Centers in dimension 3

Isaac A. García
University of Lleida, Spain

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PART I:

Centers on center manifolds

The Hopf points in $\mathbb{R}^3$

We consider the analytic three-dimensional system

$$\begin{aligned} \dot{x} &= -y + \mathcal{F}_1(x,y,z), \\ \dot{y} &= \ \ x + \mathcal{F}_2(x,y,z), \\ \dot{z} &= \lambda\, z + \mathcal{F}_3(x,y,z), \end{aligned} \tag{1}$$

- $\lambda \in \mathbb{R}\backslash\{0\}$;
- $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \in C^w(\mathcal{U})$ with $\mathcal{U} \subset \mathbb{R}^3$ neighborhood of 0;
- $\mathcal{F}(0) = 0$ and $D\mathcal{F}(0) = 0$.

Hopf singular point

The origin is a *Hopf singularity* of system (1): it possesses the eigenvalues $\pm i \in \mathbb{C}$ and $\lambda \in \mathbb{R}\backslash\{0\}$.

Local center manifolds $\mathcal{W}^c$

Let $\mathcal{W}^c$ be a local center manifold at the origin of system (1):

- $\mathcal{W}^c$ is an invariant surface, tangent to the (x, y) plane at the origin.
- $\mathcal{W}^c = \{z = h(x, y) \ : \ \text{for } (x, y) \text{ around } (0, 0)\}$ with $h(0, 0) = 0$ and $Dh(0, 0) = 0$.
- For any $k \geq 1$ there exists a C^k local center manifold.
- The local center manifold need not be unique.

Local dynamics on a center manifold $\mathcal{W}^c$

- The origin is a center of (1) if all the orbits on $\mathcal{W}^c$ are periodic;
- Otherwise, the origin is a saddle-focus: a focus on $\mathcal{W}^c$.

The center problem in $\mathbb{R}^3$

To decide when the origin of (1) is a center or not.

The Lyapunov solution to the center problem

Lyapunov Center Theorem

The origin is a center for the analytic system (1) if and only if (1) admits a real analytic local first integral of the form

$$H(x, y, z) = x^2 + y^2 + \cdots$$

in a neighborhood of the origin in $\mathbb{R}^3$.

Remark

Moreover, when there is a center, the local center manifold $\mathcal{W}^c$ is unique and analytic.

Inverse Jacobi multipliers (1844)

$\mathcal{X}$ will denote the associated vector field to system (1), that is,

$$\mathcal{X} = (-y+\mathcal{F}_1(x,y,z))\frac{\partial}{\partial x}+(x+\mathcal{F}_2(x,y,z))\frac{\partial}{\partial y}+(\lambda z+\mathcal{F}_3(x,y,z))\frac{\partial}{\partial z}.$$

Inverse Jacobi multiplier

A C^1 function $V : \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$ is an *inverse Jacobi multiplier* of $\mathcal{X}$ if it is not locally null and it satisfies the linear first-order partial differential equation

$$\mathcal{X}V = V \,\mathrm{div}\mathcal{X},$$

where $\mathrm{div}\mathcal{X}$ is the divergence of $\mathcal{X}$.

Remark

For the rescaled vector field $\mathcal{X}/V$ on $\mathcal{U}\backslash V^{-1}(0)$: $\mathrm{div}(\mathcal{X}/V) \equiv 0$.

A new solution to the center problem

Theorem 1

System (1) has a center at the origin if and only if it admits a local analytic inverse Jacobi multiplier of the form

$$V(x, y, z) = z + \cdots$$

in a neighborhood of the origin in $\mathbb{R}^3$.

Sketch of the proof of Theorem 1 (necessary condition)

$\Rightarrow$ Assume that (1) has a center at the origin.

