# When theorems meets computers

## An encounter: the parameterization method

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# Some historical encounters (in dynamical systems)

# Some computer assisted proofs

Computer aided proofs in dynamical systems back more than 30 years ago, and have lead to proofs of long-standing problems:

- Feigenbaum conjecture [Lanford 82][Lanford 84];
- Existence of heteroclinic intersections in the Hénon mapping [Franceschini,Russo 81];
- Rigorous (and realistic) KAM bounds [Celletti, Chierchia 88][Llave, Rana 91];
- Existence of Lorenz attractor [Tucker 99, 02];
- Existence of critical invariant tori [Koch 08].

Nowadays, there are research groups developing software:

- COSY INFINITY (M. Berz, K. Makino);
- CAPD, Computer Assisted Proofs in Dynamics (P. Zgliczynski);
- CHOMP, Computational Homology Project (K. Mischaikov).

# A quotation

*Computer assisted proofs is a very interesting area in which it is possible to find a meaningful collaboration between Mathematicians (proving theorems of the right kind), Computer Scientists (developing good software tools that relieve the tedium of programming the variants required) and Applied Scientists (that have challenging real life problems).*

[R. de la Llave 01]

# A new paradigm for invariant manifolds

The long term behavior of a dynamical system is organized by invariant manifolds that serve as landmarks that organize the traffic.

Two main theorems, established around 40-60 years ago, concern the **persistence** of invariant manifolds under small perturbations: KAM tori and Normally Hyperbolic Invariant Manifolds.

In recent times there have been **constructive proofs** of these results which lead to **effective algorithms** that allow us to explore what happens in the border of the **applicability** of these theorems, producing reliable computations that can be **validated**.

This new paradigm is the **parameterization method**.
(R. de la Llave and collaborators).

# Planning new encounters

We use the parameterization method to cover the program

*theorems* – *algorithms* – *applications* – *validations*

The rigorous validation of a computation is many times performed with the aid of computers (computer assisted proofs).

We plan to review the basic concepts and present the results in different contexts obtained by many people (Marta Canadell, Jordi-Lluís Figueras, Alejandra González, Àngel Jorba, Alejandro Luque, Rafael de la Llave, Jordi Villanueva)

A. Haro, M. Canadell, J.Ll. Figueras, A. Luque, J.M. Mondelo, *The parameterization method for invariant manifolds: from rigorous results to efective computations*, `www.maia.ub.es/~alex/review/review.pdf`

# The parameterization method for invariant manifolds

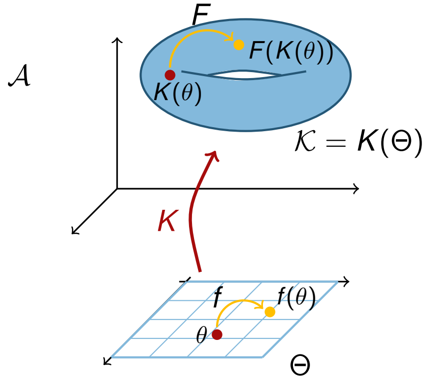
# Invariant manifolds

## Functional equations

Let $F : \mathcal{A} \to \mathcal{A}$ be a diffeomorphism on an ambient manifold $\mathcal{A}$.

A submanifold $\mathcal{K} \subset \mathcal{A}$ parameterized by $K : \Theta \to \mathcal{A}$ is $F$-invariant if there exists a diffeomorphism $f : \Theta \to \Theta$ (the internal dynamics) such that

$$F \circ K = K \circ f.$$



# Invariant manifolds

## Styles of parameterizations

The invariance equation is underdetermined:

> *if* $(K, f)$ *is a solution of the invariance equation and h is a diffeomorphism in* $\Theta$*, then* $(K \circ h, h^{-1} \circ f \circ h)$ *is also a solution.*

There are different *styles* of parameterizations, among them:

- the **graph style**: the manifold is the graph of a function in a selected system of coordinates.
- the **normal form style**: the parameterization is adapted to the shape of the manifold, and the internal dynamics is “simple”.

# Setting

## The dynamical system

The ambient manifold is an annulus $\mathcal{A} \subset \mathbb{T}^d \times \mathbb{R}^{m-d}$: $\mathcal{A}$ is homotopic to $\mathbb{T}^d \times \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^{m-d}$ is open.

The coordinates on $\mathcal{A}$ are $z = (x, y)$, with $x \in \mathbb{T}^d$, $y \in \mathbb{R}^{m-d}$.

