PERIODIC SOLUTIONS OF RESONANT HAMILTONIAN SYSTEMS THROUGH SINGULAR REDUCTION

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Goal

Main achievement

Detemining periodic solutions, stability and bifurcation of resonant Hamiltonians

Methodology

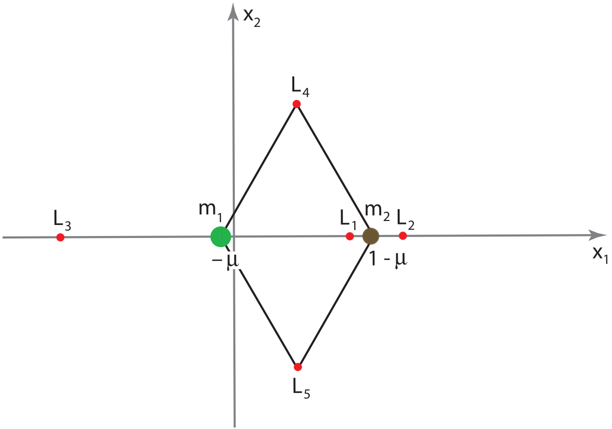
(i) Apply normal forms to introduce approximate symmetries of the Hamiltonians.

(ii) Use regular and singular reduction theory reducing out the continuous symmetries obtaining a Hamiltonian system on the corresponding reduced (orbit) space.

(iii) Investigate the dynamics of the reduced problem: existence, stability and bifurcations of the relative equilibria in terms of some parameters.

(iv) Get conclusions about the full system by reconstructing the flow of the original problem. We obtain families of periodic solutions and invariant tori.

Hamiltonian in a Rotating Frame



The Hamiltonian has five equilibria:

- L_1, L_2, L_3 unstable (Euler),
- L_4, L_5 linearly stable iff $0 < \mu < (1 - \sqrt{69}/9)/2$ (Lagrange).

L_4 and L_5 in the Planar Circular Restricted Three Body Problem

The Hamiltonian written in a rotating frame $x_1x_2x_3$ is given by

$$\mathcal{H}_R = \frac{1}{2}(y_1^2 + y_2^2) - (x_1y_2 - x_2y_1) - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2}} - \frac{1-\mu}{\sqrt{(x_1 + \mu)^2 + x_2^2}}.$$

- $\mu = m_1/(m_1 + m_2)$,
- assuming that $m_1 \geq m_2$ then $\mu \in (0, 1/2)$,
- the masses m_1 and m_2 are located at the points $(-\mu, 0)$ and $(1 - \mu, 0)$ of the coordinate space, respectively.

Coordinates of L_4 and L_5: $(1/2 - \mu\ ,\ \pm\sqrt{3}/2\ ,\ \mp\sqrt{3}/2\ ,\ 1/2 - \mu)$

Finding Periodic Solutions

A. Deprit and J. Henrard, A manifold of periodic orbits, *Adv. Astron. Astrophys.* 6, 1968, 2–124: It reveals a very rich dynamics

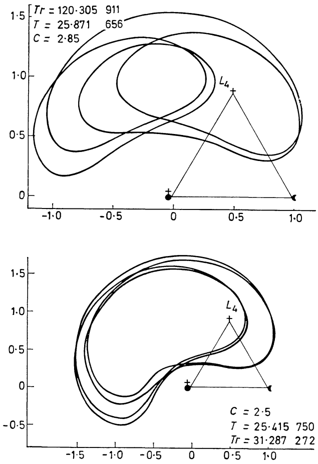


FIG. 4. *Evolution of the branch* $\mathscr{L}_4{}^l$.

Initial Conditions

E. Rabe and A. Schanzle, Periodic librations about the triangular solutions of the restricted Earth-Moon problem and their orbital stabilities, *Astron. J.* 67, 1962, 732–739

