A singular phenomenom: conjectures, experiments, and theorems

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Outline

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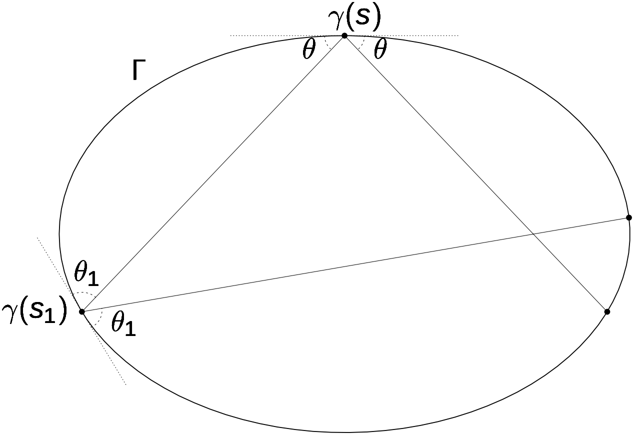
Convex billiards

- Let $\Gamma \subset \mathbf{R}^2$ be an analytic strictly convex closed curve.
- Let $\gamma(s)$ be an arc-length parametrization of Γ.
- Strictly convex means that the curvature $\kappa(s)$ is positive.
- We assume, without loss of generality, that the curve Γ has length one, so $s \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$.
- Let $\theta =$ angle of incidence/reflection $\in (0, \pi)$.
- Let $(s, \theta) \in \mathbf{T} \times (0, \pi)$ be so-called Birkhoff coordinates.
- The billiard map is

$$f : \mathbf{T} \times (0, \pi) \to \mathbf{T} \times (0, \pi), \qquad f(s, \theta) = (s_1, \theta_1).$$

Billiard map



Periodic billiard trajectories

A (p, q)-periodic billiard trajectory forms a closed polygon with q sides that makes p turns inside Γ.

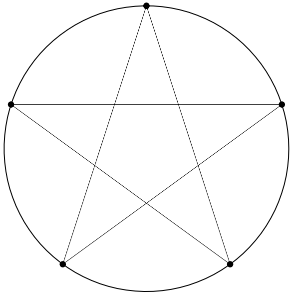


Figure: A $(2, 5)$-periodic (or $(3, 5)$-periodic) billiard trajectory.

Length differences

Periodic sets

$T^{(p,q)} =$ set of (p,q)-periodic billiard trajectories.

Theorem (Birkhoff)

$\#T^{(p,q)} \geq 2$ for any relatively prime integers $1 \leq q \leq p$.

Lenght differences

$\Delta^{(p,q)} = \sup \{\mathrm{Length}(T^{(p,q)})\} - \inf \{\mathrm{Length}(T^{(p,q)})\}$.

Ellipses and circles

Definition

Caustics are curves with the property that a billiard trajectory, once tangent to one, stays tangent after every reflection.

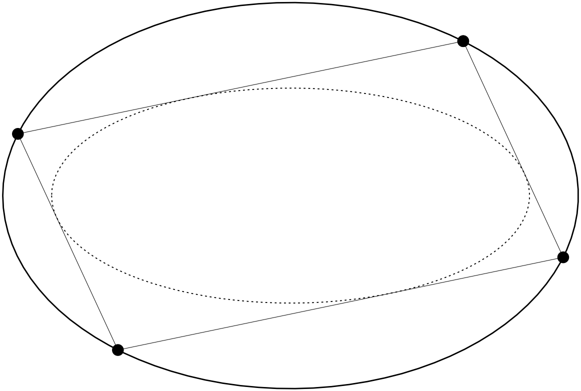
Theorem (Poncelet)

Any billiard trajectory inside an ellipse has a caustic which is a (maybe singular) confocal conic. If a billiard trajectory inside an ellipse is (p, q)-periodic, all the trajectories sharing its caustic are (p, q)-periodic and have the same length.