1. Using normal form theory, system (1) having a center is real analytically conjugated to the normal form

$$\dot{\xi} = -\eta F(\xi^2+\eta^2)\,,\ \dot{\eta} = \xi F(\xi^2+\eta^2)\,,\ \dot{w} = \lambda w + w\, G(\xi^2+\eta^2)\,.$$

2. It has the inverse Jacobi multiplier $\hat{V}(\xi, \eta, w) = w$.
3. Going back we get $V(x, y, z) = z + \cdots$ $\square$

Sketch of the proof of Theorem 1 (sufficient condition)

$\Leftarrow$ Assume that (1) possesses $V(x,y,z) = z + \cdots$.

1. Using the Implicit Function Theorem for $V(x,y,z) = 0$: there exists a unique analytic function $h(x,y)$ such that $h(0,0) = 0$, $Dh(0,0) = 0$ and $V(x,y,h(x,y)) \equiv 0$.
2. Hence, from the flow–invariance of the surface $V = 0$, we have $\mathcal{W}^c = \{z = h(x,y)\}$ is an analytic local center manifold for (1).
3. *** We prove that $V(x,y,z) = (z - h(x,y))W(x,y,z)$ such that $W|_{\mathcal{W}^c}(x,y) = W(x,y,h(x,y)) \not\equiv 0$.
4. *** We prove that $W|_{\mathcal{W}^c}$ is an analytic inverse integrating factor of $\mathcal{X}|_{\mathcal{W}^c}$ that is non-vanishing at the origin.
5. The Reeb Criterium assures that the origin is a center for $\mathcal{X}|_{\mathcal{W}^c}$.

Relations between $\mathcal{W}^c$ and $V^{-1}(0)$ around centers

Remark: Non-uniqueness of $V \in C^w$ around a center

For any $k \geq 0$, there are analytic inverse Jacobi multipliers $\hat{V}$ at a center of the form

$$\hat{V} = VH^k = (z + \cdots)(x^2 + y^2 + \cdots)^k = z(x^2 + y^2)^k + \cdots$$

A consequence of the proof of Theorem 1

When system (1) has a center, then the $V(x, y, z) = z + \cdots$ predicted by Theorem 1 satisfies $\mathcal{W}^c \subset V^{-1}(0)$

Theorem 2

When system (1) has a center, then any local C^∞ inverse Jacobi multiplier V of system (1) must satisfy $\mathcal{W}^c \subset V^{-1}(0)$.

An application: classification of centers in the Lü system

For $(a, b, c) \in \mathbb{R}^3$, consider the 3-parametric Lü family

$$\dot{x} = a(y - x), \;\; \dot{y} = cy - xz, \;\; \dot{z} = -bz + xy.$$

- The singularities $Q_{\pm} = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ when $c = (a + b)/3$ and $ab > 0$ are Hopf points.
- Invariance under the symmetry $(x, y, z) \mapsto (-x, -y, z)$.
- The first three Lyapunov constants of $Q_{\pm}$ vanish if and only if $(a, b, c) \in L = \{(a, b, c) \in \mathbb{R}^3 : a \neq 0, b = 2a, c = a\}$.

Theorem 3. (The centers in the Lü system)

The singularities $Q_{\pm}$ are centers if and only if $(a, b, c) \in L$.

Proof: When $(a, b, c) \in L$, $V(x, y, z) = x^2 - 2az$ is an inverse Jacobi multiplier.

Existence and smoothness of V and $\mathcal{W}^c$ around the saddle-focus

Theorem 4

Assume that the origin is a saddle-focus for the analytic system (1). Then the following holds:

- There exists a local C^∞ and non-flat inverse Jacobi multiplier of (1) having the expression

$$V(x, y, z) = z(x^2 + y^2)^n + \cdots$$

  for some $n \geq 2$.
- For the former V, there is a local C^∞ center manifold $\mathcal{W}^c$ such that $\mathcal{W}^c \subset V^{-1}(0)$.

