The center-focus problem in piecewise systems

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http://www.gsd.uab.cat

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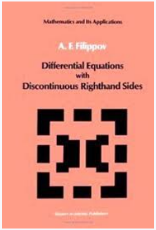
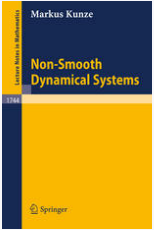
Piecewise differential systems

- A. F. Filippov, *Differential equations with discontinuous righthand sides*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers–Dordrecht, 1988.



- M. Kunze, *Non-smooth dynamical systems*. Lecture Notes in Mathematics, 1744. Springer-Verlag, Berlin, 2000.



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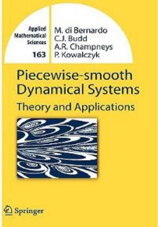
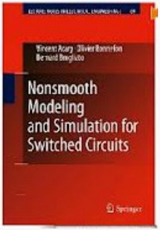
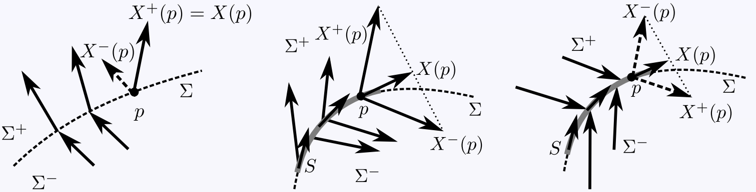
- M. di Bernardo, C. J. Budd, A. R. Champneys & P. Kowalczyk, *Piecewise-smooth dynamical systems. Theory and applications*, Applied Mathematical Sciences, 163. Springer-Verlag London, Ltd., London, 2008.

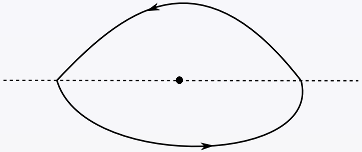


- V. Acary, O. Bonnefon & B. Brogliato, *Nonsmooth modeling and simulation for switched circuits*. Lecture Notes in Electrical Engineering, 69. Springer, Dordrecht, 2011.



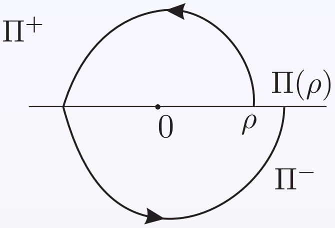


sewing scaping sliding

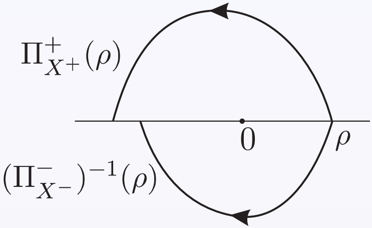


Sewing (crossing) limit cycle

Return map / Difference map



Return (Composition) map



Difference map

Periodic orbits crossing Σ

$$\Updownarrow$$

$$\{\Pi(\rho) = \rho\} = \{\Pi^+_{X^+}(\rho) - (\Pi^-_{X^-})^{-1}(\rho) = 0\}$$

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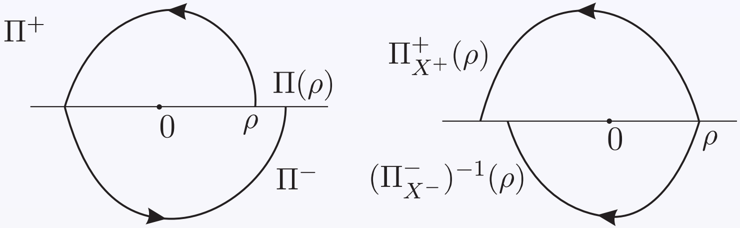

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The center-focus problem and related problems

Definition

If $V_n \neq 0$ and

$$\Pi(\rho) - \rho = V_n \rho^n + O(\rho^{n+1})$$

for $\rho > 0$ close to zero, then V_n is called the n-th Lyapunov constant.