TABLE IV

Initial conditions for the branch $\mathscr{L}_4^l$

No.	x_0	y_0	$\dot{x}_0$	$\dot{y}_0$
1	0·494050076	0·977910758	0·149215754	−0·054919858
2	0·507396097	1·012800883	0·201524769	−0·083454244
3	0·512405341	1·024756987	0·217237020	−0·095921375
4	0·516360218	1·033606182	0·227836282	−0·106473701
5	0·519836776	1·040954368	0·235814421	−0·116433037
6	0·523163185	1·047597479	0·242133500	−0·126965897
7	0·528098017	1·056630699	0·249221205	−0·145591939
8	0·576620485	1·026776464	0·232584018	−0·160936225
9	0·581416004	1·033339813	0·238024091	−0·178807439
10	0·584891378	1·037839987	0·244030785	−0·188659522
11	0·588245748	1·042106229	0·250458033	−0·196976479
12	0·591544917	1·046255473	0·257104080	−0·204454802
13	0·594776822	1·050319927	0·263849839	−0·211344653
14	0·597953704	1·054286280	0·270599528	−0·217780353
15	0·601066536	1·058815424	0·277296887	−0·223841425
16	0·604108706	1·061923017	0·283903324	−0·229583137
17	0·609965983	1·069162922	0·296746251	−0·240266164
18	0·623265885	1·085595193	0·326219938	−0·263503507
19	0·644975332	1·112518834	0·374215068	−0·300992247
20	0·653999012	1·123737498	0·393858443	−0·317007259
21	0·694011593	1·173095324	0·477517243	−0·393441011
22	0·735500418	1·222745972	0·558728889	−0·483601882
23	0·763105804	1·254389866	0·610428854	−0·551426370
24	0·783188373	1·276460734	0·647373839	−0·606714169
25	0·798162512	1·292521080	0·675368101	−0·653646257
26	0·809429240	1·304216355	0·697086978	−0·694571133
27	0·817698589	1·312555495	0·714095400	−0·730849058
28	0·823378278	1·318135836	0·727307801	−0·763354355
29	0·827485503	1·322262941	0·744203619	−0·819073126
30	0·820762807	1·316328790	0·747429619	−0·863928747
31	0·810304875	1·307355436	0·740516843	−0·881462901
32	0·784650823	1·286181796	0·714541006	−0·889300818
B_1	0·770867229	1·275071808	0·697984995	−0·885130287

Many Contributions: The Trojan Web

1. Lyapunov centre theorem and power series:
   - Lyapunov (1892)
   - Moulton (1920)
   - Buchanan (1941)

2. Use of computers:
   - Strömgen and coworkers (1930), but they applied numerical methods by hand!
   - Rabe (1961)

3. Normal forms:
   - Deprit & Henrard (1968)
   - Meyer, Palmore & Schmidt (1970)

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Two Fundamental Results

- Lyapunov Centre Theorem:

  Consider the smooth Hamiltonian system

$$\dot{z} = Az + \cdots = JSz + \cdots$$

  defined in a neighborhood of the origin in $\mathbb{R}^4$*, let the eigenvalues of the Hamiltonian matrix A be the pure imaginaries* $\pm i\omega_1$, $\pm i\omega_2$, $\omega_1, \omega_2 \neq 0$*, then if* ω_1/ω_2 *is not an integer the system has a one parameter family of periodic solutions emanating from the origin of period near* $2\pi/|\omega_1|$.

- Weinstein Theorem:

  Two periodic solutions are found in each small energy level (H constant) provided the symmetric matrix S is definite, positive or negative.

Resonant Hamiltonians

Hamiltonian around L_4 when $0 < \mu < (1 - \sqrt{69}/9)/2$ reads as

$$H = H_2 + H_3 + \dots,$$

where $H_2 = \frac{1}{2}[\omega_1(x_1^2 + y_1^2) - \omega_2(x_2^2 + y_2^2)]$, and we consider $\omega_i > 0$ and such that ω_1/ω_2 is rational.

In general we shall consider systems such that

$$H_2 = \frac{1}{2}[q(x_1^2 + y_1^2) + p(x_2^2 + y_2^2)],$$

with q, p non-null integers such that $q > 0$ and $\gcd(q, p) = 1$.

There are many approaches: Schmidt, Duistermaat, Golubitskii and Stewart, Kummer, etc.

Invariants #1

Higher-order terms are put in normal form, i.e., $\{H_k, H_2\} = 0$ for $k = 3, 4, \ldots$.

Invariants associated to the $q : p$ resonance:

$$\begin{aligned}
a_1 &= I_1 = x_1^2 + y_1^2, \\
a_2 &= I_2 = x_2^2 + y_2^2, \\
a_3 &= I_1^{|p|/2} I_2^{q/2} \cos(q\,\theta_2 - p\,\theta_1) = \mathrm{Re}\,[(x_1 - \mathrm{sign}(p)\,y_1 i)^{|p|}(x_2 + y_2 i)^q], \\
a_4 &= I_1^{|p|/2} I_2^{q/2} \sin(q\,\theta_2 - p\,\theta_1) = \mathrm{Im}\,[(x_1 - \mathrm{sign}(p) y_1 i)^{|p|}(x_2 + y_2 i)^q],
\end{aligned}$$

subject to the constraints

$$a_3^2 + a_4^2 = a_1^{|p|}\, a_2^q, \quad a_1 \geq 0, \; a_2 \geq 0.$$

Then

- $H_2 = q a_1 + p a_2 = h$,
- $H_k \equiv H_k(a_1, a_2, a_3, a_4)$ where H_k are polynomials in a_i.

