rössler system

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + ay + w, \\ \dot{z} = b + xz, \\ \dot{w} = -cz + dw, \end{cases}$$

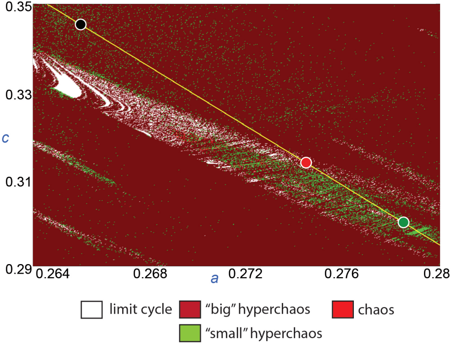
where we fix the values of parameters *b*=3.0 and *d*=0.05, and we allow the values of *a* and *c* change.





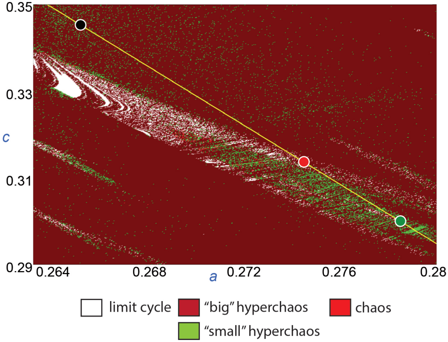
Playing with hyperchaos

Biparametric study of the 4D Rössler system (Lyapunov exponents based)

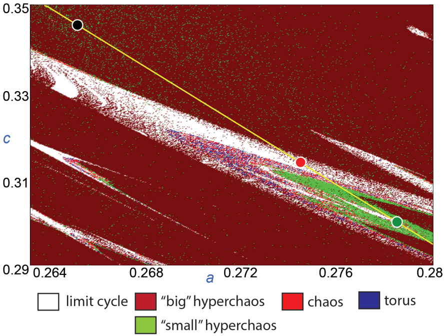


Playing with hyperchaos

Biparametric study of the 4D Rössler system (Lyapunov exponents based)



... and now with a long transient time + larger precision +
.... + 189 days



STILL NOISY !!!!

... and now some theorems (CAP)

Extensive numerical studies yield us to find many approximate periodic orbits for P. For some pairs of these orbits we could find approximate heteroclinic connections on which we will built chaotic dynamics.

$$\begin{aligned}
p_1^8 &= (-104.32937253702462, 0.028756669726685443, 44.645081351998819),\\
p_2^8 &= (-104.26664163365506, 0.028773972266421831, 44.640115482927115),\\
p_3^8 &= (-104.42324539012806, 0.028730815749171541, 44.678254866134068),\\
p_4^{16} &= (-104.39575243552828, 0.028738382959034744, 44.666264617071981),\\
q_1^{12} &= (-103.69667754570543, 0.028932144798038389, 44.407870627484129),\\
q_2^{14} &= (-103.37098255164607, 0.029023312473829044, 44.284349486019579).
\end{aligned}$$

The points p_i^j are approximate periodic points for P of period j with one-dimensional unstable manifold. The points q_i^j are approximate periodic points of period j with two-dimensional unstable manifold.

... and now some theorems (CAP)

This numerical study $\longrightarrow$ generates hypothesis and conjectures

Now it is time to state results :-)

- Defining the Poincaré section

$$\Pi = \{(x, 0, z, w) \in \mathbb{R}^3; \dot{y} = x + z < 0\}$$

  and $P : \Pi \to \Pi$, the associated Poincaré map, and fixing the parameter values $a = 0.27857$, $b = 3$, $c = 0.3$ and $d = 0.05$.
- Using rigorous ODE solvers for the systems and variational equations (CAPD library[¶] of the CAPD group (Krakow))

[¶] Free-software: `http://http://capd.ii.uj.edu.pl/`

... and now some theorems (CAP)

Theorem

For each $u^j \in \{p_1^8, p_2^8, p_3^8, p_4^{16}, q_1^{12}, q_2^{14}\}$ there is a unique periodic orbit v for P of the principal period j in the ball $B(u^j, 10^{-8})$ in the maximum norm. Moreover, the resulting periodic points close to p_i^j have one-dimensional unstable manifold and those corresponding to q_i^j have two-dimensional invariant manifold.