Related Problems

- *Characterization of centers*
- *Maximum order of a weak focus*
- *Cyclicity*

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Lyapunov constants (Smooth case)

For differential systems, a singular point is of center-focus type if $\mathrm{tr} DX(x_0) = 0$ and $\det DX(x_0) < 0$. Then after a translation and a change of time the system writes as:

$$(x', y') = (-y + P(x, y), x + Q(x, y))$$

and, in complex coordinates ($z = x + iy$),

$$z' = i\,z + \sum_{k+\ell=m} r_{k,\ell} z^k \bar{z}^\ell.$$

- $V_{2n} = 0$ for all n.
- Quasihomogeneity and zero weight: $V_{2n+1}(\lambda^{-k+\ell+1} r_{k,\ell}, \lambda^{k-\ell-1} \bar{r}_{k,\ell}) = V_{2n+1}(r_{k,\ell}, \bar{r}_{k,\ell})$.
- Quasihomogeneity and quasidegree: $V_{2n+1}(\lambda^{k+\ell-1} r_{k,\ell}, \lambda^{k+\ell-1} \bar{r}_{k,\ell}) = \lambda^{2n} V_{2n+1}(r_{k,\ell}, \bar{r}_{k,\ell})$.
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First Lyapunov constants

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\begin{aligned}
V_3 =& \operatorname{Re}(r_{2,1}) - \operatorname{Im}(r_{2,0} r_{1,1}).\\
V_5 =& \operatorname{Re}(r_{3,2}) + \frac{1}{3}\operatorname{Im}(-\bar{r}_{1,3} r_{0,2} - 3\bar{r}_{2,0} r_{3,1} - 3\bar{r}_{2,2} r_{1,1} - 4 r_{0,2} r_{4,0}\\
& - 6 r_{1,1} r_{3,1} - 3 r_{1,2} r_{3,0}) + \frac{1}{3}\operatorname{Re}(2\bar{r}_{0,2} r_{0,3} r_{2,0} + 3\bar{r}_{0,2} r_{1,1} r_{1,2}\\
& + \bar{r}_{0,3} r_{0,2} r_{1,1} + 5\bar{r}_{1,1} r_{0,2} r_{3,0} - 15\bar{r}_{1,1} r_{1,1} r_{2,1} + 3\bar{r}_{1,1} r_{1,2} r_{2,0}\\
& + 2\bar{r}_{1,2} r_{0,2} r_{2,0} - 3\bar{r}_{2,0} r_{1,1} r_{3,0} - 30\bar{r}_{2,0} r_{2,0} r_{2,1} - 21\bar{r}_{2,1} r_{1,1} r_{2,0}\\
& - 2 r_{0,2} r_{2,0} r_{3,0} - 6 r_{1,1}^2 r_{3,0} - 24 r_{1,1} r_{2,0} r_{2,1})\\
& + \frac{1}{3}\operatorname{Im}(4\bar{r}_{0,2}\bar{r}_{1,1}\bar{r}_{2,0} r_{0,2} - 2\bar{r}_{0,2} r_{1,1}^3 + 3\bar{r}_{1,1}^2 r_{0,2} r_{2,0} - 2\bar{r}_{1,1} r_{0,2} r_{2,0}^2\\
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Number of monomials $N_3 = 4$, $N_5 = 54$, $N_7 = 526(0.2s)$,
$N_9 = 3800(9s)$, $N_{11} = 23442(14m)$,

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Centers, weak-foci and cyclicity

Center Problem

$$\{\Pi(\rho) \equiv \rho\} \Leftrightarrow \{V_3 = 0, V_5 = 0, \ldots, V_{2n+1} = 0, \ldots\}$$

General Problems / Family Problems

- Finiteness problem $\Leftrightarrow$ Hilbert's Basis Theorem
- Computational difficulties:
  - Explicit computation
  - Solution of polynomial system of equations of high degree
  - Radicality
  - $\mathbb{R}$ versus $\mathbb{C}$
- Why is it a center? (First integral, Hamiltonian, Darboux, reversible, symmetry,...)

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Order of a weak-focus

For a given family of polynomial vector fields which is the highest value of n such that $\Pi(\rho)-\rho = V_n\rho^n + O(\rho^{n+1})$?

General Problems / Family Problems

Problem (Gasull-Giné-Torregrosa 2014)

There exists c such that the origin of

$$z' = i\,z + z^{2d+1} + c\,z^{2d}\bar{z}$$

is a weak-focus of order $k = 2d^2+3d$ $(V_{2k+1}\neq 0)$.

True up to $d=44$. (2 days of CPU time).