A simple example of saddle-focus shows the possibilities

The following system has, $\forall a \in \mathbb{R}$,

$$\dot{x} = -y - x(x^2 + y^2)\ ,\ \dot{y} = x - y(x^2 + y^2)\ ,\ \dot{z} = -z,$$

$$\mathcal{W}_a^c = \begin{cases} \{z = 0\} & \text{(analytic)} \\ \left\{z = a \exp\left(-\frac{1}{2(x^2+y^2)}\right)\right\} & (C^\infty \text{ flat}) \end{cases}$$

$$V_a(x, y, z) = \begin{cases} z(x^2 + y^2)^2 & \text{(analytic)} \\ \left(z - a \exp\left(-\frac{1}{2(x^2+y^2)}\right)\right)(x^2 + y^2)^2 & (C^\infty \text{ non-flat}) \end{cases}$$

$$\hat{V}(x, y, z) = V_0(x, y, z) - V_1(x, y, z) = \exp\left(-\frac{1}{2(x^2+y^2)}\right)(x^2 + y^2)^2$$

is C^∞ flat and $\hat{V}^{-1}(0) = \{(0, 0, 0)\}$.

More properties of $\mathcal{W}^c$ and $V^{-1}(0)$ around a saddle-focus

Theorem 5

Assume that the origin is a saddle-focus for system (1).

- Any two locally C^∞ and non-flat at the origin linearly independent inverse Jacobi multipliers of (1) have the same Taylor expansion at the origin.
- Let V be a locally C^∞ and non-flat at the origin inverse Jacobi multiplier of (1). Then there is exactly one smooth center manifold $\mathcal{W}^c$ of (1) such that $\mathcal{W}^c \subset V^{-1}(0)$.

PART II:

Characterizing centers on center manifolds via Lie symmetries

Normal forms near a Hopf singularity

Analytical normal form near a center

If the origin is a center for $\mathcal{X}_\lambda$ then there is a real analytic near-identity diffeomorphism Φ such that

$$\begin{aligned}\widehat{\mathcal{X}}_\lambda &= \Phi^* \mathcal{X}_\lambda = -y\left(1 + F(x^2+y^2)\right)\partial_x + x\left(1 + F(x^2+y^2)\right)\partial_y \\ &\quad + z\left(\lambda + G(x^2+y^2)\right)\partial_z\end{aligned}$$

where F and G are real analytic on a neighborhood of zero in $\mathbb{R}$ and $F(0) = G(0) = 0$.

Normal forms near a Hopf singularity

C^∞ normal form near a saddle-focus

If the origin is a saddle-focus for $\mathcal{X}_\lambda$ then there is a C^∞ near-identity diffeomorphism Φ such that $\widehat{\mathcal{Y}}_\lambda = \Phi^* \mathcal{X}_\lambda$ where

$$\begin{aligned}\widehat{\mathcal{Y}}_\lambda &= \Big(-y + \frac{1}{2}[(x+iy)A(x^2+y^2) + (x-iy)B(x^2+y^2)] \Big) \partial_x \\ &\quad + \Big(x + \frac{1}{2}[(y-ix)A(x^2+y^2) + (y+ix)B(x^2+y^2)] \Big) \partial_y \\ &\quad + z[\lambda + C(x^2+y^2)] \partial_z\end{aligned}$$

where $(i^2 = -1)$ and the symmetry conjugation $\overline{B(x^2+y^2)} = A(x^2+y^2)$ holds (so the normal form is real), $A(x^2+y^2) + B(x^2+y^2) \not\equiv 0$, and $\mathrm{Re}(A) \not\equiv 0$.

Normal forms near a Hopf singularity

Notation:

$$\mathcal{X}_\lambda = \mathcal{X}_A + \cdots$$

where $\mathcal{X}_A$ is the linear vector field with associated matrix A.

Linearizable and orbitally linearizable centers

- $\mathcal{X}_\lambda = \mathcal{X}_A + \cdots$ is *analytically orbitally linearizable* in $\mathcal{U}$ if there exists an analytic change of coordinates Φ on $\mathcal{U}$ such that $\Phi^* \mathcal{X}_\lambda = f(x, y, z)\mathcal{X}_A$ for some analytic function $f : \mathcal{U} \to \mathbb{R}$ on a neighborhood of the origin with $f(0, 0, 0) = 1$.
- In the particular case that $f(x, y, z) \equiv 1$ we say that $\mathcal{X}_\lambda$ is *analytically linearizable* in $\mathcal{U}$.