Proof Let us fix $u^j \in \{p_1^8, p_2^8, p_3^8, p_4^{16}, q_1^{12}, q_2^{14}\}$ and define $F : \Pi^j \to \Pi^j$ by

$$F_j(v_1, v_2, \ldots, v_j) = (v_1 - P(v_j), v_2 - P(v_1), \ldots, v_j - P(v_{j-1})).$$

Solutions to $F_j(v_1, v_2, \ldots, v_j) = 0$ correspond to j-periodic orbits for P provided $v_i \neq v_c$ for $i \neq c$.
Put

$$z_0 = (v_1, v_2, \ldots, v_j) = (u^j, \hat{P}(u^j), \hat{P}^2(u^j), \ldots, \hat{P}^{j-1}(u^j)),$$

where by $\hat{P}(u)$ we denote an approximate value of $P(u)$ obtained by nonrigorous numerical method.
Let $Z = B(z_0, 10^{-8})$ be the ball centered at z_0 in the maximum norm. Using rigorous solvers for ODEs and variational equations from the CAPD library we computed the interval Newton operator

$$N(F_j, Z, z_0) = z_0 - [DF_j(Z)]^{-1} \cdot F_j(z_0)$$

and obtained that $N(F_j, Z, z_0) \subset \mathrm{int}(Z)$. This proves that F_j has unique zero $(v_1, \ldots, v_j)$ in Z. Moreover, this zero belongs to $N(F_j, Z, z_0)$ which in most cases had diameter less than 10^{-9}. From these estimation we could conclude that u^j has principal period j.

... and now some theorems (CAP)

Then we computed rigorous bounds for $DP^j(v_1) \in DP^j(B(u^j, 10^{-10}))$ and we could check the hyperbolicity type of u^j by analysis of the spectrum of the obtained interval matrix. The actual bounds for eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ are the following

orbit	λ_1	λ_2	λ_3	return time
p_1^8	2.7^{6}_{4}	-0.16^{6}_{4}	$[-1,1]\cdot 10^{-11}$	56.585531937^{57}_{00}
p_2^8	-3.9^{91}_{89}	-0.54^{7}_{4}	$[-5,5]\cdot 10^{-12}$	56.59832252^{73}_{61}
p_3^8	$-1.^{91}_{88}$	0.36^{6}_{4}	$[-1,1]\cdot 10^{-11}$	56.593931^{91}_{89}
p_4^{16}	$[-3.30, -2.96]$	$[-0.05, 0.26] + [-0.13, 0.13]i$		113.1853^{61}_{59}
q_1^{12}	$-22.^{31}_{28}$	$-3.^{14}_{06}$	$[-8,8]\cdot 10^{-11}$	85.0883812^{36}_{29}
q_2^{14}	$-4.^{47}_{39} + 5.^{48}_{36}i$	$-4.^{47}_{39} - 5.^{48}_{36}i$	$[-5,5]\cdot 10^{-11}$	99.2973738^{53}_{34}

∎

... and now some theorems (CAP)

We have also proved that:

- there is an explicitly given trapping region $B \subset \Pi$ for P, i.e. $P(B) \subset B$,
- the maximal invariant set $A =$inv(P, B) contains three invariant sets, say S_1, S_2, S_3, on which the dynamics is Σ_2 chaotic, i.e. it is semiconjugated to the Bernoulli shift on two symbols,
- S_1 is hyperchaotic set with two positive Lyapunov exponents,
- S_2 and S_3 are chaotic sets with one positive Lyapunov exponent,
- there is a countable infinity of heteroclinic connections linking S_1 with S_2, S_2 with S_3 and S_1 with S_3,
- there is countable infinity of periodic orbits and heteroclinic/homoclinic orbits inside each horseshoe.