Invariants #2

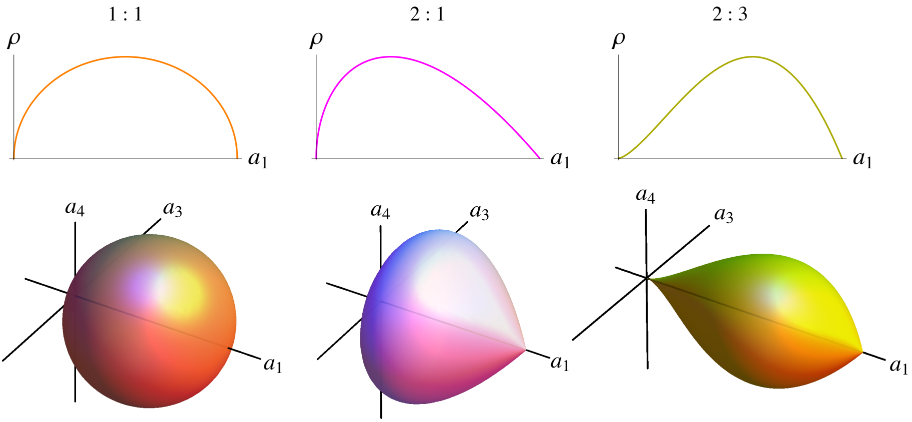
Reduced space, e.g. orbit space is an orbifold

- If (i) $q = p = 1$; (ii) $p = -1$ and $h < 0$; (iii) $q = 1$ and $h > 0$, we are in the case of regular reduction, i.e. the reduced space is a manifold;
- In the rest of situations we are in the case of singular reduction with one or two singularities.

Poisson structure

$\{\,,\,\}$	a_1	a_2	a_3	a_4
a_1	0	0	$2p\,a_4$	$-2p\,a_3$
a_2	0	0	$-2q\,a_4$	$2q\,a_3$
a_3	$-2p\,a_4$	$2q\,a_4$	0	$a_1^{\lvert p\rvert-1}\,a_2^{q-1}(q^2\,a_1 - p\,\lvert p\rvert\,a_2)$
a_4	$2p\,a_3$	$-2q\,a_3$	$-a_1^{\lvert p\rvert-1}\,a_2^{q-1}(q^2\,a_1 - p\,\lvert p\rvert\,a_2)$	0

Orbit Spaces: $p > 0$



(a) Orange 1 : 1 (b) Turnip 2 : 1 (c) Lemon 2 : 3

Figure: Case $p > 0$. Above: ρ versus a_1. Below: Orbit spaces

Orbit Spaces: $p < 0$

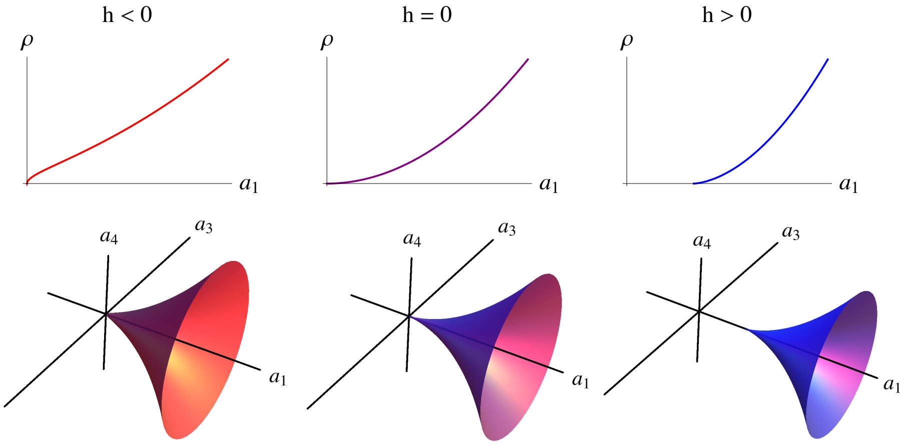


Figure: Case $q = 3$, $p = -1$. Above: ρ versus a_1. Below: Orbit spaces

Symplectic Smoothing #1

For instance, the case $(a_1, a_2, a_3, a_4) = (h/q, 0, 0, 0)$ and $h > 0$, the singularity occurs at $a_1 = h/q$.