Background on Centralizers and Normalizers in $\mathbb{R}^n$

$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$ denotes the usual Lie bracket of vector fields $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{V} \subset \mathbb{R}^n$ regarded as derivations.

Definitions and notations

- The set of analytic centralizers of $\mathcal{X}$ on $\mathcal{V}$ will be denoted $\mathfrak{C}(\mathcal{X}, \mathcal{V}) = \{\mathcal{Z} \in C^w(\mathcal{V}) : [\mathcal{X}, \mathcal{Z}] = 0\}$.
- The set of linear centralizers of $\mathcal{X}$ is $\mathcal{L}(\mathcal{X})$.
- The set of analytic normalizers of $\mathcal{X}$ on $\mathcal{V}$ will be denoted $\mathfrak{N}(\mathcal{X}, \mathcal{V}) = \{\mathcal{Z} \in C^w(\mathcal{V}) : [\mathcal{X}, \mathcal{Z}] = \Lambda\mathcal{X}\}$ where $\Lambda : \mathcal{V} \to \mathbb{R}$ is a meromorphic function.
- The set of real analytic first integrals (including constants) of $\mathcal{X}$ on $\mathcal{V}$ will be denoted $\mathcal{I}(\mathcal{X}, \mathcal{V})$.

Background on Centralizers and Normalizers in $\mathbb{R}^n$

Several related algebraic structures

- The sets $\mathfrak{C}(\mathcal{X}, \mathcal{V})$ and $\mathfrak{N}(\mathcal{X}, \mathcal{V})$ are Lie algebra over the field $\mathbb{R}$ which are, in general, infinite-dimensional.
- The set $\mathcal{L}(\mathcal{X}_M)$ is a finite-dimensional real vector space.
- The set $\mathcal{I}(\mathcal{X}, \mathcal{V})$ is a ring.

Certain very interesting cases

One has in some cases the interesting fact that the Lie algebra $\mathfrak{C}(\mathcal{X}, \mathcal{V})$ has dimesion $\dim \mathfrak{C}(\mathcal{X}, \mathcal{V}) = \infty$ but $\mathfrak{C}(\mathcal{X}, \mathcal{V})$ is a finitely generated module over $\mathcal{I}(\mathcal{X}, \mathcal{V})$.

Background on Centralizers and Normalizers in $\mathbb{R}^n$

Note

We emphasize that there is no algorithmic procedure for determining if $\mathfrak{N}(\mathcal{X}, \mathcal{V})$ is nontrivial, that is, $\mathfrak{N}(\mathcal{X}, \mathcal{V}) \neq \mathbb{R}\,\mathcal{X}$.

The formal counterpart

- $\mathfrak{C}_{\rm for}(\mathcal{X})$, $\mathfrak{N}_{\rm for}(\mathcal{X})$ and $\mathcal{I}_{\rm for}(\mathcal{X})$ denotes the set of formal centralizers, normalizers, and first integrals of a given formal vector field $\mathcal{X}$.
- If $\mathcal{X}$ is analytic on $\mathcal{V}$ we have:

$$\mathfrak{C}(\mathcal{X}, \mathcal{V}) \subset \mathfrak{C}_{\rm for}(\mathcal{X}) \,,\; \mathfrak{N}(\mathcal{X}, \mathcal{V}) \subset \mathfrak{N}_{\rm for}(\mathcal{X}) \,,\; \mathcal{I}(\mathcal{X}, \mathcal{V}) \subset \mathcal{I}_{\rm for}(\mathcal{X}).$$

Centralizers and dynamics of $\mathcal{X}_\lambda$

Recall that $\mathcal{X}_\lambda = \mathcal{X}_A + \cdots$ with

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The vector space $\mathcal{L}(\mathcal{X}_A)$