Outline

1. Numerical techniques to explore complex behaviors

2. Experimental mathematics: high-precision and more ...

3. Need to mixing techniques (so collaborations): “bailar juntos”

4. Need to prove some results: CAP

5. Need to use “problem”-specific techniques: kneading plots

References

- R. Barrio, S. Serrano, *Bounds for the chaotic region in the Lorenz model*, Phys. D 238 (2009) 1615–1624.
- A. Shilnikov, L.P. Shilnikov, R. Barrio, *Symbolic dynamics and spiral structures due to the saddle-focus bifurcations*, “Chaos, CNN, Memristors ...”, (2012).
- R. Barrio, A. Shilnikov, L.P. Shilnikov, *Kneadings, symbolic dynamics and painting Lorenz chaos. A tutorial*, Internat. J. Bifur. Chaos, 22 (2012), no. 4.
- T. Xing, J. Wojcik, R. Barrio and A. Shilnikov, *Symbolic toolkit for chaos exploration*, in “Theory and Applications in Nonlinear Dynamics,” Springer, (2014).
- R. Barrio, F. Blesa, S. Serrano, T. Xing and A. Shilnikov, *Homoclinic spirals: theory and numerics*. in “Progress and Challenges in Dynamical Systems,” Springer Proceedings in Mathematics & Statistics, v. 54, (2013).
- T. Xing, R. Barrio, A. Shilnikov, *Symbolic quest into homoclinic chaos*, Internat. J. Bifur. Chaos, 24 (2014), no. 8.

The Lorenz model

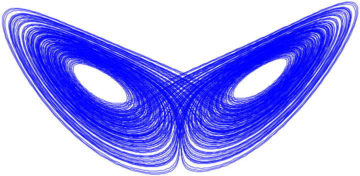
The Lorenz model

$$\frac{dx}{dt} = -\sigma\, x + \sigma\, y, \quad \frac{dy}{dt} = -xz + r\, x - y, \quad \frac{dz}{dt} = xy - b\, z,$$

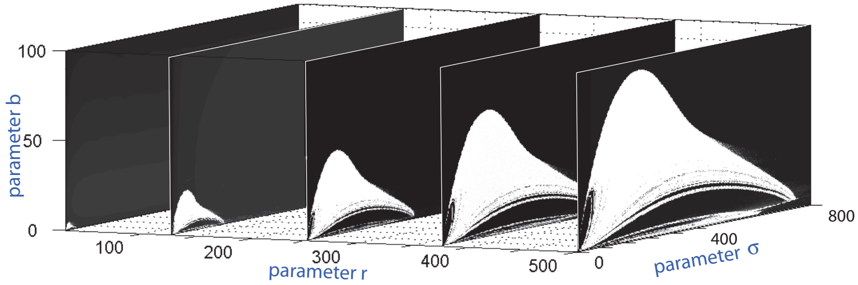
Three dimensionless control parameters:

- σ Prandtl number,
- b a positive constant,
- r relative Rayleigh number.

The Saltzman values: $\sigma = 10,\ b = 8/3,\ r = 28$



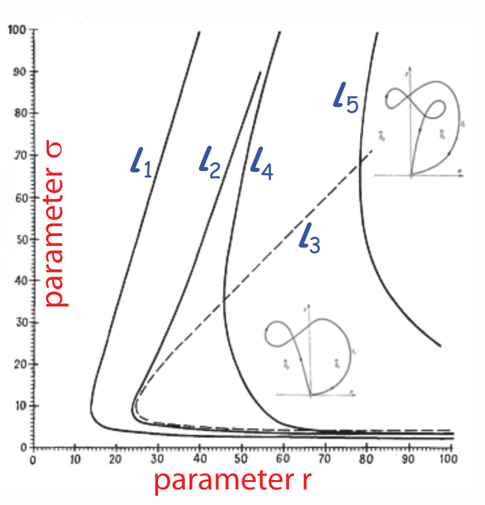
The Lorenz model: Three-parametric analysis

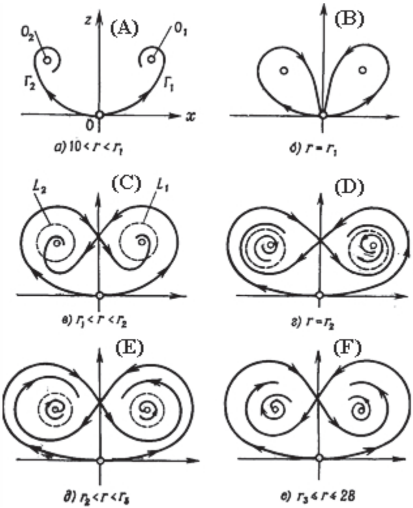


Theorem

For a given fixed r > 1 the region where chaos is possible is bounded in b, and if b $\geq \epsilon > 0$ then the region is bounded in σ too. To be precise, outside a bounded region every positive semiorbit of the Lorenz system converges to an equilibrium.