The dynamical system is a diffeomorphism $F : \mathcal{A} \to \mathcal{A}$, homotopic to $A \times 0 : \mathbb{T}^d \times \mathbb{R}^{m-d} \to \mathbb{T}^d \times \mathbb{R}^{m-d}$, where $A \in \mathrm{GL}_d(\mathbb{Z})$. That is:

$$F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ 0 \end{pmatrix} + F_p \begin{pmatrix} x \\ y \end{pmatrix},$$

where $F_p : \mathcal{A} \to \mathbb{R}^m$ is 1-periodic in $x$.

# Setting

## The invariant tori

The model manifold is a $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

A torus $\mathcal{K}$ is given by a parameterization $K : \mathbb{T}^d \to \mathcal{A}$ that is homotopic to the zero-section on $\mathbb{T}^d \times \mathbb{R}^{m-d}$. That is:

$$K(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + K_p(\theta),$$

where $K_p : \mathbb{T}^d \to \mathbb{R}^m$ is 1-periodic in $\theta$.

If the torus $\mathcal{K}$ is $F$-invariant, the internal dynamics $f : \mathbb{T}^d \to \mathbb{T}^d$ is homotopic to the torus automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$. That is:

$$f(\theta) = A\theta + f_p(\theta),$$

where $f_p : \mathbb{T}^d \to \mathbb{R}^d$ is 1-periodic in $\theta$.

# A meta-algorithm
# (One step of a Newton-like method)

# First step: evaluation of the error

Let $(K, f)$ be an approximate solution of the invariance equation. That is, assume the error function $E : \mathbb{T}^d \to \mathbb{R}^m$

$$E(\theta) = F(K(\theta)) - K(f(\theta)),$$

is "small" (using appropriate norms).

**Remark**: The fact that $E$ is 1-periodic in $\theta$ has to do with the fact the homotopy classes of $F$, $K$ and $f$ do match.

One step of Newton method consists in finding the correction $(\Delta K, \Delta f)$ of $(K, f)$, with $\Delta K : \mathbb{T}^d \to \mathbb{R}^m$ and $\Delta f : \mathbb{T}^d \to \mathbb{R}^d$, solving

$$\mathrm{D}F(K(\theta))\Delta K(\theta) - \Delta K(f(\theta)) - \mathrm{D}K(f(\theta))\Delta f(\theta) = -E(\theta).$$

**Remark**: it is sufficient to solve approximately the linearized equation so that the error function $\bar{E}$ for the new approximation $(\bar{K}, \bar{f}) = (K + \Delta K, f + \Delta f)$ is **quadratically small** with respect to $E$.

# Second step: construction of an adapted frame

The columns of the matrix valued map $L : \mathbb{T}^d \to \mathbb{R}^{m \times d}$ defined as

$$L(\theta) = \mathrm{D}K(\theta),$$

generate the **tangent directions** to the torus,

We construct a matrix valued map $N : \mathbb{T}^d \to \mathbb{R}^{m \times (m-d)}$ whose columns generate complementary **normal directions** to the torus.

The matrix valued map $P : \mathbb{T}^d \to \mathbb{R}^{m \times m}$, defined as

$$P(\theta) = \begin{pmatrix} L(\theta) & N(\theta) \end{pmatrix},$$

is invertible for each $\theta \in \mathbb{T}^d$.

We refer to $P$ to as an **adapted frame**.

# Third step: approximate reducibility

The tangent bundle to $\mathcal{K}$ is approximately invariant:

$$\mathrm{D}F(K(\theta))\mathrm{D}K(\theta) - \mathrm{D}K(f(\theta))\mathrm{D}f(\theta) = \mathrm{D}E(\theta),$$

The normal bundle $\mathrm{N}\mathcal{K}$ is (possibly) twisted:

$$\mathrm{D}F(K(\theta))N(\theta) = L(f(\theta))T(\theta) + N(f(\theta))\Lambda_N(\theta).$$

The **torsion** $T$ measures how much the normal bundle is twisted.

The linearized dynamics around the approximately invariant torus is approximately reduced to a block-triangular skew-product $(f, \Lambda)$ , where

$$\Lambda(\theta) = \begin{pmatrix} \mathrm{D}f(\theta) & T(\theta) \\ O & \Lambda_N(\theta) \end{pmatrix}.$$

That is the error in the reducibility,

$$E_{\mathrm{red}}(\theta) = P(f(\theta))^{-1}\mathrm{D}F(K(\theta))P(\theta) - \Lambda(\theta),$$

is “small”.

# Fourth step: cohomological equation

We decompose the correction $\Delta K$ in tangent and normal components:

$$\Delta K(\theta) = L(\theta)\xi^L(\theta) + N(\theta)\xi^N(\theta) = P(\theta)\xi(\theta),$$

where $\xi : \mathbb{T}^d \to \mathbb{R}^m$ is the new unknown.