A basis of $\mathcal{L}(\mathcal{X}_A)$ is $\{\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2\}$ where

$$\mathcal{Z}_0 = \mathcal{X}_A = -y\partial_x + x\partial_y + \lambda z\partial_z \ , \ \mathcal{Z}_1 = x\partial_x + y\partial_y \ , \ \mathcal{Z}_2 = z\partial_z.$$

Centralizers and dynamics of $\mathcal{X}_\lambda$

Theorem: Module structure of Centralizers of $\widehat{\mathcal{X}}_\lambda$ in the center case. Application to linearizable centers

Suppose $\mathcal{X}_\lambda$ has a center on the center manifold W^c at the origin and let $\widehat{\mathcal{X}}_\lambda$ be its analytic normal form.

- For any open neighborhood $\mathcal{U}^* \subset \mathcal{U}$ of the origin and any $\mathcal{Z} \in \mathfrak{C}(\widehat{\mathcal{X}}_\lambda, \mathcal{U}^*)$ there exist $\mu_i(x^2+y^2) \in \mathcal{I}(\mathcal{X}_A, \mathcal{U}^*)$, $0 \leq i \leq 2$, such that

$$\mathcal{Z} = \mu_0(x^2+y^2)\mathcal{Z}_0 + \mu_1(x^2+y^2)\mathcal{Z}_1 + \mu_2(x^2+y^2)\mathcal{Z}_2.$$

- If $\mathcal{X}_\lambda$ is not analytically linearizable then $\mu_1 \equiv 0$.

Centralizers and dynamics of $\mathcal{X}_\lambda$

Theorem: The formal Centralizers of $\mathcal{X}_\lambda$ and $\widehat{\mathcal{Y}}_\lambda$ in the saddle-focus case. Finite-dimensional Lie algebras

Suppose $\mathcal{X}_\lambda$ has a saddle-focus at the origin and let $\widehat{\mathcal{Y}}_\lambda$ be its C^∞ normal form.

- The Lie algebras $\mathfrak{C}_{\text{for}}(\mathcal{X}_\lambda)$ and $\mathfrak{C}_{\text{for}}(\widehat{\mathcal{Y}}_\lambda)$ satisfy

$$\dim \mathfrak{C}_{\text{for}}(\mathcal{X}_\lambda) = \dim \mathfrak{C}_{\text{for}}(\widehat{\mathcal{Y}}_\lambda) = 3.$$

- A basis of the Lie algebra $\mathfrak{C}_{\text{for}}(\widehat{\mathcal{Y}}_\lambda)$ is $\{\mathcal{Z}_0, \mathcal{Z}_2, \widehat{\mathcal{Y}}_\lambda\}$.

Centralizers and dynamics of $\mathcal{X}_\lambda$

Corollary 1: A solution of the center problem in $\mathbb{R}^3$ in terms of $\dim \mathfrak{C}_{\text{for}}(\mathcal{X}_\lambda)$

- The origin is a center for $\mathcal{X}_\lambda$ if and only if $\dim \mathfrak{C}_{\text{for}}(\mathcal{X}_\lambda) = \infty$, which is true if and only if $\dim \mathfrak{C}(\mathcal{X}_\lambda, \mathcal{U}) = \infty$.
- The origin is a saddle-focus for $\mathcal{X}_\lambda$ if and only if $\dim \mathfrak{C}_{\text{for}}(\mathcal{X}_\lambda) = 3$.

Corollary 2: $\dim \mathfrak{C}(\mathcal{X}_\lambda, \mathcal{U})$ and analytic normalization in the saddle-focus case

Assume the origin is a saddle-focus for $\mathcal{X}_\lambda$. Then:

- $\dim \mathfrak{C}(\mathcal{X}_\lambda, \mathcal{U}) \leq 3$.
- If $\mathcal{X}_\lambda$ is analytically normalizable then $\dim \mathfrak{C}(\mathcal{X}_\lambda, \mathcal{U}) = 3$.