The Lorenz model: bifurcation analysis





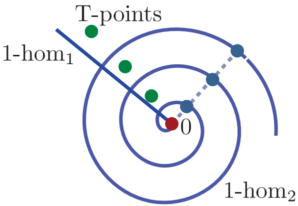
(L.P. Shilnikov, 1980)

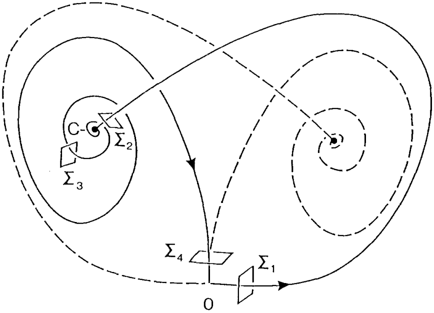
The Lorenz model: but that's all?

More T-points: the Lorenz model

Location of T-points in the Lorenz system is a quite complex task.
And, what about locating *all* the T-points automatically?

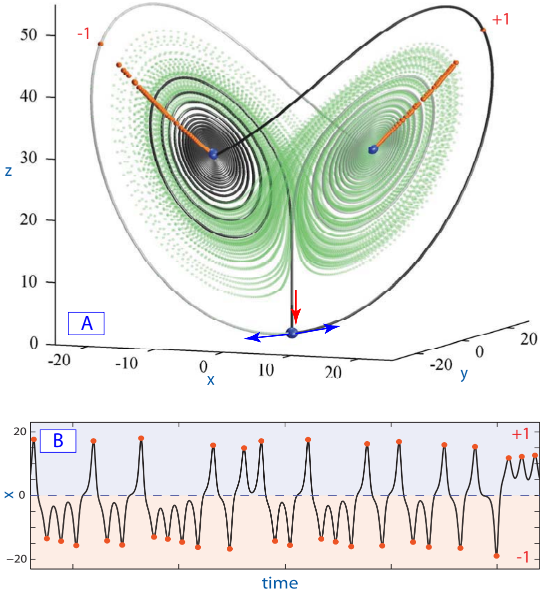
In this case, the T-points are a kind of codimension-two heteroclinic loop. It connects a homoclinic curve with another spiral homoclinic curve.





The Lorenz model: T-points

A new computer technique: more and more T-points



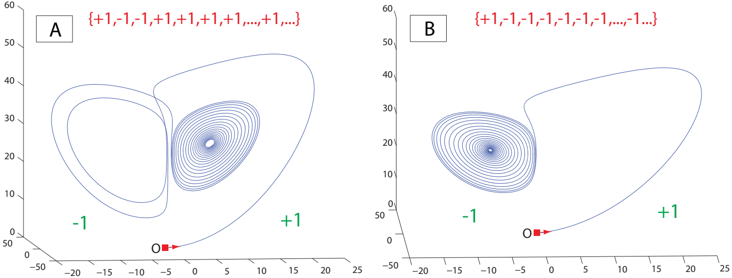
The Lorenz model: T-points

A new computer technique: more and more T-points

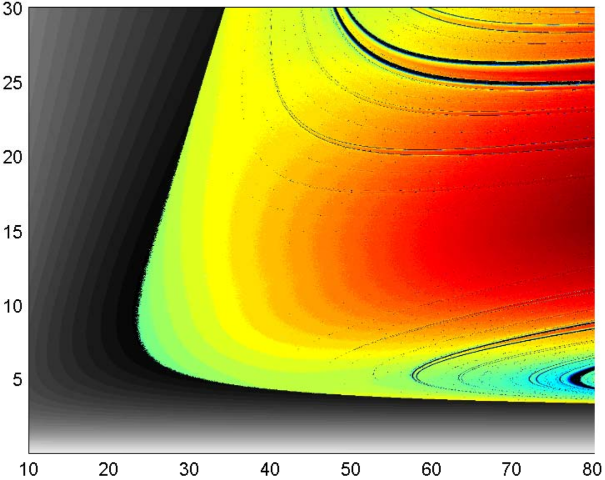
Kneading sequence $\{\kappa_n(O^+)\}$ defined (Milnor and Thurston, 1980)