Corollary

If Γ is a circle, then $\Delta^{(p,q)} = 0$ for all $0 < p/q \leq 1/2$. If Γ is an ellipse, then $\Delta^{(p,q)} = 0$ for all $0 < p/q < 1/2$.

A $(1,4)$-periodic trajectory and its caustic



Action = Length

- If $r = -\cos\theta$ and $r_1 = -\cos\theta_1$, then the billiard map $f : \mathbf{T} \times (-1,1) \to \mathbf{T} \times (-1,1)$, $f(s,r) = (s_1, r_1)$, is an area-preserving twist map whose generating function is the distance between consecutive impact points. That is,

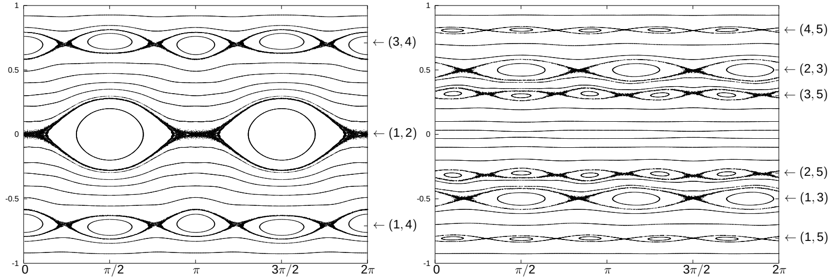
$$f(s,r) = (s_1, r_1) \Leftrightarrow \begin{cases} r = -\partial_1 h(s,s_1) \\ r_1 = \partial_2 h(s,s_1) \end{cases}$$

  where $h(s,s_1) = |\gamma(s) - \gamma(s_1)|$.
- Hence, the action of a (p,q)-periodic billiard orbit is equal to the length of its associated closed polygonal trajectory.

Dynamical interpretation of $\Delta^{(p,q)}$

MacKay-Meiss-Percival Action Principle (1984)

The action (length) difference $\Delta^{(p,q)}$ is equal to the flux along the (p,q)-resonance of the twist map $f(s,r) = (s_1, r_1)$.



Known result in the smooth case

Let L_q be the length of any $(1,q)$-periodic billiard trajectory.

Marvizi & Melrose (1982)

If the curve Γ is smooth and strictly convex, then

$$L_q \asymp \sum_{j \geq 0} \frac{c_j}{q^{2j}}, \qquad \text{as } q \to +\infty,$$

for some asymptotic coefficients $c_j = c_j(\Gamma) \in \mathbf{R}$. For instance,

- $c_0 = \mathrm{Length}(\Gamma) = \int_\Gamma ds > 0$, and
- $c_1 = -\frac{1}{24}\left(\int_\Gamma \kappa^{2/3} ds\right)^3 < 0$, where κ is the curvature of Γ.

Outline

A conjecture

The length difference $\Delta^{(1,q)}$ tends to zero as $q \to +\infty$ faster than any power. That is, $\Delta^{(1,q)}$ is beyond all orders in q.

General principle in conservative systems

If a dynamical quantity is beyond all orders in the smooth class, then it may be exponentially small in the analytic class.

Conjecture

If Γ is analytic and strictly convex, then there exists a constant $N = N(\Gamma) > 0$ and an exponent $r = r(\Gamma) > 0$ such that

$$\Delta^{(1,q)} \leq N \mathrm{e}^{-rq}, \qquad \text{as } q \to +\infty.$$

Outline

Model curves

Model curves

We consider the “perturbed” ellipses and circles

$$\Gamma = \Gamma_{b,\epsilon,n} = \left\{ (x,y) \in \mathbf{R}^2 : x^2 + y^2/b^2 + \epsilon y^n \leq 1 \right\}$$

for some semiaxis length $0 < b \leq 1$, some exponent $n \in \mathbf{N}$, and some perturbation strength $\epsilon \in \mathbf{R}$.

We shall take exponents $n \geq 3$. Otherwise $\Gamma_{b,\epsilon,n}$ is an ellipse and $\Delta^{(1,q)} = 0$ for all $q \geq 3$.