PART III:

Multiple Hopf bifurcations from a saddle-focus

Analytic perturbation of a saddle-focus

We perturb analytically system (1) of the form

$$\begin{aligned} \dot{x} &= -y + \mathcal{G}_1(x, y, z; \varepsilon) \ , \\ \dot{y} &= x + \mathcal{G}_2(x, y, z; \varepsilon) \ , \\ \dot{z} &= \lambda z + \mathcal{G}_3(x, y, z; \varepsilon) \ , \end{aligned} \tag{2}$$

- $\varepsilon \in \mathbb{R}^p$, $0 < \|\varepsilon\| << 1$ and $\mathcal{G}_i(x, y, z; 0) \equiv \mathcal{F}_i(x, y, z)$.
- $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ is analytic for both (x, y, z) and ε in a neighborhood of the origin.
- We keep the monodromic nature of the origin: $\mathcal{G}_i(0, 0, 0; \varepsilon) = 0$ and $D\mathcal{G}(0, 0, 0; \varepsilon) = 0$.

Multiple Hopf bifurcation

We study the existence of periodic orbits of (2) bifurcating from $(x, y, z) = (0, 0, 0)$ in a multiple Hopf bifurcation for $\|\varepsilon\| << 1$.

Theorem 6

Assume that the origin of (1) is a saddle-focus. Let $V(x, y, z)$ be a C^∞ and non-flat at the origin inverse Jacobi multiplier of the unperturbed analytic system (1). Then:

- $V(x, y, z) = z(x^2 + y^2)^n + \cdots$ with $n \geq 2$ fixed.
- The maximum number of limit cycles that can bifurcate from the origin in the perturbed system (2) with $\|\varepsilon\|$ sufficiently small is $n - 1$.

Sketch of the proof of Theorem 6

1. We perform the polar blow–up $(x, y, z) \mapsto (\theta, r, w)$ defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = rw$ and the new time $t \mapsto \theta$ bringing system (2) into

$$\frac{dr}{d\theta} = R(\theta, r, w; \varepsilon) \ , \ \frac{dw}{d\theta} = \lambda w + W(\theta, r, w; \varepsilon) \ , \qquad (3)$$

defined for $|r|$ sufficiently small on the cylinder $C = \{(\theta, r, w) \in \mathbb{S}^1 \times \mathbb{R}^2\}$ where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$.

2. We define for system (3):

   (a) Poincaré map: the 2π–time flow $\Pi(r_0, w_0; \varepsilon) = (r(2\pi; r_0, w_0; \varepsilon), w(2\pi; r_0, w_0; \varepsilon))$.

   (b) displacement map: $d(r_0, w_0; \varepsilon) = \Pi(r_0, w_0; \varepsilon) - \mathrm{Id}(r_0, w_0) = (d_1(r_0, w_0; \varepsilon), d_2(r_0, w_0; \varepsilon))$.

Sketch of the proof of Theorem 6

Problem:

Look for zeroes of $d(r_0, w_0; \varepsilon)$ around $(r_0, \varepsilon) = (0, 0)$ and with $r_0 > 0$.

3 Lyapunov-Schmidt reduction to $d(r_0, w_0; \varepsilon)$: there exists one unique analytic function $\bar{w}(r_0, \varepsilon)$ defined near $(r_0, \varepsilon) = (0, 0)$ such that $d_2(r_0, \bar{w}(r_0, \varepsilon); \varepsilon) \equiv 0$.

Reduced Problem:

Look for zeroes of the analytic reduced displacement map $\Delta(r_0; \varepsilon) = d_1(r_0, \bar{w}(r_0, \varepsilon); \varepsilon)$ around $(r_0, \varepsilon) = (0, 0)$ with $r_0 > 0$.

Sketch of the proof of Theorem 6

4 Define $\delta(r_0) = \Delta(r_0; 0)$ with Taylor expansion at $r_0 = 0$: $\delta(r_0) = \sum_{i \geq k} c_i r_0^i$ with $c_k \neq 0$.
k is the order at the origin of $\delta(r_0)$.