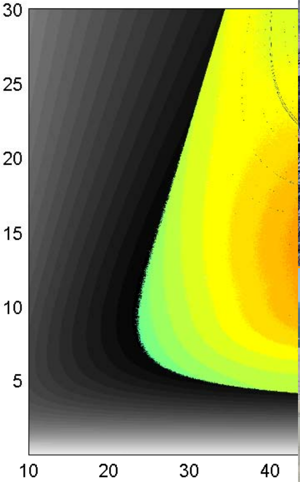
$$\kappa_n(O^+) = \begin{cases} +1, & \text{if } \ T^n(O^+) > 0, \\ -1, & \text{if } \ T^n(O^+) < 0, \\ 0, & \text{if } \ T^n(O^+) = 0; \end{cases}$$

here $T^n(O^+)$ is the n-th iterate of the right separatrix O^+ of the origin. The condition $T^n(O^+) = 0$ is interpreted as a homoclinic loop, i.e. the separatrix returns to the origin after n steps.



The kneading invariant for the separatrix is defined in the form of a formal


30
25
20
15
10
5
10
20
30
40
50
60
70
80


30
25
20
15
10
5
10
20
30
40

The Shimizu-Morioka model

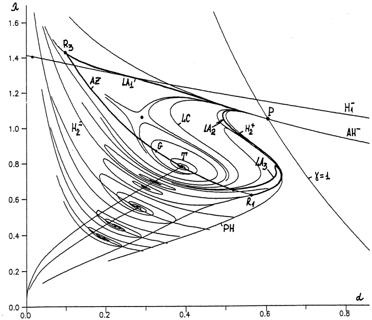
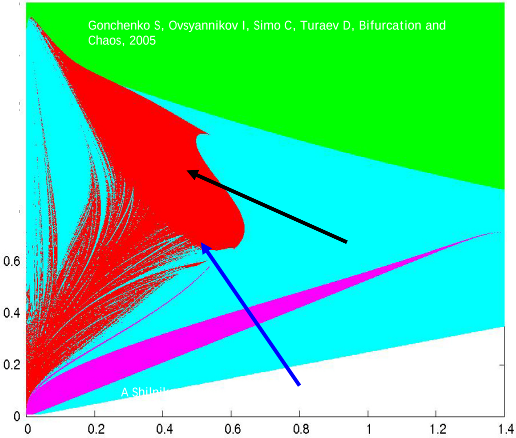
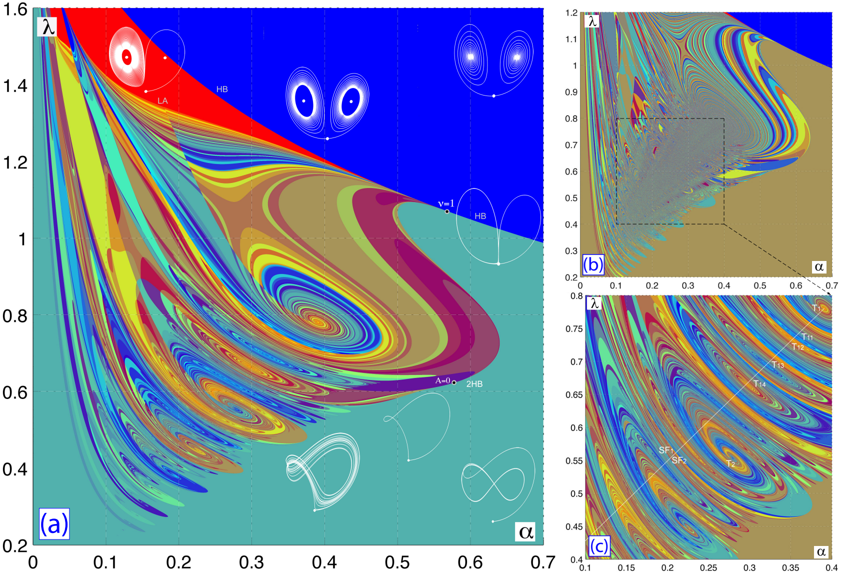


Fig. 13. The (α, λ) bifurcation diagram for $B = 0$.