(a) For q odd, we build

$$\begin{aligned} a_1 &= \frac{h - pq(x^2 + y^2)}{q}, \\ a_2 &= q(x^2 + y^2), \\ a_3 &= q^{(q-|p|)/2} x (x^2 + y^2)^{(q-1)/2} \left[h - pq(x^2 + y^2)\right]^{|p|/2}, \\ a_4 &= q^{(q-|p|)/2} y (x^2 + y^2)^{(q-1)/2} \left[h - pq(x^2 + y^2)\right]^{|p|/2}, \end{aligned}$$

with inverse

$$x = q^{-1/2} a_1^{-|p|/2} a_2^{(1-q)/2} a_3, \quad y = q^{-1/2} a_1^{-|p|/2} a_2^{(1-q)/2} a_4.$$

The transformed surface in the $a_2\, x\, y$-space is

$$a_2 = q(x^2 + y^2).$$

(b) For q even, we get

$$
\begin{aligned}
a_1 &= \frac{2h - pq(x^2+y^2)}{2q},\\
a_2 &= \frac{q}{2}(x^2+y^2),\\
a_3 &= 2^{-(|p|+q)/2} q^{(q-|p|)/2} (x^2-y^2)(x^2+y^2)^{q/2-1}\left[2h - pq(x^2+y^2)\right]^{|p|/2},\\
a_4 &= 2^{1-(|p|+q)/2} q^{(q-|p|)/2} xy(x^2+y^2)^{q/2-1}\left[2h - pq(x^2+y^2)\right]^{|p|/2},
\end{aligned}
$$

and when $a_3 \geq 0$ the inverse is

$$x = (2/q)^{1/2} a_2^{1/2} \cos\left[\frac{1}{2}\tan^{-1}\left(\frac{a_4}{a_3}\right)\right], \quad y = (2/q)^{1/2} a_2^{1/2} \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{a_4}{a_3}\right)\right],$$

and similarly for $a_3 < 0$.

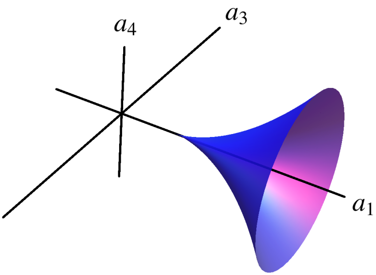
The transformed surface in the $a_2\,x\,y$-space is

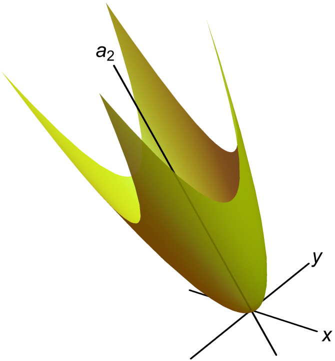
$$a_2 = \frac{q}{2}(x^2+y^2).$$

Symplectic Smoothing #3

In all cases $\{x, y\} = 1$.

The change is local.





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Cherry's Example: $2 : -1$ Resonance

$$H = \frac{1}{2}\lambda(x_1^2 + y_1^2) - \lambda(x_2^2 + y_2^2) + \frac{1}{2}\alpha[x_2(x_1^2 - y_1^2) - 2x_1y_1y_2],$$

with λ and α as arbitrary parameters.

It is an integrable system. Whittaker provided the solution:

$$x_1 = \frac{\sqrt{2}}{\alpha(t+\epsilon)}\sin(\lambda t + \gamma), \qquad y_1 = \frac{\sqrt{2}}{\alpha(t+\epsilon)}\cos(\lambda t + \gamma),$$

$$x_2 = \frac{1}{\alpha(t+\epsilon)}\sin\left(2(\lambda t + \gamma)\right), \quad y_2 = \frac{-1}{\alpha(t+\epsilon)}\cos\left(2(\lambda t + \gamma)\right),$$

where ϵ and γ are constants of integration.

Applying Reduction

1. Written in invariants:

$$H = H_2 + H_3 \quad \text{with} \;\; H_2 = 2a_1 - a_2 \quad \text{and} \;\; H_3 = a_3.$$

2. The orbit space, when $2a_1 - a_2 = h$, is the surface

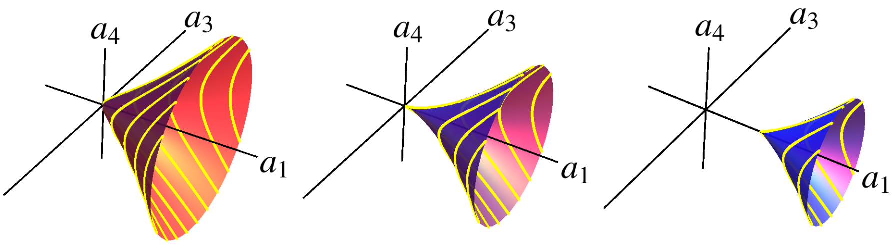
$$a_3^2 + a_4^2 = a_1(h - 2a_1)^2, \qquad a_1 \geq 0, \quad a_1 \geq h/2.$$