- If n is even, $\Gamma_{b,\epsilon,n}$ is symmetric with regard to both axis.
- If n is odd, $\Gamma_{b,\epsilon,n}$ is symmetric with regard to the y-axis.

On the perturbation strength ϵ

We can take a relatively big "perturbation", provided $\Gamma_{b,\epsilon,n}$ remains strictly convex.

- If n is even, $\Gamma_{b,\epsilon,n}$ is strictly convex for all $\epsilon > 0$.
- If n is odd, $\Gamma_{b,\epsilon,n}$ is strictly convex for $|\epsilon| < \epsilon_n$, where

$$\epsilon_n = \epsilon_n(b) = 2(n-2)^{n/2-1} n^{-n/2} b^{-n}$$

is the smallest positive value for which equation

$$y^2/b^2 + \epsilon y^n = 1$$

has a double root. For instance, $\epsilon_3(1) \simeq 0.384$, $\epsilon_5(1) \simeq 0.186$, and $\epsilon_7(1) \simeq 0.123$.

Axisymmetric periodic trajectories

- Once fixed an exponent $n \geq 3$ and a period $q \geq 3$, there are exactly two kinds of axisymmetric $(1, q)$-periodic billiard trajectories inside $\Gamma_{b,\epsilon,n}$.
- To compute them, we must deal with five scenarios:
  1. n even and $q = 2k + 1$ odd;
  2. n even and $q = 4k$ multiple of four;
  3. n even and $q = 4k + 2$ even but not multiple of four;
  4. n odd and $q = 2k + 1$ odd; and
  5. n odd and $q = 2k$ even.
- All the axisymmetric $(1, q)$-periodic billiard trajectories of the same kind have the same length.

Regular study versus singular study

- Let $D_q = D_q(\epsilon)$ be the signed difference between the lengths of the axisymmetric $(1, q)$-periodic trajectories inside $\Gamma_{b,\epsilon,n}$.
- We will prove later on that q is a singular parameter.
- On the contrary, the perturbation strength ϵ is a regular parameter. That is, $D_q(\epsilon)$ is analytic at $\epsilon = 0$. Besides, $D_q(0) = 0$ for all $q \geq 3$.
- Thus, we can carry out two different kind of studies:
  - *Singular:* To study $D_q(\epsilon)$ for $q \to +\infty$ and fixed $\epsilon \in \mathbf{R}$.
  - *Regular:* To study $D_q(\epsilon)$ for $\epsilon \to 0$ and fixed $q \geq 3$.

Result 1: Behavior of the asymptotic coefficients c_j

We consider the normalized coefficients $\hat{c}_j = c_j/(2j)!$, where $c_j = c_j(\Gamma_{b,\epsilon,n})$ are the asymptotic coefficients introduced in the Marvizi-Melrose theorem.

- If $0 < b \leq 1$, $3 \leq n \leq 8$ and $\Gamma_{b,\epsilon,n}$ is strictly convex, then:
  - The asymptotic series $\sum_{j\geq 0} c_j q^{-2j}$ is always divergent; and
  - Its Borel transform $\sum_{j\geq 0} \hat{c}_j z^{2j}$ has a positive radius of convergence $\rho = \rho(\Gamma_{b,\epsilon,n}) > 0$. Indeed,

$$\left|\hat{c}_j/\hat{c}_{j+1}\right|^{1/2} = \rho + \mathrm{O}(1/j) \qquad \text{as } j \to +\infty.$$

- If $b = 1$, then $\lim_{\epsilon\to 0} \rho = +\infty$.
- If $0 < b < 1$, then $\exists \lim_{\epsilon\to 0} \rho \in (0, +\infty)$.