By multiplying

$$\mathrm{D}F(K(\theta))\Delta K(\theta) - \Delta K(f(\theta)) - \mathrm{D}K(f(\theta))\Delta f(\theta) = -E(\theta).$$

by $P(f(\theta))^{-1}$ and skipping second order error terms, we obtain the **cohomological equation**

$$\Lambda(\theta)\xi(\theta) - \xi(f(\theta)) - \begin{pmatrix} I_d \\ O \end{pmatrix} \Delta f(\theta) = \eta(\theta),$$

where $\eta(\theta) = -P(f(\theta))^{-1}E(\theta)$.

# Fifth step: styles of parameterizations

The equation for $\xi^N$ is the normal cohomological equation

$$\Lambda_N(\theta)\xi^N(\theta) - \xi^N(f(\theta)) = \eta^N(\theta),$$

The equation for $\xi^L$ is the tangent cohomological equation

$$\Lambda_L(\theta)\xi^L(\theta) - \xi^L(f(\theta)) - \Delta f(\theta) = \eta^L(\theta) - T(\theta)\xi^N(\theta),$$

where $\Lambda_L(\theta) = \mathrm{D}f(\theta)$.

The method the tangent cohomological equation is solved gives rise to a particular **style** of parameterization.

The method depends of the context of the problem.

# Contexts

In the following, we will consider three contexts:

- Response tori in non-autonomous dynamical systems
- KAM tori in Hamiltonian systems
- Normally hyperbolic invariant manifolds

Each context has its own peculiarities, both in the theory (functional and geometrical setting) and in the algorithms and their implementations (data structures, methods for solving the equations).

# A first date: response tori in skew-product systems

# Response tori in skew-product systems

The dynamical system is a bundle map

$$F = (f, F^y) : \mathcal{A} \subset \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{T}^d \times \mathbb{R}^n,$$

where

$$f : \mathbb{T}^d \to \mathbb{T}^d$$

is a diffeomorphism.

A torus $\mathcal{K}$ that is **copy of the base** is parameterized by a section $K = (\mathrm{id}, K^y) : \mathbb{T}^d \to \mathcal{A}$.

The internal dynamics is prescribed by $f$, and we say that an invariant torus is the response to the forcing $f$.

# Newton method

The error function for a section

$$K(\theta) = \begin{pmatrix} \theta \\ K^y(\theta) \end{pmatrix}$$

is

$$E(\theta) = \begin{pmatrix} 0 \\ E^y(\theta) \end{pmatrix}$$

where $E^y(\theta) = F^y(\theta, K^y(\theta)) - K^y(f(\theta))$.

A natural normal bundle is *the* normal bundle. An adapted frame $P = (L \mid N)$ is of the form

$$P(\theta) = \begin{pmatrix} I_d & O \\ \mathrm{D}K^y(\theta) & P^{yy}(\theta) \end{pmatrix},$$

where $P^{yy} : \mathbb{T}^d \to \mathbb{R}^{(m-d)\times(m-d)}$ is invertible.

# Newton method

The reduced matrix is

$$\Lambda(\theta) = \begin{pmatrix} \mathrm{D}f(\theta) & O \\ O & \Lambda_N(\theta) \end{pmatrix}$$

where $\Lambda_N(\theta) = P^{yy}(f(\theta))^{-1}\mathrm{D}_y F^y(\theta, K^y(\theta))P^{yy}(\theta)$.

The normal cohomological equation is

$$\Lambda_N(\theta)\xi^N(\theta) - \xi^N(f(\theta)) = \eta^N(\theta),$$

where $\eta^N(\theta) = -P^{yy}(f(\theta))^{-1}E^y(\theta)$. It has a unique solution $\xi^N$ for any $\eta^N$, under **hyperbolicity** properties of $(f, \Lambda_N)$.

One can prove a **Kantorovich-like theorem** for the convergence of the Newton-like method.

On can use this a posteriori theorem to **validate** numerical computations.

(See Jordi-Lluís Figueras talk).

# Algorithms

For *f* being a rotation on the torus (quasi-periodic skew-products):

- **Large matrix method**: Based on the direct discretization of the linearized equationin in terms of the Fourier coefficients of the expansions.
- **Projection method**: Based on the (hyperbolic) reducibiliy of the linearized dynamics to a block diagonal linear skew-product, splitting the normal cohomological equations into stable and unstable components.
- **Reducibility method**: Based on reducibility of the linearized dynamics to a constant cocycle, trivializing the linearized equations.

We use two implementations of the methods:

- Implementation with Automatic Differentation (AD);
- Implementation with Fast Fourier Transform (FFT).