Standard Arguments:

- Upper bound of # zeroes: From the Weierstrass Preparation Theorem, the number of zeros of $\Delta(r_0; \varepsilon)$ near $(r_0, \varepsilon) = (0, 0)$ is at most k.
- Symmetry: System (3) is invariant under $(r, \theta, w) \mapsto (-r, \theta + \pi, -w)$. Hence the zeroes of $\Delta(r_0; \varepsilon)$ near $(r_0, \varepsilon) = (0, 0)$ appear in pairs of opposite sign except the trivial one $r_0 = 0$. Thus the maximum number of limit cycles (associated with the zeros with $r_0 > 0$) is $(k-1)/2$.

Sketch of the proof of Theorem 6

5 FINAL STEP: prove that $(k-1)/2 = n-1$

An inverse Jacobi multiplier $\hat{V}(\theta, r, w)$ of system $(3)_{\varepsilon=0}$ in $r \neq 0$ is given by

$$\hat{V}(\theta, r, w) = \frac{V(r\cos\theta, r\sin\theta, rw)}{r^2(1+\Theta(\theta, r, w; 0))} .$$

THE FUNDAMENTAL RELATION

Let $\hat{V}(\theta, r, w)$ be an inverse Jacobi multiplier of system $(3)_{\varepsilon=0}$ and $\Pi(r_0, w_0)$ its Poincaré map. Then

$$\hat{V}(2\pi, \Pi(r_0, w_0)) = \hat{V}(0, r_0, w_0) \det(D\Pi(r_0, w_0)) . \qquad (4)$$

PART IV:

The 3-dimensional center problem at the zero-Hopf singularity

Families of 3-dimensional analytic zero-Hopf singularities

Take $\lambda = 0$!!!

We consider an analytic three-dimensional family of system

$$\begin{aligned} \dot{x} &= -y + F_1(x, y, z; \mu) \\ \dot{y} &= \quad x + F_2(x, y, z; \mu) \\ \dot{z} &= F_3(x, y, z; \mu), \end{aligned} \tag{5}$$

where $\mu \in \Lambda \subset \mathbb{R}^p$ are the parameters of the family and:

- The functions $F_i : U \subset \mathbb{R}^3 \to \mathbb{R}$ are $\mathcal{C}^w(U)$, $F_i(0, 0, 0; \mu) = 0$ and $\nabla F_i(0, 0, 0; \mu) = (0, 0, 0)$;
- The eigenvalues associated to the singularity of at the origin of (5) are $\{\pm i, 0\}$.
- The origin of (5) is called a *zero-Hopf* or a *fold-Hopf* singularity.

The linear part is completely integrable

- The linearization $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$ of (5) has two first integrals $H_1(x, y, z) = x^2 + y^2$ and $H_2(x, y, z) = z$.
- The orbits are the intersection of the level sets of H_1 and H_2: cylinders and planes.

Definition: 3-dimensional center

The origin of the nonlinear system (5) is a *3-dimensional center* if there is a neighborhood of it completely foliated by periodic orbits of (5), including continua of equilibriums as trivial periodic orbits.

The analytic Poincaré return map

The polar-directional blow-up

Doing first the rescaling $(x, y, z) \mapsto (x/\varepsilon, y/\varepsilon, z/\varepsilon)$ and later the polar blow–up $(x, y, z) \mapsto (\theta, r, w)$ defined by

$$x = r\cos\theta \ , \ \ y = r\sin\theta \ , \ \ z = rw \ , \tag{6}$$

system (5) can be written for $|\varepsilon|$ sufficiently small into the analytic system

$$\frac{dr}{d\theta} = \varepsilon R(\theta, r, w; \mu, \varepsilon) \ , \quad \frac{dw}{d\theta} = \varepsilon W(\theta, r, w; \mu, \varepsilon) \ , \tag{7}$$

around its invariant set $\{r = 0\}$ and defined on the cylinder $\{(\theta, r, w) \in \mathbb{S}^1 \times \mathcal{K}\}$ where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{K} \subset \mathbb{R}^2$ is an arbitrary compact set.

