Strong mixing measures and invariant sets in linear dynamics

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Contenido

Let $T : X \to X$ be a linear operator on a topological vector space (in short, tvs) X.

Definitions

- Given two operators (Y, S) and (X, T), we will say that T is quasi-conjugated to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$. If ϕ can be chosen to be a homeomorphism, then S and T are conjugated.
- (X, T) is called topologically transitive if for any pair of nonempty open sets $U, V \subset X$ there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$.

- (X, T) is called mixing if for any pair of nonempty open sets $U, V \subset X$ there exists some $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for every integer $n \geq n_0$.
- (X, T) is weakly mixing if $T \times T$ is transitive.
- (X, T) is called hypercyclic if there is some $x \in X$ whose orbit $Orb(x, T)$ is dense in X.
- (X, T) is called chaotic if it is hypercyclic and the set of periodic points of T is dense in X.

The corresponding notions for a sequence of operators $T_n : X \to X$ are defined by considering the sequence $(T_n)_n$.

Definitions

A Borel probability measure μ, has full support if for all non-empty open set $U \subset X$ we have $\mu(U) > 0$. T is ergodic if $T^{-1}(A) = A$ for $A \in \mathfrak{B}$ implies $\mu(A)(1 - \mu(A)) = 0$. T is strongly mixing with respect to μ if

$$\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B}),$$

Definition

An operator T on a t.v.s space X is called frequently hypercyclic is there is some $x \in X$ such that, for any nonempty open subset U of X,

$$\liminf_{N \to \infty} \frac{card\{0 \leq n \leq N; T^n x \in U\}}{N+1} > 0.$$

Under the same hypothesis of the Frequently Hypercycliclity Criterion, given by Bonilla and Grosse-Erdmann, we derive a stronger result by showing that a T-invariant mixing measure can be obtained.

Theorem

Let T be an operator on a **separable Banach space** *X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X_0$ such that, for each $x \in X_0$,*

(i) *$\sum_{n=0}^{\infty} T^n x$ converges unconditionally*
(ii) *$\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and*
(iii) *$T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$.*

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

Sketch of the proof

The idea behind the proof is to construct

1. a "model" probability space $(Z, \overline{\mu})$ and
2. a Borel measurable map $\Phi : Z \to X$, where

- $Z \subset \mathbb{N}^{\mathbb{Z}}$ is such that $\sigma(Z) = Z$ for the Bernoulli shift $\sigma(\dots, n_{-1}, n_0, n_1, \dots) = (\dots, n_0, n_1, n_2, \dots)$,
- $\overline{\mu}$ is a σ^{-1}-invariant strongly mixing measure,
- $\Phi\sigma^{-1} = T\Phi$,

As a consequence the Borel probability measure μ on X defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is T-invariant and strongly mixing.

Corollary

Let $B_w : X \to X$ be a bilateral weighted backward shift on $X = \ell_p(\mathbb{Z})$ defined as

$$B_w(x_n)_{n\in\mathbb{Z}} = (w_{n+1}x_{n+1})_{n\in\mathbb{Z}},$$

such that

$$\sum_{n=-\infty}^{0}\left(\prod_{\nu=n+1}^{0} w_\nu\right) e_n + \sum_{n=1}^{\infty}\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$

converges unconditionally in X. Then there exists a T-invariant strongly mixing Borel probability measure on X with full support.

Definition

A one-parameter family $(T_t)_{t\geq 0}$ of operators on a Banach space X is called a C_0-**semigroup** if the following three conditions are satisfied:

(i) $T_0 = I$,

(ii) $T_t T_s = T_{t+s}$ for all $t, s \geq 0$

(iii) $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$

Mangino and Peris gave a continuous version of the Frequently Hypercyclicity Criterion. Under the same hypothesis we derive a stronger result, the existence of a strongly mixing measure with full support.

Theorem

Let $(T_t)_t$ be a C_0-semigroup on a **separable Banach space** X. If there exists $X_0 \subset X$ dense in X and maps $S_t : X_0 \to X_0$, $t > 0$, such that :

(i) $T_t S_t x = x, T_t S_r x = S_{r-t} x, t > 0, r > t > 0,$

(ii) $t \to T_t x$ is Pettis integrable in $[0, \infty)$ for all $x \in X_0$,

(iii) $t \to S_t x$ is Pettis integrable in $[0, \infty)$ for all $x \in X_0$.

then there is a $(T_t)_t$-invariant strongly mixing Borel probability measure μ on X with full support.