Result 2: Singular behaviour of D_q

- Let D_q be the signed difference between the lengths of the axisymmetric $(1, q)$-periodic trajectories inside $\Gamma_{b,\epsilon,n}$.
- In general, $\Delta^{(1,q)} \geq |D_q|$. However, in most cases there are no more $(1, q)$-periodic trajectories, so $\Delta^{(1,q)} = |D_q|$.
- We consider the normalized quantities $\hat{D}_q = q^m \mathrm{e}^{rq} D_q$.
- If $0 < b < 1$, then:

q	r	m	Behavior of $\hat{D}_q$ as $q \to +\infty$
even	$\rho/2$	3	Tends to a non-zero constant
odd	ρ	2	Tends to another non-zero constant

The case of perturbed circles

If $b = 1$, then:

n	q	r	m	Behavior of $\hat{D}_q$ as $q \to +\infty$
4	even	$\rho/2$	3	Tends to a non-zero constant
4	odd	ρ	2	Tends to another non-zero constant
6	even	$\rho/2$	3	Oscillates periodically, mean $= 0$
6	odd	ρ	2	Oscillates periodically, mean $\neq 0$
even ≥ 8	even	$\rho/2$	3	Oscillates quasiperiodic., mean $= 0$
even ≥ 8	odd	ρ	2	Oscillates quasiperiodic., mean $\neq 0$
odd	any	$\rho/2$	3	Oscillates quasiperiodic., mean $= 0$

Questions (first round)

- Can we prove that the length differences $\Delta^{(1,q)}$ are exponentially small in q as $q \to +\infty$?
- Can we compute (or guess) “analytically” ρ and r?
- Why are ellipses and circles so different?

Outline

Dynamics close to the border

- The billiard map can be analytically extended to a complex neighborhood of the boundary $C_- = \mathbf{T} \times \{0\}$ of the phase space $\mathbf{T} \times (0, \pi)$ in the Birkhoff coordinates (s, θ).
- If $\varrho(s)$ is the radius of curvature of Γ, then the billiard map $(s, \theta) \mapsto (s_1, \theta_1)$ has the Taylor expansion

$$s_1 = s + \sum_{j \geq 1} \alpha_j(s) \theta^j, \qquad \theta_1 = \theta + \sum_{j \geq 2} \beta_j(s) \theta^j,$$

where $\alpha_1 = 2\varrho$, $\alpha_2 = 4\varrho\varrho'/3$, $\alpha_3 = 2\varrho^2\varrho''/3 + 4\varrho(\varrho')^2/9$, $\beta_2 = -2\varrho'/3$, $\beta_3 = 4(\varrho')^2/9 - 2\varrho\varrho''/3$, etcetera.

Lazutkin's coordinates

The billiard map has the form

$$x_1 = x + y + \mathrm{O}(y^3), \qquad y_1 = y + \mathrm{O}(y^4),$$

in the Lazutkin's coordinates (x, y) defined by

$$x = C \int_0^s \varrho^{-2/3}(t)\mathrm{d}t, \qquad y = 4C\varrho^{1/3}(s)\sin(\theta/2),$$

where the constant $C > 0$ is determined in such a way that x is an angular variable defined modulus one: $x \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$. That is,

$$C^{-1} = \int_\Gamma \varrho^{-2/3}\mathrm{d}s = -2\sqrt[3]{3c_1(\Gamma)}.$$

Theorem for the analytic case

Since the billiard map $f(x, y) = (x_1, y_1)$ is real analytic at $y = 0$, it can be extended to a complex domain of the form

$$D_{a_\star, b_\star} := \{(x, y) \in \mathbf{C}/\mathbf{Z} \times \mathbf{C} : |\Im x| < a_\star, |y| < b_\star\}.$$

Martín, Tamarit-Sariol & RRR (2014)

If Γ is analytic and strictly convex and $\alpha \in (0, 2\pi a_\star)$, then there exist $N, q_\star > 0$ such that

$$\Delta^{(1,q)} \leq N\mathrm{e}^{-\alpha q}, \qquad \forall q \geq q_\star.$$

The constant N may explode when $\alpha \to 2\pi a_\star$, so, in general, we can not take $\alpha = 2\pi a_\star$.