# Example: saddle tori in a qp forced standard map

The *quasi-periodically forced standard map* is $(R_\omega, F) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$ defined as

$$\left\{ \begin{array}{rcl} \bar{\theta} & = & \theta + \omega \\ \bar{x} & = & x + \bar{y} \\ \bar{y} & = & y - \frac{b}{2\pi} \sin(2\pi x) - \varepsilon \sin(2\pi\theta) \end{array} \right. ,$$

where we fix $\omega = \frac{1}{2}(\sqrt{5} - 1)$.

In the following, we also fix $b = 1.3$ and move $\varepsilon$.

For $\varepsilon = 0$, the torus $K^y(\theta) = (\frac{1}{2}, 0)$ is fiberwise hyperbolic.

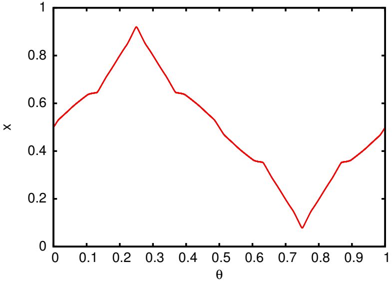
We continue this saddle torus w.r.t. $\varepsilon$, up to breakdown.

# Benchmark

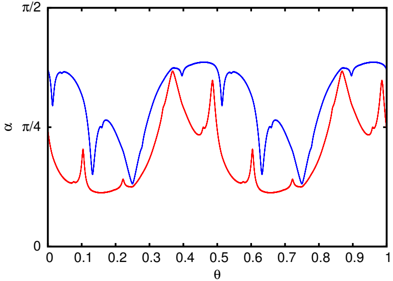
| | | Large Matrix | | Reducibility + AD | | | Reducibility + FFT | | |
|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | $N$ | $E$ | s/s | $E$ | $E_{\rm red}$ | s/s | $E$ | $E_{\rm red}$ | s/s |
| 1.0 | 64 | 4.3e-15 | 0.035 | 1.6e-14 | 3.9e-13 | 0.023 | 1.6e-14 | 4.0e-13 | 0.061 |
| 1.2 | 128 | 1.7e-15 | 0.155 | 3.8e-15 | 1.8e-13 | 0.066 | 1.3e-15 | 3.0e-15 | 0.075 |
| 1.23 | 256 | 5.4e-14 | 1.068 | 6.9e-14 | 1.3e-10 | 0.082 | 7.7e-13 | 1.4e-09 | 0.096 |
| 1.232 | 512 | 3.3e-16 | 9.866 | 1.8e-14 | 2.5e-14 | 0.129 | 1.0e-14 | 8.6e-14 | 0.123 |
| 1.235 | 1024 | 1.2e-13 | 79.380 | 1.5e-11 | 4.4e-09 | 0.445 | 1.6e-11 | 7.3e-09 | 0.258 |
| 1.2352 | 2048 | 3.7e-11 | 627.761 | 3.4e-12 | 8.6e-09 | 1.701 | 5.3e-12 | 7.3e-09 | 0.335 |
| 1.23522 | 4096 | NC | NC | 3.7e-12 | 4.8e-11 | 6.714 | 1.1e-12 | 1.5e-10 | 0.922 |
| **1.23527** | 8192 | NC | NC | 1.6e-11 | 1.2e-07 | 26.713 | 1.6e-11 | 1.2e-07 | 1.604 |
| **1.235273** | 16384 | NC | NC | 5.6e-11 | 3.2e-08 | 106.627 | 7.0e-13 | 5.2e-12 | 6.887 |
| **1.235275** | 32768 | NC | NC | 9.9e-12 | 2.0e-07 | 425.805 | 1.2e-11 | 2.0e-07 | 15.332 |
| 1.2352755 | 65536 | NC | NC | 2.8e-09 | 2.0e-04 | 1706.804 | 3.2e-09 | 5.5e-03 | 45.354 |