Gonchenko S, Ovsyannikov I, Simo C, Turaev D, Bifurcation and Chaos, 2005

The Shimizu-Morioka model



The Shimizu-Morioka model

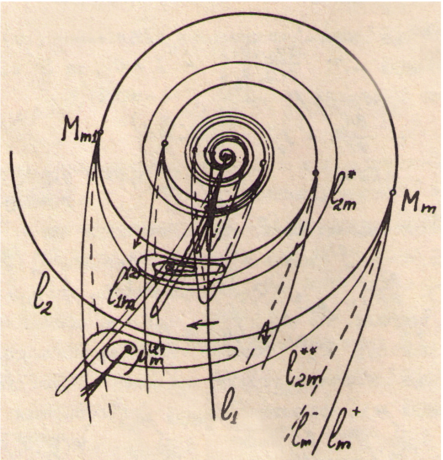
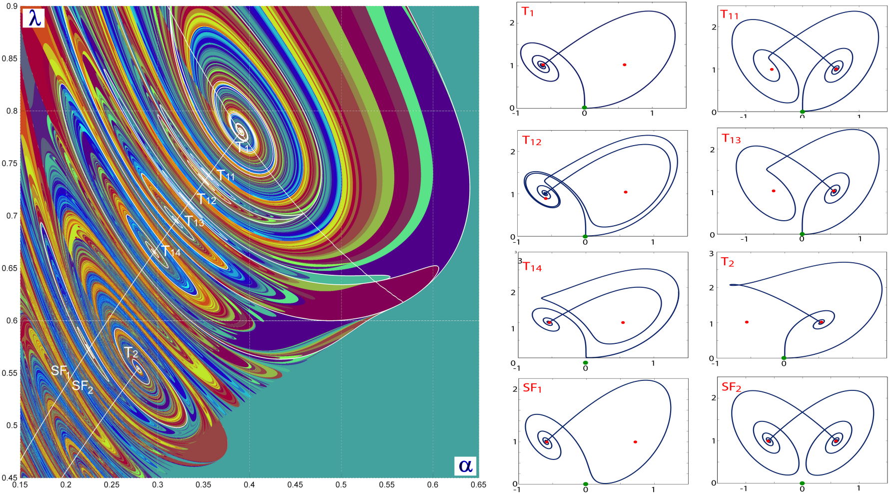
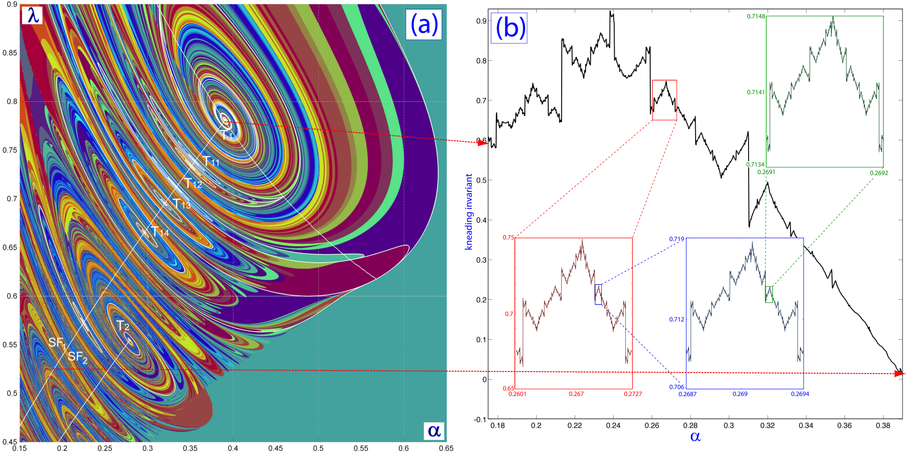


Figure 2: Sketch of a partial bifurcation unfolding of a Bykov T-point (from [Bykov, 1980]) corresponding to a codimension-two heteroclinic connection between a saddle of the (2,1)-type and a saddle-focus of the (1,2)-type. It features the characteristic spirals corresponding to homoclinic bifurcations of the saddle. Turning points (labeled by M's) on the spiral are codimension-two points of inclination-switch bifurcations giving rise to stable periodic orbits through saddle-node and period-doubling bifurcations (l_m-curves) and subsequent spiral structures of smaller scales between spiral's scrolls.