3. Using the Poisson structure with $q = 2$ and $p = -1$, the vector field reads as

$$\dot{a}_1 = \{a_1, H_3\} = -2a_4, \;\; \dot{a}_3 = \{a_3, H_3\} = 0, \;\; \dot{a}_4 = \{a_4, H_3\} = -a_2(4a_1 + a_2).$$

4. It is easy to achieve that there is an equilibrium at $(a_1, a_3, a_4) = (h/2, 0, 0)$ only when $h \geq 0$.

Flow of the Reduced Space



$h < 0$ $h = 0$ $h > 0$

1. When $h = 0$ the origin is the origin in $\mathbb{R}^4$: There are two solutions spiraling to the origin as $t \to \pm\infty$, the ones given by Whittaker.

2. When $h > 0$ the equilibrium point gives rise to a periodic solution of period $T \sim \pi$ for each $h \geq 0$ (short periodic family given by Lyapunov centre theorem. These solutions are unstable.

The $3 : -1$ Resonance

In action-angle coordinates: $I_j = x_j^2 + y_j^2$, $\theta_j = \tan^{-1} y_j/x_j$, we write:

$$H = 3I_1 - I_2 + \frac{\varepsilon^2}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \varepsilon^2 G I_1^{1/2} I_2^{3/2} \cos(\theta_1 + 3\theta_2),$$

where A, B, C, G are constants.

1. Introduce the constants $D = \frac{1}{2}(A + 6B + 9C)$ and $R = B + 3C$.
2. Pass to the averaged system we get

$$H_3 = Da_1^2 - Rha_1 + Ga_3$$

which is defined on the orbit space:

$$a_3^2 + a_4^2 = a_1(3a_1 - h)^3, \qquad a_1 \geq 0, \quad 3a_1 \geq h.$$

Associated Vector Field

$$
\begin{aligned}
\dot{a}_1 &= \{a_1, H_3\} = -2Ga_4, \\
\dot{a}_3 &= \{a_3, H_3\} = 2a_4(2Da_1 - Rh), \\
\dot{a}_4 &= \{a_4, H_3\} = -2a_3(2Da_1 - Rh) - Ga_2^2(9a_1 + a_2) \\
&= -2a_3(2Da_1 - Rh) - G(3a_1 - h)^2(12a_1 - h),
\end{aligned}
$$

The analysis is a bit delicate since there are many but there are basically two scenarios.

Bifurcation lines:

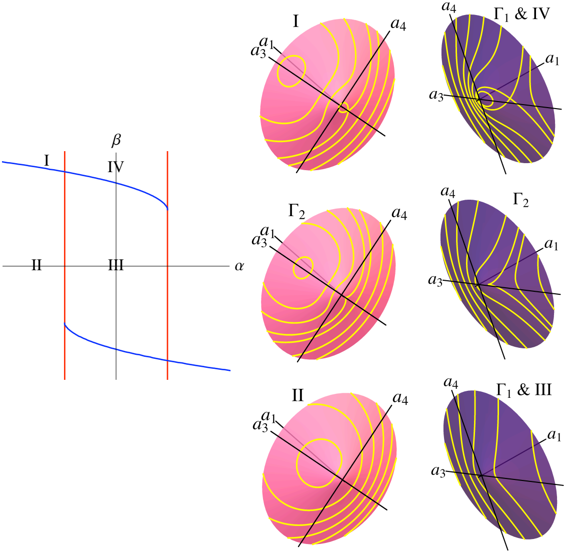
$$\alpha = D/G, \qquad \beta = R/G$$

Γ_1: $\alpha^2 - 27 = 0$ (red lines);

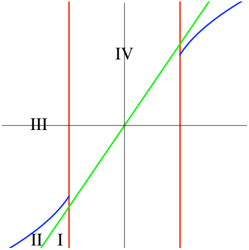
Γ_2: $729 + 108\alpha^2 + 648\beta^2 - 48\beta^4 + 8\alpha\beta(-81 + 4\beta^2) = 0$ (blue curves);

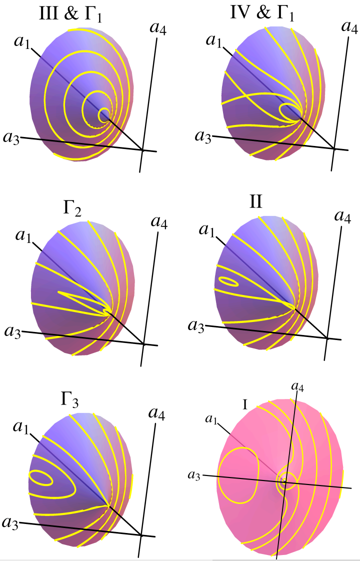
Γ_3: $2\alpha - 3\beta = 0$ (green curve).