# Saddle tori on the verge of the breakdown

$\varepsilon = 1.235$



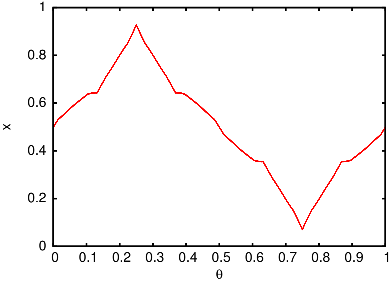

(a) x-projection of the invariant torus



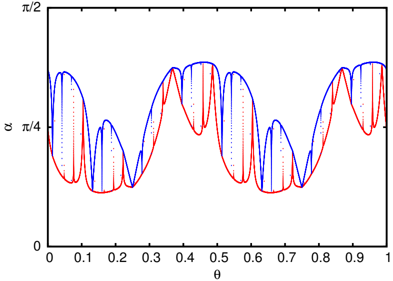

(b) Projectivized invariant subbundles

# Saddle tori on the verge of the breakdown

$\varepsilon = 1.235275$

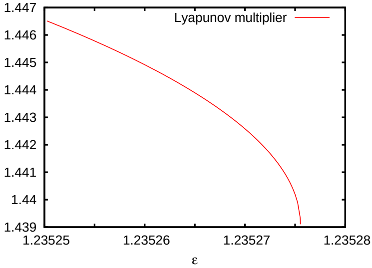



(c) x-projection of the invariant torus

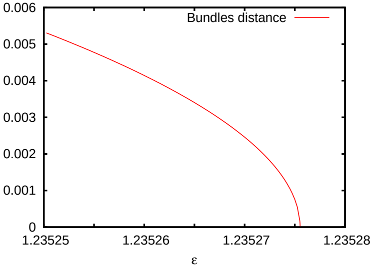



(d) Projectivized invariant subbundles

# Observables near the breakdown




(a) Maximal Lyapunov multiplier




(b) Bundles distance

# A second date:
# KAM tori in Hamiltonian systems

# The Hamiltonian setting

In Hamiltonian dynamics, the phase space is endowed a symplectic structure, the dynamical systems preserve such symplectic structure and the invariant tori carry quasi-periodic motions.

$m = 2n$ and the annulus $\mathcal{A} \subset \mathbb{T}^d \times \mathbb{R}^{2n-d}$ is endowed an exact symplectic structure $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$ ($\boldsymbol{\alpha}$ is the Liouville form).

$F : \mathcal{A} \to \mathcal{A}$ preserves the symplectic structure, i.e $F^*\boldsymbol{\omega} = \boldsymbol{\omega}$, and, moreover $F^*\alpha - \alpha = dS$ ($F$ is exact symplectic).

Given a rigid rotation $R_\omega(\theta) = \theta + \omega$, where $\omega \in \mathbb{R}^d$ satisfies a Diophantine condition, we look for an invariant torus $K : \mathbb{T}^d \to \mathcal{A}$ with internal dynamics $f = R_\omega$.

From now on, we consider the special case $d = n$.

# Approximate reducibility

The internal dynamics is prescribed, and the unknown in the invariance equation $F \circ K - K \circ R_\omega = 0$ is $K$.

The adapted frame $P(\theta)$ is selected to be approximately symplectic.

Notice that $\Lambda_L(\theta) = \mathrm{D}f(\theta) = I_n$.

The linearized dynamics $\mathrm{D}F(K(\theta))$ is approximately reduced to an approximately symplectic block triangular matrix with $\Lambda_N(\theta) = I_n$:

$$\Lambda(\theta) = \begin{pmatrix} I_n & T(\theta) \\ O & I_n \end{pmatrix}.$$

# Small divisors equations

The equation for the correction on the normal directions is

$$\xi^N(\theta) - \xi^N(\theta + \omega) = \eta^N(\theta).$$

By using Fourier series, if

$$\eta^N(\theta) = \sum_{k \in \mathbb{Z}^n} \hat{\eta}_k^N e^{2\pi \mathbf{i} k \cdot \theta} \;,\; \xi^N(\theta) = \sum_{k \in \mathbb{Z}^n} \hat{\xi}_k^N e^{2\pi \mathbf{i} k \cdot \theta},$$

one obtains, for $k \neq 0$,

$$\hat{\xi}_k^N = \frac{\hat{\eta}_k^N}{1 - e^{2\pi \mathbf{i} k \cdot \omega}},$$

and $\hat{\xi}_0^N$ is free, provided that $\hat{\eta}_0^N = 0$.

Although $\hat{\eta}_0^N$ could be non-zero, exactness property of $F$ implies that $\hat{\eta}_0^N$ is quadratically small with respect to the error $E$, so we can skip it!

# The twist condition

The equation for the correction in the tangent directions is

$$\xi^L(\theta) - \xi^L(\theta + \omega) = \eta^L(\theta) - T(\theta)\xi^N(\theta)$$

(notice that $\Delta f(\theta) = 0$).

A sufficient condition to solve this equation is the **twist condition** that $\langle T \rangle$ is an invertible matrix, so one can make the average of the r.h.s. equal to zero.

The convergence of the Fourier expansions is guaranteed by the Diophantine properties of $\omega$ and regularity properties of $\eta$.