$V_3 = V_5 = \ldots = V_{7831} = 0,\ V_{7833} = D_1(E_1 c\bar{c} - E_2)(c^{44}+\bar{c}^{44})\pi,$
$V_{7835} = \ldots = V_{8007} = 0,\ V_{8009} = -D_2(c^{44}+\bar{c}^{44})\pi.$
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For a given family of polynomial vector fields which is the maximum number of limit cycles that bifurcate from a singular point?

General Problems / Family Problems

Theorem

For a general system, the number of limit cycles that bifurcate from a weak-focus of order k $(V_{2k+1} \neq 0)$ *is k.*

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Maximum order of a weak-focus and cyclicity problem

Theorem

Consider a one-parameter family of differential systems of the form

$$\begin{cases} x' = -y + a^k x(x^2+y^2) + aP(x,y,a), \\ y' = -x + a^k y(x^2+y^2) + aQ(x,y,a), \end{cases}$$

where P and Q are analytic functions, starting at least with terms of degree 4 in x and y, and $k \geq 1$ is an integer number. Then:

- *The first Lyapunov constant is $V_3 = 2\pi a^k$ and the origin is a center if and only $a = 0$.*
- *The cyclicity of the origin is at most $k-1$ and there are analytic functions, P and Q, for which this upper bound is sharp.*

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Smooth / nonsmooth

- Tangential points, real and virtual singular points,...
- $V_k \neq 0$ for every k (in general)
- Polynomial in parameters and exp (trace).

If the origin is a singular point of focus-focus type, and for a system that writes

$$z' = (i + \lambda^{\pm})z + \dots$$

in $y > 0$ ($y < 0$), we have

$$V_1 = e^{\pi(\lambda^{+}+\lambda^{-})} - 1.$$

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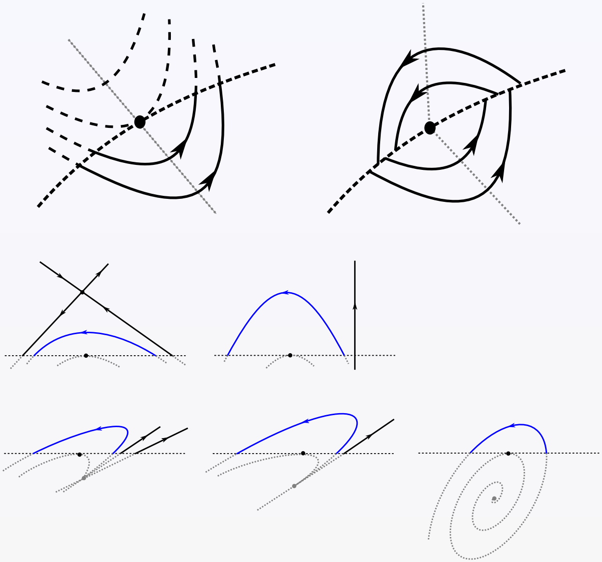
Tangential points



Centers in piecewise linear systems without sliding

Proposition

If $\mu_0^{\pm} \neq 0$, *the origin of system*

$$(x', y') = \begin{cases} (\mu_0^+ + \mu_1^+ x + \mu_2^+ y, x) & \text{if } y > 0, \\ (\mu_0^- + \mu_1^- x + \mu_2^- y, x) & \text{if } y < 0 \end{cases}$$

is a center if and only if

$$\mu_1^- \mu_0^+ - \mu_1^+ \mu_0^- = \mu_1^+ (\mu_2^+ (\mu_0^-)^2 - \mu_2^- (\mu_0^+)^2) = 0.$$

Proof.

$$V_1 = 0, \qquad V_2 = \frac{2}{3} \frac{\mu_1^- \mu_0^+ - \mu_1^+ \mu_0^-}{\mu_0^- \mu_0^+}, \qquad V_3 = 0,$$

$$V_4 = \frac{2}{15} \frac{\mu_1^+}{(\mu_0^-)^2 (\mu_0^+)^3} (-\mu_2^+ (\mu_0^-)^2 + \mu_2^- (\mu_0^+)^2), \text{ when } V_2 = 0.$$

If $V_2 = V_4 = 0$ the system is reversible ($(x, y, t) \to (x, -y, -t)$). □

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If $V_2 = V_4 = 0$ the system is reversible $((x, y, t) \to (x, -y, -t))$. □

Proposition

When $\mu_0^{\pm} \neq 0$, *the maximal cyclicity of the origin of the vector field*

$$(x', y') = \begin{cases} (\mu_0^+ + \mu_1^+ x + \mu_2^+ y, x) & \text{if } y > 0, \\ (\mu_0^- + \mu_1^- x + \mu_2^- y, x) & \text{if } y < 0 \end{cases}$$

is one.