Bifurcations and Flows When $h < 0$



Bifurcations and Flows When $h > 0$





Non-semisimple 1 : −1 Resonance

Consider the Hamiltonian $H = H_2 + H_3 + \ldots$ where

$$H_2 = x_2 y_1 - x_1 y_2 + \frac{\delta}{2}(x_1^2 + x_2^2)$$

where $\delta = \pm 1$.

The linear system of equations is $\dot{z} = Az$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -\delta & 0 & 0 & 1 \\ 0 & -\delta & -1 & 0 \end{bmatrix}, \qquad z = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}.$$

A has repeated eigenvalues $\pm i$, but not all solutions are 2π periodic, since there are secular terms like $t \sin t$, $t \cos t$.

Normal Form and Invariants

The four invariants usually associated with this Hamiltonian are just

$$b_1 = x_2y_1 - x_1y_2, \quad b_2 = \frac{1}{2}(x_1^2 + x_2^2), \quad b_3 = \frac{1}{2}(y_1^2 + y_2^2), \quad b_4 = x_1y_1 + x_2y_2,$$

with the constraint

$$b_1^2 + b_4^2 = 4\, b_2 b_3.$$

The non-zero Poisson brackets are

$$\{b_2, b_3\} = b_4, \quad \{b_2, b_4\} = 2b_2, \quad \{b_4, b_3\} = 2b_3.$$

We start with H_2 such that $H_2 = b_1 + \delta b_2$ and the rest of terms

$$H = b_1 + \delta b_2 + (\alpha b_1^2 + 2\beta b_1 b_3 + \gamma b_3^2) + \cdots .$$

Use an adequate scaling, so the Hamiltonian becomes

$$H = b_1 + \varepsilon(\delta b_2 + \gamma b_3^2) + O(\varepsilon^2).$$

Equations of Motion

1. Thus the equation of the orbit space is

$$h^2 + b_4^2 = 4\, b_2\, b_3.$$

2. The reduced averaged Hamiltonian is

$$H_4 = \delta b_2 + \gamma b_3^2 = \bar{h}.$$

3. The equations of motion are

$$\dot{b}_2 = \{b_2, H_4\} = 2\gamma b_3 b_4, \quad \dot{b}_3 = \{b_3, H_4\} = 0, \quad \dot{b}_4 = \{b_4, H_4\} = -2\delta b_2 + 4\gamma b_3^2.$$

In the case of $1 : -1$ *resonance there are two families of nearly* 2π *elliptic periodic solutions emanating from the origin when* $\delta\gamma > 0$. *One family exists for* $\mathcal{H} > 0$ *and one for* $\mathcal{H} < 0$. *There are no nearby* 2π *periodic solutions when* $\delta\gamma < 0$.

Flows

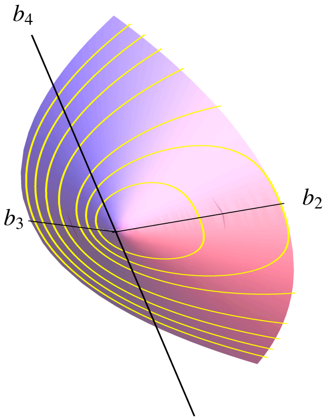


Figure: Flows in the $1 : -1$ resonance. On the left: $\delta\gamma > 0$. On the right: $\delta\gamma < 0$.

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Resonant Hamiltonians

We deal with Hamiltonians

$$H = H_2 + H_3 + \dots$$

such that the Hamiltonian matrix A of H_2 is semi-simple with pure imaginary eigenvalues:

$$\pm k_1\,\omega i,\ \pm k_2\,\omega i,\ \dots,\ \pm k_n\,\omega i$$

where ω is positive real, $k_i \in \mathbb{Z}$, with $\gcd(k_1, k_2, \dots, k_n) = 1$.

By a change of the time scale we may take $\omega = 1$. In this case the Hamiltonian can be put into the form

$$H_2(x,y) = \frac{1}{2}[k_1\,(x_1^2 + y_1^2) + k_2\,(x_2^2 + y_2^2) + \cdots + k_n\,(x_n^2 + y_n^2)],$$

where $z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

H_2 is a super-integrable system since there are $2n-1$ independent integrals, namely

$$I_1,\ I_2,\ \ldots,\ I_n,\ k_1\,\theta_n - k_n\,\theta_1,\ k_1\,\theta_{n-1} - k_{n-1}\,\theta_1,\ \ldots,\ k_1\,\theta_2 - k_2\,\theta_1.$$