The convergence of the Newton-like method is a matter of KAM theory.
(see Alejandro Luque's talk)

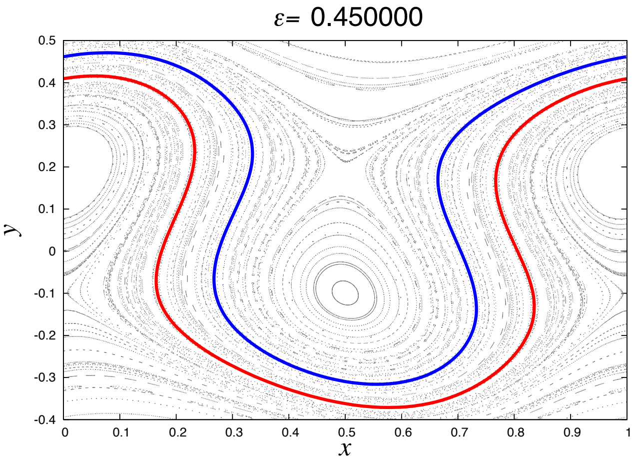
# Example: computation of meandering curves

Consider a quadratic standard family of symplectomorphisms $f_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, defined by:

$$\begin{cases} \bar{x} &= x + (\bar{y} + 0.1)(\bar{y} - 0.2)\,, \\ \bar{y} &= y - \dfrac{\varepsilon}{2\pi} \sin(2\pi x)\,. \end{cases}$$

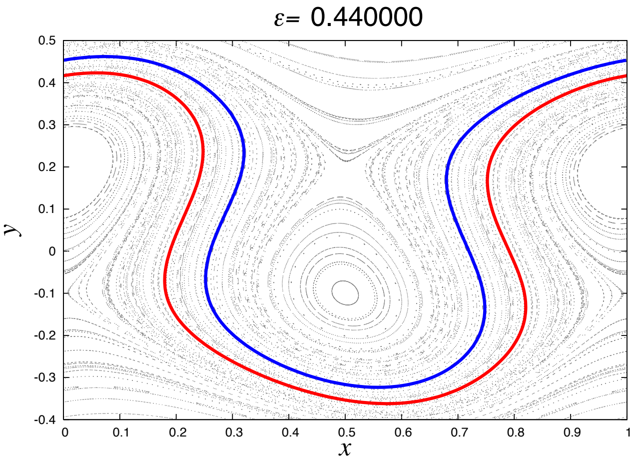
Problem: Look for invariant tori with frequency $\omega = \frac{\sqrt{5}-1}{32}$, with respect to parameter $\varepsilon$.

# Computation of meandering curves




$T = -0.0309003$
$T = \quad 0.0309003$

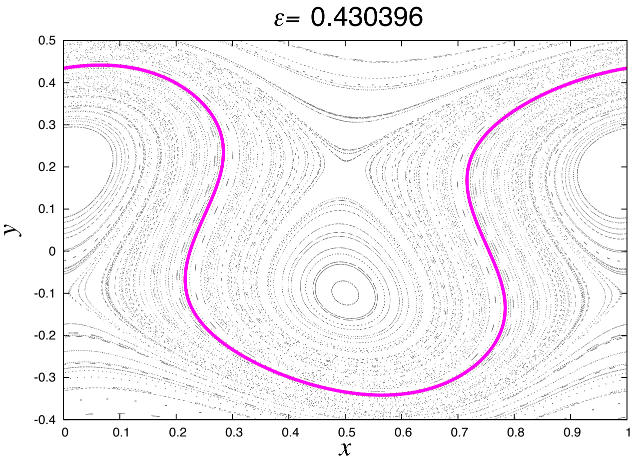
# Computation of meandering curves




$T = -0.0211805$

$T = \quad 0.0211805$

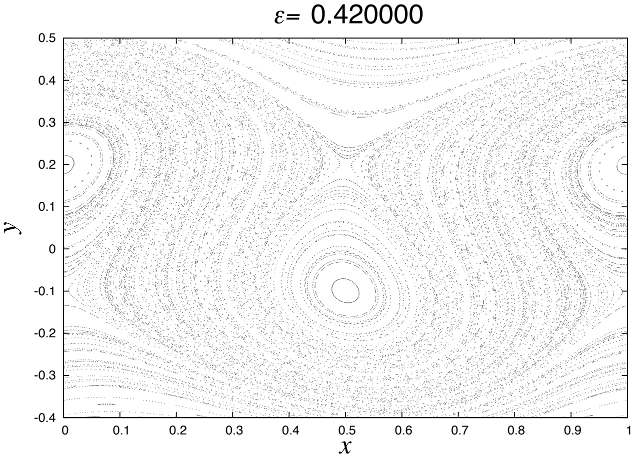
# Computation of meandering curves




$T = -1.6261 \cdot 10^{-7}$

$T = \quad 1.6261 \cdot 10^{-7}$

# Computation of meandering curves



# A third date: normally hyperbolic invariant manifolds

# The normal cohomology equation and hyperbolicity

In the dissipative context, the **normal hyperbolicity** property of an invariant manifold ensures its robustness under small perturbations.