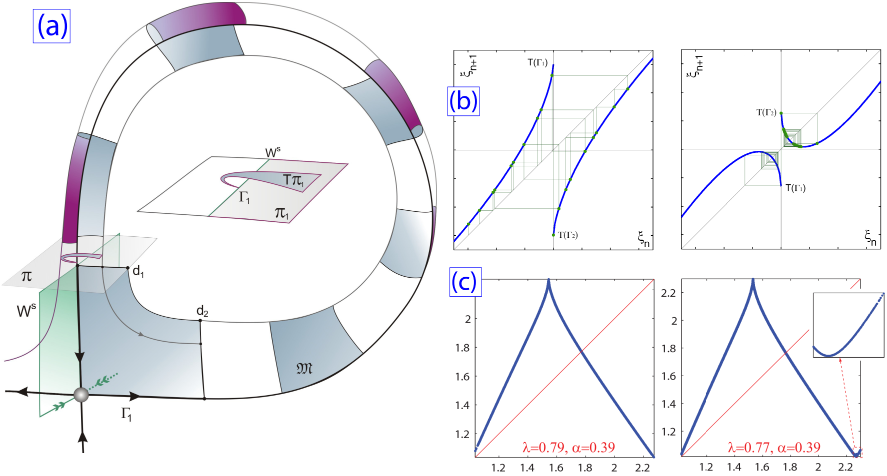
The Shimizu-Morioka model: more and more T-points



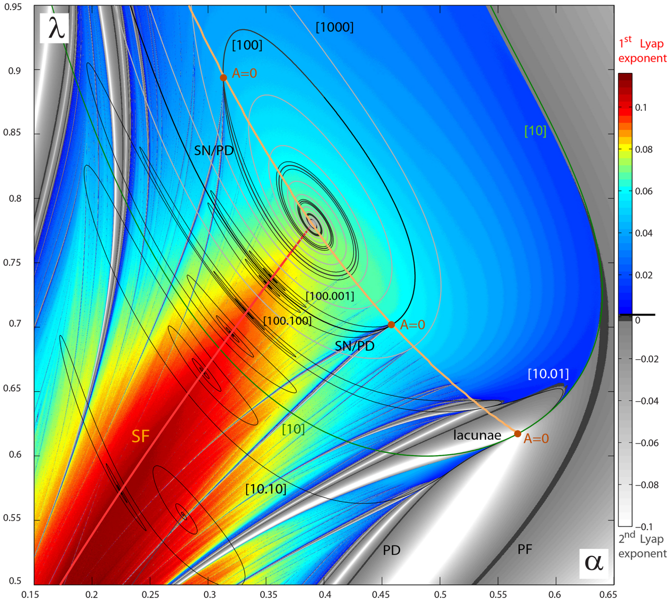
The Shimizu-Morioka model: fractal structure (Bykov)



The Shimizu-Morioka model

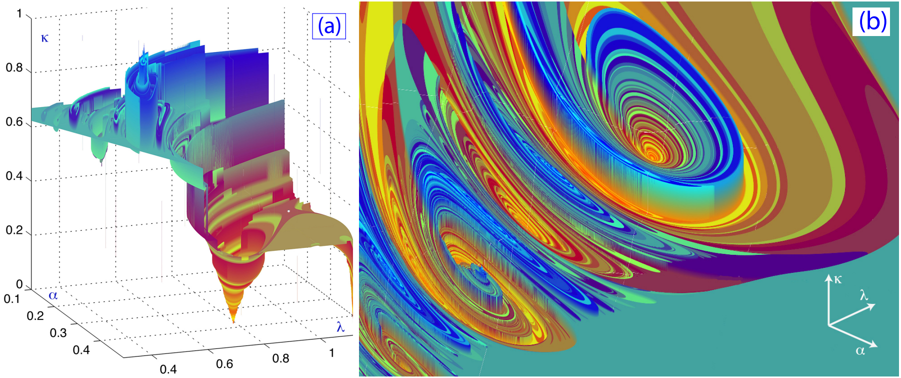


The Shimizu-Morioka model



The Shimizu-Morioka model

- Fractal structure: our “open-air” homoclinic mines




That's all Folks!