Example

Let us consider the death model with variable coefficients

$$\begin{cases} \frac{\partial f_n}{\partial t} = -\alpha_n f_n + \beta_n f_{n+1}, & n \geq 1, \\ \\ f_n(0) = a_n, & n \geq 1 \end{cases} \tag{1}$$

with $(\alpha_n)_n$, $(\beta_n)_n$ bounded positive sequences and $(a_n)_n \in \ell^1$ is a real sequence. Let $X = \ell^1$, and

$$Af = (-\alpha_n f_n + \beta_n f_{n+1})_n \text{ for } f = (f_n)_n \in X,$$

It generates a C_0-semigroup $(T_t)_{t\geq 0}$ which is solution of (1). If $\sup_{n\geq 1} \alpha_n < \liminf_{n\to\infty} \beta_n$ the semigroup $(T_t)_{t\geq 0}$ admits an invariant mixing measure with full support on X.

Recently Bayart and Ruzsa (2013), characterized frequently hypercyclic weighted shifts on $\ell_p(\mathbb{Z})$ and $c_0(\mathbb{Z})$.

Definition

Let $L_p^\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\ ;\ f \text{ is measurable and } ||f||_p < \infty\}$, where $||f||_p = (\int_{-\infty}^{\infty} |f(t)|^p \rho(t) dt)^{\frac{1}{p}}$ and $C_0^\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\ ;\ f \text{ is continuous and } \lim_{x\to\infty} f(x)\rho(x) = 0\}$, with $||f||_\infty = \sup_{t\in\mathbb{R}} f(t)\rho(t)$.
If X is any of the spaces above, the translation semigroup $(T_t)_{t\geq 0}$ defined by $T_t f(x) = f(x+t)$ is a well defined C_0-semigroup.

Proposition

Let $(T_t)_{t\geq 0}$ be a mixing (equivalently chaotic) translation C_0-semigroup on $C_0^\rho(\mathbb{R})$. Then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

The converse of the previous proposition does not hold.

Theorem

Let $(T_t)_{t\geq 0}$ be the translation semigroup defined on $L_p^\rho(\mathbb{R})$. The following assertions are equivalent:

(1) $(T_t)_{t\geq 0}$ *is frequently hypercyclic.*

(2) $\sum_{k\in\mathbb{Z}} \rho(k) < \infty$.

(3) $\int_{-\infty}^{\infty} \rho(t)dt < \infty$.

(4) $(T_t)_{t\geq 0}$ *is chaotic.*

(5) $(T_t)_{t\geq 0}$ *satisfies the Frequently Hypercyclicity Criterion.*

Proposition

Let $(K_n)_n$ be an increasing sequence of T-invariant sets, and $Y = \overline{\bigcup_{n=1}^{\infty} K_n}$. We have:

(i) If $T|_{K_n}$ is transitive for all $n \in \mathbb{N}$ then $T : Y \to Y$ is transitive.

(ii) If $T|_{K_n}$ is mixing for all $n \in \mathbb{N}$ then $T : Y \to Y$ is mixing.

(iii) If $T|_{K_n}$ is weakly-mixing for all $n \in \mathbb{N}$ then $T : Y \to Y$ is weakly-mixing.

(iv) If $T|_{K_n}$ is chaotic for all $n \in \mathbb{N}$ then $T : Y \to Y$ is chaotic.

Corollary

Let K be an absolutely convex T-invariant set such that $T|_K$ is transitive (respectively weakly-mixing, mixing, chaotic, topologically ergodic), then $T|_{\overline{span(K)}}$ is transitive (respectively $\dots$). In particular, if $\overline{span(K)} = X$, then the property is inherited by T on X.

Theorem

Let $T : X \to X$ be an operator and $(K_n)_n$ an increasing sequence of T-invariant compact sets such that $T|_{K_n}$ is transitive and $\overline{\bigcup_{n=1}^{\infty} K_n} = X$. Then T is weakly mixing.