Given an approximate solution $(K, f)$ of the invariance equation, and an adapted frame $P = (L|N)$, the correction is written $\Delta K = L\xi^L + N\xi^N$.

The normal cohomological equation

$$\Lambda_N(\theta)\xi^N(\theta) - \xi^N(f(\theta)) = \eta^N(\theta),$$

is solvable under hyperbolicity properties of the skew-product $(f, \Lambda_N)$.

# Styles of parameterizations

The tangent cohomological equation

$$\Lambda_L(\theta)\xi^L(\theta) - \xi^L(f(\theta)) - \Delta f(\theta) = \eta^L(\theta) - T(\theta)\xi^N(\theta),$$

where $\Lambda_L(\theta) = \mathrm{D}f(\theta)$, has infinitely many solutions, and the choice of a particular one gives rise to a particular **style**.

- The graph style: take $\xi^L(\theta) = 0$ and $\Delta f(\theta) = -\eta^L(\theta) + T(\theta)\xi^N(\theta)$.
- The normal form style: under hyperbolicity properties of $(f, \Lambda_L)$, we take $\Delta f(\theta) = 0$, and $\xi^L(\theta)$ solving the cohomological equation

$$\Lambda_L(\theta)\xi^L(\theta) - \xi^L(f(\theta)) = \eta^L(\theta) - T(\theta)\xi^N(\theta).$$

**Remark:** We can compute a normal bundle for which $T(\theta) = 0$, that is, a normal invariant bundle. In fact, one can design a method that computes the stable and the unstable bundles.

(See Marta Canadell's thesis)

# Example: computation of an invariant cylinder

The Froeschlé map is an exact symplectic map which consists in two coupled standard maps with fixed parameters $\kappa_1, \kappa_2$, and a coupling parameter $\varepsilon$.

We consider $F_\varepsilon : \mathbb{T} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R} \times \mathbb{R}^2$ of the form

$$F_\varepsilon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - \frac{\kappa_1}{2\pi} \sin(2\pi x_1) - \frac{\varepsilon}{2\pi} \sin(2\pi(x_1 + x_2)) \\ y_1 - \frac{\kappa_1}{2\pi} \sin(2\pi x_1) - \frac{\varepsilon}{2\pi} \sin(2\pi(x_1 + x_2)) \\ x_2 + y_2 - \frac{\kappa_2}{2\pi} \sin(2\pi x_2) - \frac{\varepsilon}{2\pi} \sin(2\pi(x_1 + x_2)) \\ y_2 - \frac{\kappa_2}{2\pi} \sin(2\pi x_2) - \frac{\varepsilon}{2\pi} \sin(2\pi(x_1 + x_2)) \end{pmatrix}$$

# Normal hyperbolicity of an invariant cylinder

For $\varepsilon = 0$ the system is uncoupled. The cylinder

$$\mathcal{C}_0 = \{(x_1, y_1, \frac{1}{2}, 0) \mid (x_1, y_1) \in \mathbb{T} \times \mathbb{R}\}$$

is invariant, and the internal dynamics on $\mathcal{C}_0$ is a standard map with parameter $\kappa_1$.

The model manifold is $\Theta = \mathbb{T} \times \mathbb{R}$.

If $\kappa_2$ is large enough, the hyperbolicity of the saddle fixed point $(\frac{1}{2}, 0)$ dominates the internal dynamics of $\mathcal{C}_0$, and the cylinder is a normally hyperbolic invariant manifold.

In such a case, the normally hyperbolic invariant cylinder persists for small coupling constant $\varepsilon$.

# Homotopy classes

By writing $z_i = (x_i, y_i)$ for $i = 1, 2$, $F : \mathbb{T} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R} \times \mathbb{R}^2$ with

$$F \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Az_1 \\ 0 \end{pmatrix} + F_p \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $F_p(z_1, z_2)$ is 1-periodic in $z_1$.

A cylinder $\mathcal{C}$ is parameterized by $C : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R} \times \mathbb{R}^2$ with

$$C(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + C_p(\theta),$$

where $C_p(\theta)$ is 1-periodic in $\theta = (\theta_1, \theta_2)$.

The dynamics on the cylinder $\mathcal{C}$ is a map $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ of the form

$$f(\theta) = A\theta + f_p(\theta),$$

where $f_p$ is 1-periodic in $\theta$.

The homotopy classes of $F$, $C$ and $f$ has to match!