The analytic Poincaré return map

Remark

Any 2π–periodic solution of (7) corresponds to a periodic orbit of (5) near $(x, y, z) = (0, 0, 0)$.

3-dimensional centers and displacement map

The origin is a 3-dimensional center of (5) with $\mu = \mu^* \in \mathbb{R}^p$ if and only if $d(r_0, w_0; \mu^*, \varepsilon) \equiv 0$.

Characterizing 3-dimensional centers via normal form

It is immediate to check that:

$$\text{complete analytical integrability} \quad \Rightarrow \quad \text{3-dimensional center.}$$

What about the converse?

Theorem (3-dimensional centers and complete integrability)

The origin of system (5) for $\mu = \mu^*$ is a 3-dimensional center if and only if one of the following statements hold:

1. *System (5) is completely analytically integrable.*
2. *System (5) is analytically orbitally linearizable.*

A 9-parameter family of quadratic vector fields

Let us consider the 9-parameter family of quadratic vector fields in $\mathbb{R}^3$

$$\begin{aligned} \dot{x} &= -y + x(a_1 x + a_2 y + a_3 z), \\ \dot{y} &= x + y(b_1 x + b_2 y + b_3 z), \\ \dot{z} &= z(c_1 x + c_2 y + c_3 z), \end{aligned} \qquad (8)$$

where the parameters of the family are

$$\mu = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) \in \mathbb{R}^9.$$

The irreducible components of the center variety

Theorem (García-Valls)

The origin of (8) *is a 3-dimensional center if and only if:*

(i) $a_3 = b_3 = c_3 = 0$, $a_2 = \pm b_1$, $b_2 = \pm a_1$ *and* $c_2 = \pm c_1$;

(ii) $a_2 = a_3 = b_2 = b_3 = c_2 = c_3 = 0$;

(iii) $a_3 = b_3 = c_3 = 0$, $a_2 = b_2$, $b_1 = a_1$, $c_2 = c_1 b_2 / a_1$ *with* $a_1 \neq 0$;

(iv) $a_3 = b_3 = c_3 = c_1 = c_2 = 0$, $a_2 = -2b_2$, $b_1 = -2a_1$;

(v) $c_3 = 0$, $a_2 = \pm b_1$, $b_2 = \pm a_1$, $c_2 = \pm c_1$;

(vi) $c_1 = c_2 = c_3 = 0$, $a_2 = \mp 2a_1$, $a_3 = -b_3$, $b_1 = -2a_1$, $b_2 = \pm a_1$;

(vii) $c_1 = c_2 = c_3 = 0$, $a_2 = b_2$, $a_3 = -b_3$, $b_1 = a_1$;

(viii) $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = c_3 = 0$;

(ix) $a_1 = a_3 = b_1 = b_2 = b_3 = c_1 = c_3 = 0$;

(x) $a_1 = a_3 = b_2 = b_3 = c_3 = 0$, $b_1 = -a_2$, $c_1 = -c_2$;

(xi) $a_1 = a_3 = b_1 = b_3 = c_1 = c_3 = 0$.

PART V:

Future work and open problems

Future work and open problems

1. The cyclicity problem for a center on W^c in some POLYNOMIAL family $\mathcal{X}_\lambda$ with $\lambda \neq 0$. We need to control the displacement map and the Bautin ideal!!
2. The nilpotent center on a center manifold: We can lose analyticity of W^c even in the center case!!

$$\begin{aligned} \dot{x} &= y + F_1(x, y, z; \lambda) \\ \dot{y} &= \quad F_2(x, y, z; \lambda) \\ \dot{z} &= \lambda z + F_3(x, y, z; \lambda), \end{aligned} \tag{9}$$

with $\lambda \neq 0$.

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Generalizations to arbitrary dimension

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