Theorem

Let X be a Banach space and let the system $(X, (T_n)_n)$, where $\{T_n : X \to X \ ; \ n \in \mathbb{N}\}$ is a sequence of operators such that $T_n(Y) \subset Y$ for every $n \in \mathbb{N}$ and for certain $Y \subset X$ with $0 \in Y$. We consider $Z := \overline{span(Y)}$.

(1) *If $(Y, (T_n|_Y)_n)$ is weakly mixing of all orders then $(Z, (T_n|_Z)_n)$ is also weakly mixing of all orders.*

(2) *If $(Y, (T_n|_Y)_n)$ is mixing then $(Z, (T_n|_Z)_n)$ is also mixing.*

Example

Logistic map: Let $p : [0,1] \to [0,1]$ be the logistic polynomial $p(x) := 4x(1-x)$, which is chaotic and mixing. We will embed $[0,1]$ in a locally convex space X via a map ϕ, and we define $T : X \to X$ such that $T \circ \phi = \phi \circ p$ and $\overline{span(\phi[0,1])} = X$. Let

$$X = \{(x_i)_i \in \mathbb{C}^{\mathbb{N}} \ ; \ \exists r > 0 \text{ such that } \sup_i |x_i| r^i < \infty\}.$$

X has its natural inductive topology.
We define $\phi : I \to X$ as $\phi(x) = (x, x^2, x^3, \dots)$, and

$$T(x_1, x_2, \dots)_k = 4^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} x_{j+k}, \quad k \in \mathbb{N}.$$

$T \circ \phi = \phi \circ p$. Let $Y := \phi[0,1]$. $span(Y)$ is dense in X, hence T is mixing and chaotic.

Definition

A lattice is a non-empty set M with an order $\leq$ such that every pair of elements $x, y \in M$ has both a supremum and an infimum.
An ordered vector space is a real vector space X which is also an ordered space such that:

- If $x, y, z \in X$ and $x \leq y$ then $x + z \leq y + z$
- If $x, y \in X$, $x \leq y$ and $0 \leq \alpha \in \mathbb{R}$, then $\alpha x \leq \alpha y$

The set $X^+ = \{x \in X \ ; \ x \geq 0\}$ is termed the positive cone in X .
An ordered vector space which is also a lattice is a vector lattice.
If X and Y are vector lattices then an operator $T : X \to Y$ is positive if $x \geq 0$ implies $Tx \geq 0$.
A Banach(Fréchet) lattice is a Banach(Fréchet) space which is also a vector lattice in which $x \leq y$ implies $\|x\| \leq \|y\|(\|x\|_n \leq \|y\|_n, n \in \mathbb{N}$, where $(||\cdot||_n)_n$ is an increasing sequence of seminorms).

Theorem

Let $(T_t)_{t\geq 0}$ be a C_0-semigroup of positive operators on a separable Banach lattice X. If there exist $X_0 \subset X^+$ dense in X and maps $S_t : X_0 \to X^+$, $t > 0$ such that

- $T_t S_t x = x, T_t S_r x = S_{r-t} x, t > 0, r > t > 0,$
- $t \to T_t x$ is Pettis integrable in $[0, \infty)$ for all $x \in X_0$,
- $t \to S_t x$ is Pettis integrable in $[0, \infty)$ for all $x \in X_0$.

then $(T_t|_{X^+})_{t\geq 0}$ is mixing, each operator $T_t|_{X^+}$ with $t > 0$ is chaotic, and there is a $(T_t)_t$-invariant strongly mixing Borel probability measure μ on X^+ whose support is X^+.

Example

Let $X = \{f \in C([0,1],\mathbb{R}) : f(0) = 0\}$ with the sup norm. We consider the following initial value problem of a partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma x \frac{\partial u}{\partial x} + h(x)u, \\ \\ u(0,x) = f(x) \end{cases} \tag{2}$$

where $\gamma < 0$, $h \in C([0,1],\mathbb{R})$ and $f \in X$. Then the solution semigroup $(T_t)_{t\geq 0}$,

$$T_t f(x) = e^{\int_0^t h(e^{\gamma(t-s)}x)ds} f(e^{\gamma t}x)$$

to the equation (2) is a strongly continuous semigroup on X. If $\min\{h(x) : x \in [0,1]\}$ is positive, then there exists a $(T_t)_t$-invariant strongly mixing Borel probability measure μ on X^+ whose support is X^+.

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