In rectangular coordinates:

$$\begin{aligned}
a_1 &= I_1 = x_1^2 + y_1^2,\ a_2 = I_2 = x_2^2 + y_2^2,\ \ldots,\ a_n = I_n = x_n^2 + y_n^2,\\
a_{n+1} &= a_n^{|k_1|/2}\,a_1^{|k_n|/2}\cos(k_1\,\theta_n - k_n\,\theta_1) = \mathrm{Re}\,[(x_n + \mathrm{sgn}(k_1)\,y_n i)^{|k_1|}(x_1 - \mathrm{sgn}(k_n)\,y_1 i)^{|k_n|}],\\
a_{n+2} &= a_n^{|k_1|/2}\,a_1^{|k_n|/2}\sin(k_1\,\theta_n - k_n\,\theta_1) = \mathrm{Im}\,[(x_n + \mathrm{sgn}(k_1)\,y_n i)^{|k_1|}(x_1 - \mathrm{sgn}(k_n)\,y_1 i)^{|k_n|}],\\
&\ \ \vdots\\
a_{3n-3} &= a_2^{|k_1|/2}\,a_1^{|k_2|/2}\cos(k_1\,\theta_2 - k_2\,\theta_1) = \mathrm{Re}\,[(x_2 + \mathrm{sgn}(k_1)\,y_2 i)^{|k_1|}(x_1 - \mathrm{sgn}(k_2)\,y_1 i)^{|k_2|}],\\
a_{3n-2} &= a_2^{|k_1|/2}\,a_1^{|k_2|/2}\sin(k_1\,\theta_2 - k_2\,\theta_1) = \mathrm{Im}\,[(x_2 + \mathrm{sgn}(k_1)\,y_2 i)^{|k_1|}(x_1 - \mathrm{sgn}(k_2)\,y_1 i)^{|k_2|}].
\end{aligned}$$

Some Relationships

Clearly $a_1 \geq 0$, $a_2 \geq 0$, $\ldots$, $a_n \geq 0$ and the identity $\cos^2\phi + \sin^2\phi = 1$ yields

$$\begin{aligned} a_{n+1}^2 + a_{n+2}^2 &= a_n^{|k_1|}\, a_1^{|k_n|}, \\ &\vdots \\ a_{3n-3}^2 + a_{3n-2}^2 &= a_2^{|k_1|}\, a_1^{|k_2|}. \end{aligned}$$

But more invariants are needed in order to express the perturbation $H_3 + \ldots$ in terms of them in terms of polynomials.

Approach

One needs to resort to computer algebra techniques and obtain the invariants, the relationships among them and to write down the Hamiltonian and the corresponding vector field in terms of the invariants

Automatic Determination of Invariants #1

Derksen and Kemper's Algorithm for Invariant of Tori (2002):

- It is more efficient than those that use Gröbner bases (Sturmfels and others).
- It performs better than the methods of Fekken.
- The computation relies on divisibility test of two monomials, it is an integer programming problem.

Automatic Determination of Invariants #2

1. Let $T = (K^*)^r$ be a torus acting diagonally on an n-dimensional vector space V
2. Identify $K[V] \equiv K[x_1, \ldots, x_n]$
3. $\omega = (\omega^{(1)}, \ldots, \omega^{(r)}) \in \mathbb{Z}^r$ is a weight, we write $t^\omega = t_1^{\omega^{(1)}} \cdot t_r^{\omega^{(r)}}$
4. For $i = 1, \ldots, n$ let ω_i the weight with which T acts on x_i: $t \cdot x_i = t^{\omega_i} x_i$
5. If $m = x_1^{a_1} \cdots x_n^{a_n}$ then T acts on m with wieght $a_1\omega_1 + \cdots a_n\omega_n$

Main idea: Choose a suitable finite set $\mathcal{C}$ of weights and produce sets S_ω with $\omega \in \mathcal{C}$ of monomials of weight ω.

These sets grow during the course of the algorithm, until upon termination we get S_0 that generates $K[V]^G$.

The algorithm performs well for systems of 4, 5 or 6 degrees of freedom.

Orbit Space: Szygyies

- For a given resonance $k_1 : k_2 : \ldots : k_n$, introducing complex variables, say u_i, v_i (instead of x_i, y_i), we get the set of invariants: $\{a_1, a_2, \ldots, a_s\}$ using the previous algorithm ($s > n$). The n first invariants are the ones $a_k = x_k^2 + y_k^2$.
- We find a Gröbner basis in terms of the a_k and u_k, v_k, eliminating the complex variables, to determine the relationships among the a_k. We also take into account the relationship $a_1 + \ldots + a_n = h$.
- The number of fundamental szygyies is $s - 2n + 2$, so that the orbit space has dimension $2(n-1)$ and a reduced Hamiltonian of $n-1$ degrees of freedom (after applying normal forms and truncating higher-order terms) lives in that space.