J. C. Medrado & J. Torregrosa "Uniqueness of limit cycles for sewing planar piecewise linear systems". *Preprint*.

A homogeneous quadratic/center family: Centers

Proposition

Consider the system

$$\begin{cases} x' = -y + p_{20}x^2 + p_{11}xy + p_{02}y^2, \\ y' = x + q_{20}x^2 + q_{11}xy + q_{02}y^2, \\ y > 0, \end{cases} \qquad \begin{cases} x' = -y, \\ y' = x, \\ y < 0. \end{cases}$$

Then, it has a center at the origin if and only if one of the following conditions holds:

(i) $p_{11} = q_{20} = q_{02} = 0$,

(ii) $p_{20} = p_{11} + q_{20} = p_{02} + q_{11} = q_{02} = 0$,

(iii) $2p_{20} + q_{11} = p_{11} + 2q_{02} = q_{20} = 0$,

(iv) $p_{20} = -p_{11} + q_{20} = q_{02} + q_{20} = p_{02} = 0$,

(v) $2p_{11}q_{20} + 3p_{20}^2 - 2q_{20}^2 = 2q_{11} + 5p_{20} = 8p_{02}q_{20}^2 - 3p_{20}^2 + 8q_{20}^2 = 4q_{02}q_{20} - 3p_{20}^2 + 4q_{20}^2 = 0$.

A. Gasull & J. Torregrosa, "Center-focus problem for discontinuous planar differential equations". *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **13** (2003), 1755–1765.

Proof: 1. Lyapunov constants

$$
\begin{aligned}
V_1 =& 0,\\
V_2 =& \frac{2}{3}(p_{11} + q_{20} + 2q_{02}),\\
V_3 =& -\frac{\pi}{8}(2p_{20}q_{02} + q_{02}q_{11} + 3p_{20}q_{20} + q_{11}q_{20} + p_{02}q_{20}),\\
V_4 =& \frac{1}{15}(2q_{20}^3 - 2p_{11}^2 q20 - 18p_{20}^2 q_{20} + 6p_{11}p_{20}q_{11} + 12p_{11}p_{20}^2\\
& - 6q_{11}p_{20}q_{20}),\\
V_5 =& \frac{\pi}{64}q_{20}p_{20}(p_{20}^2 - 2q_{11}p_{20} + 4p_{11}q_{20} - 4q_{20}^2),\\
V_6 =& \frac{8}{105}q_{20}(p_{11} - q_{20})(p_{11} + q_{20})(-5p_{11}q_{20} + 5q_{20}^2 + 3q_{11}p_{20}).
\end{aligned}
$$

Solving the system $\{V_2 = V_3 = V_4 = V_5 = V_6 = 0\}$ we obtain the families of the statement.

Proof: 2. Why are they centers?

The first family is invariant with respect the change

$$(x, y, t) \to (-x, y, -t).$$

The other families satisfy $H_i(x, 0) = H_i(-x, 0)$ where $H_i = H_i(x, y)$ are their first integrals:

$$\begin{aligned}
H_2 =& x^2 + y^2, \\
H_3 =& \frac{1}{2}(x^2 + y^2) + \frac{q_{11}}{2}x^2 y + q_{02}xy^2 - \frac{p_{02}}{3}y^3, \\
H_4 =& (q_{20}x - 1)\left(q_{20}x + \frac{(q_{11} - \gamma)}{2}y + 1\right)^{\alpha}\left(q_{20}x + \frac{(q_{11} + \gamma)}{2}y + 1\right)^{(1-\alpha)}, \\
H_5 =& (-2q_{20}x + p_{20}y + 2)^2\Big(4(q_{20}x + 1)^2 - (4p_{20} + 12p_{20}q_{20}x)y \\
& + (3p_{20}^2 - 8q_{20}^2)y^2\Big),
\end{aligned}$$

with $\alpha = 4q_{20}^2(\gamma(\gamma + q_{11}))^{-1}$ and $\gamma = \sqrt{q_{11}^2 + 8q_{20}^2}$.