# Continuation with respect to $\varepsilon$ ($\kappa_1 = 0.1$, $\kappa_2 = 1.5$)

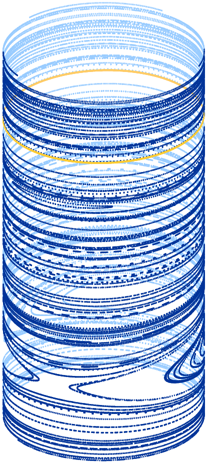
Implementation of graph style parameterization method using exponential trichotomy: use of 2-dimensional grids and (local) interpolation methods.

| $\varepsilon$ | $E$ | $E_{\text{red}}$ |
|---|---|---|
| 0.00 | 2.96e-17 | 6.07e-18 |
| 0.05 | 4.63e-10 | 4.36e-07 |
| 0.10 | 4.46e-10 | 9.96e-08 |
| 0.15 | 5.47e-10 | 6.55e-07 |
| 0.20 | 5.46e-10 | 9.93e-07 |
| 0.25 | 5.60e-09 | 9.90e-06 |
| 0.30 | 8.52e-09 | 1.55e-05 |
| 0.35 | 6.40e-09 | 6.12e-05 |

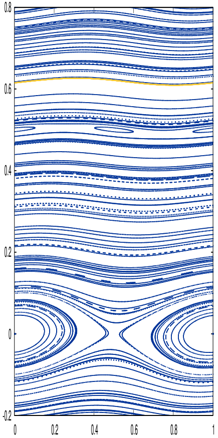
In this implementation, the grids are $512 \times 512$.

# The invariant cylinder and its internal dynamics

$\varepsilon = 0.00$



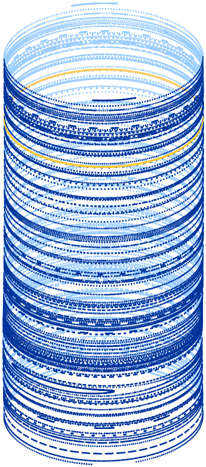
(a) invariant cylinder



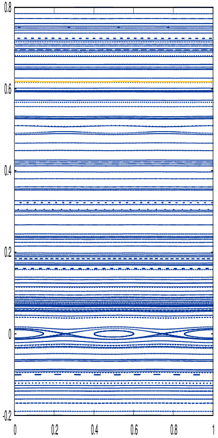

(b) internal dynamics

# The invariant cylinder and its internal dynamics

$\varepsilon = 0.10$



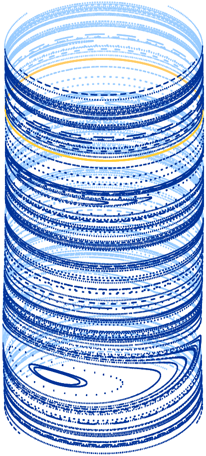
(a) invariant cylinder



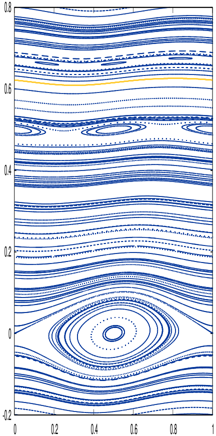

(b) internal dynamics

# The invariant cylinder and its internal dynamics

$\varepsilon = 0.20$



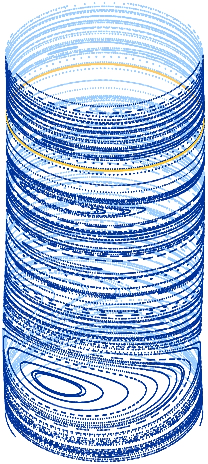
(a) invariant cylinder



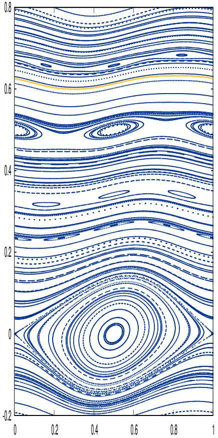

(b) internal dynamics

# The invariant cylinder and its internal dynamics

$\varepsilon = 0.30$



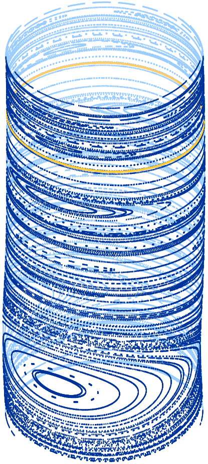
(a) invariant cylinder



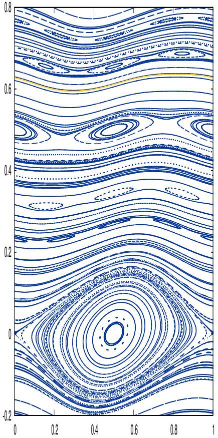

(b) internal dynamics

# The invariant cylinder and its internal dynamics

$\varepsilon = 0.35$



(a) invariant cylinder




(b) internal dynamics

# To be continued ...