Vector Fields and Dynamics

- In order to express the normal form (written in the complex coordinates u_k, v_k) in terms of the a_i one performs the division algorithm for multivariate polynomials (using the Gröbner basis) and the remainder of the division yields the desired expression (a polynomial in terms of a_i). We express it by $\bar{H}$.
- The Poisson structure of the a_k, $k = 1, \ldots s$ is obtained using the division algorithm with respect to the Gröbner basis. One obtains $\{a_j, a_k\}$ as a polynomial in a_l.
- The associated vector field is computed through $\dot{a}_k = \{a_k, \bar{H}\}$.

Once the equations of motion are computed one can obtain relative equilibria, bifurcations and so on.

For $n = 3$ several examples have been carried out: $1 : 1 : 1$, $1 : 1 : -1$, $1 : 3 : 5$, etc.

Contents

Rigorous Results #1

- Equilibria on the orbit space with non-degenerate Morse function corresponds to (families) of periodic solutions.
- Circulation about these equilibria (e.g., the elliptic points) correspond to families of invariant n-tori of the full system.
- Bifurcations of relative equilibria (centre-saddle, Hamiltonian pitchfork, and others) correspond to bifurcations of periodic solutions of the full system.

But, do these families of periodic solutions and KAM tori really exist? Do these bifurcations of periodic solutions take place in the full system?

The answer is YES if some non-degenerate conditions are fulfilled.

Rigorous Results #2

On the *plateau* (regular points):
A critical point of $\bar{H}$ at $d \in L$ (i.e. $\partial\bar{H}/\partial y(d) = 0$) is *nondegenerate* if the Hessian at the critical point, $\partial^2\bar{H}/\partial y^2(d)$, is nonsingular. The linearization about the critical point is

$$\dot{v} = \bar{A}\,v = J\frac{\partial^2\bar{H}}{\partial y^2}(d)\,v.$$

Let the eigenvalues of $\bar{A}$ be $\nu_1, \ldots, \nu_{2n-2}$.

Theorem

If $\bar{H}$ has a nondegenerate critical point at d, then there are smooth functions $\tilde{d}(\varepsilon) = \tilde{d} + O(\varepsilon)$ and $T(\varepsilon) = 2\pi + O(\varepsilon)$ for ε small, and the solution of H through $\tilde{d}(\varepsilon)$ is $T(\varepsilon)$-periodic. The multipliers are
$1, 1, 1 + \varepsilon\,\nu_1 + O(\varepsilon^2), \ldots, 1 + \varepsilon\,\nu_{2n-2} + O(\varepsilon^2)$.

At *peaks* (singular points):

Theorem

Let d be a peak of the reduced space with frequency k_s and $z \in \Pi^{-1}(d)$. If k_j/k_s is not an integer for $j \neq s$ then the solution through z of the full system for $\varepsilon = 0$ is periodic with period $2\pi/k_s$ and characteristic multipliers

$$e^{\pm(k_1/k_s)\,2\pi i}, \ldots, e^{\pm(k_s/k_s)\,2\pi i}, \ldots, e^{\pm(k_n/k_s)\,2\pi i},$$

$e^{\pm(k_s/k_s)\,2\pi i} = \pm 1$ as one expects from a periodic solution of a Hamiltonian system, but all the others are not equal to $+1$.
For $\varepsilon > 0$ and small, the full system has an elliptic periodic solution near z of period $2\pi/k_s + O(\varepsilon)$ and characteristic multipliers

$$e^{\pm(k_1/k_s)\,2\pi i} + O(\varepsilon), \ldots, e^{+(k_s/k_s)\,2\pi i} = 1, e^{-(k_s/k_s)\,2\pi i} = 1, \ldots, e^{\pm(k_n/k_s)\,2\pi i} + O(\varepsilon).$$

Some Remarks

- For $n \geq 3$ there are other types of singular points, not just peaks. We have called them *ridges* and the theory is not fully understood yet.
- The approach is analytical and combines tools of dynamical systems theory with computer algebra.
- One can use the relative equilibria (possibly computed with higher-order normal forms) and reverse the normal-form transformation to obtain initial conditions for getting numerically the periodic solutions.
- Of course there are many other techniques (numerical and semianalytical) to get periodic and quasi-periodic solutions of resonant Hamiltonians, but this approach renders particularly useful when there are many parameters, bifurcations, and so on. The accurate computations of the periodic solutions and tori has to be achieved using these techniques, or within our approach, obtaining initial conditions that must be refined numerically.