A homogeneous quadratic/center family: Cyclicity

Proposition

Consider the system

$$(x',y') = \begin{cases} (-y+w_1x+x^2+p_{11}xy+p_{02}y^2, & \\ \quad x+w_1y+x^2+q_{11}xy+q_{02}y^2) & \text{if } y>0, \\ (-y,x) & \text{if } y<0, \end{cases} \qquad (1)$$

where $p_{11} = \frac{7}{5} + \alpha$, $p_{02} = -\frac{17}{50} + \frac{3}{20}\alpha - \frac{99}{40}w_2 + \frac{32}{25}w_5 + \frac{16}{5}\alpha w_5 + \frac{3}{2}w_4 - \frac{3}{2}\alpha w_2 + 24w_2w_5 - 8w_3$, $q_{11} = \frac{13}{10} + 2\alpha - 32w_3$, *and* $q_{02} = -\frac{6}{5} - \frac{1}{2}\alpha + \frac{3}{4}w_2$, *being* $\alpha = \alpha(w_4, w_5)$ *the solution of the quadratic equation* $50\alpha^2 + (-960w_5 + 95)\alpha - 75w_4 - 384w_5 = 0$, *such that* $\alpha(0,0) = 0$. *Then, if we choose* w_1, w_2, w_3, w_4, *and* w_5 *such that* $w_1 < 0$, $w_2 > 0$, $w_3 < 0$, $w_4 > 0$, $w_5 < 0$ *and* $|w_1| \ll |w_2| \ll |w_3| \ll |w_4| \ll |w_5| \ll 1$, *the system has five small amplitude limit cycles.*

Proof

If $w_1 = 0$, from the Lyapunov constants, we get that $V_i = w_i$ for $i = 2, 3, 4, 5$. Hence the return map close to the origin is

$$\begin{aligned}
\Pi(\rho, w_1, w_2, w_3, w_4, w_5) &= e^{w_1 \pi} \rho \\
&+ (w_2 + f_2(w_1, w_2, w_3, w_4, w_5))\rho^2 \\
&+ (w_3 + f_3(w_1, w_2, w_3, w_4, w_5))\rho^3 \\
&+ (w_4 + f_4(w_1, w_2, w_3, w_4, w_5))\rho^4 \\
&+ (w_5 + f_5(w_1, w_2, w_3, w_4, w_5))\rho^5 \\
&+ (\frac{608}{4375} + f_6(w_1, w_2, w_3, w_4, w_5))\rho^6 + O(\rho^7),
\end{aligned}$$

where f_i, $i = 2, \ldots, 6$, are continuous functions satisfying $f_2(0, w_2, w_3, w_4, w_5) \equiv 0$, $f_3(0, 0, w_3, w_4, w_5) \equiv 0$, $f_4(0, 0, 0, w_4, w_5) \equiv 0$, $f_5(0, 0, 0, 0, w_5) \equiv 0$ and $f_6(0, 0, 0, 0, 0) \equiv 0$. Choosing the parameters adequately the function $\Pi(\rho) - \rho$ changes sign six times and Π has at least five fix points.

Centers in discontinuous Liénard

Consider the Liénard systems

$$(x', y') = \begin{cases} \left(-y + \sum_{i=2}^{n} a_i x^i, x\right) & \text{if } y > 0, \\ \left(-y + \sum_{i=2}^{n} b_i x^i, x\right) & \text{if } y < 0. \end{cases}$$

If

(i) $a_{2k+1} = b_{2k+1} = 0$ or

(ii) $a_k + b_k = 0$

for all $k \in \mathbb{N}$, the system has a center at the origin.

B. Coll, R. Prohens & A. Gasull. "The center problem for discontinuous Liénard differential equation." *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9** (1999), no. 9, 1751–1761.

If for the particular systems

$$(x', y') = \begin{cases} \left(-y + x^{2j+1} + x^{2(k-j)}, x\right) & \text{if } y > 0, \\ \left(-y - x^{2j+1}, x\right) & \text{if } y < 0, \end{cases}$$

for $1 \leq j < k$, the Lyapunov constant $V_{2k} = C_{k,j}$ is not zero, then the above two families are the only centers for the Liénard discontinuous system.