Questions (second round)

- Can we compute $a_\star = a_\star(\Gamma)$ in some case?
  Guess: The distance δ of the set of singularities and zeros of $\varrho(x)$ to the real axis may be related to the quantity $a_\star$.
- Is there some relation between the quantity $a_\star$, the distance δ, the radius of convergence ρ of the Borel transforms, and the exponent r of the exponentially small formulas computed in the numerical experiments?
  Guess: Sometimes $\rho = 4\pi a_\star = 4\pi\delta$ and $r \in \{\rho/2, \rho\}$.
- Can we find (and prove) an asymptotic exponentially small formula in q for $\Delta^{(1,q)}$ in some case?

Let $\delta = \delta_\epsilon = \delta(\Gamma_{b,\epsilon,n}) > 0$ be the distance of the set of singularities and zeros of the function $\varrho(x)$ to the real axis.

Circles

If $b = 1$ and $n \geq 3$, then there exists a constant $\eta_n \in \mathbf{R}$ such that

$$4\pi\delta_\epsilon = 2|\log\epsilon|/n + \eta_n + \mathrm{o}(1), \qquad \text{as } \epsilon \to 0.$$

Ellipses

Let $K(m) = \int_0^1 (1 - mt^2)^{-1/2}(1 - t^2)^{-1/2} dt$ be the complete elliptic integral of the first kind. If $0 < b < 1$ and $n \geq 3$, then

$$4\pi\delta_\epsilon = \pi\frac{K(b^2)}{K(1 - b^2)} + \mathrm{O}(\epsilon), \qquad \text{as } \epsilon \to 0.$$

Outline

On small perturbations of circles

- Once fixed the exponent $n \geq 3$ and the period $q \geq 3$, there exists some order $k = k(n,q) \in \mathbf{N}$ and some coefficient $d = d(n,q) \neq 0$ such that

$$D_q(\epsilon) = d\epsilon^k + \mathrm{O}(\epsilon^{k+1}).$$

- If we set $\rho = 2|\log \epsilon|/n + \mathrm{O}(1)$ and fix $q \geq 3$, then

$$\mathrm{e}^{-\rho q/2} = \mathrm{O}(\epsilon^{q/n}), \quad \mathrm{e}^{-\rho q} = \mathrm{O}(\epsilon^{2q/n}), \qquad \text{as } \epsilon \to 0.$$

- We have numerically checked that:
  - $|k - q/n| \leq 1$ when n is odd or when both n and q are even;
  - $|k - 2q/n| \leq 1$ when n is even and q is odd.

Outline

Details on the numerical computations (I)

- If $b = 1$, $\epsilon = 1$, $n = 4$, and $q = 201$, then the lengths of the two kinds of axisymmetric $(1, q)$-periodic trajectories are

$$\begin{aligned} L_q &\approx 5.76862620243699331641976071893775948656991580653133826425\textcolor{red}{7503}, \\ l_q &\approx 5.76862620243699331641976071893775948656991580653133826425\textcolor{red}{5034}. \end{aligned}$$

- Therefore, 57 decimal digits are lost when we compute

$$D_q = L_q - l_q \approx 2.4690108977323491947687904121408 \times 10^{-57}.$$

- The above cancellation is beyond single, double, and quadruple precisions. The use of a multiple precision arithmetic (MPA) is mandatory. We have implemented the MPA provided by the PARI/GP system.

Details on the numerical computations (II)

- All nonlinear equations are solved with Newton's method in variable precision. We double the precision after each Newton's iteration.
- We have computed the first 500 coefficients of the asymptotic series $\sum_{j\geq 0} c_j q^{-2j}$ in several cases.
- We have also computed the differences D_q for periods $q \leq 24000$ or $q \leq 12000$, depending on the scenario.
- If $b = 1$, $\epsilon = 1$, $n = 4$, $q = 12001$, and the MPA uses 4000 digits, the computation of $D_q \approx 4.2112173963 \times 10^{-3148}$ takes 131 seconds in my office desktop.