Manifolds on the verge of a hyperbolicity breakdown

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Introduction

Persistence of invariant manifolds

- **The long term behavior of dynamical systems is organized by the invariant objects.**
- **It is important to understand which invariant objects persist under modifications of the system.**
- An invariant manifold persists under perturbations if and only if it is normally hyperbolic.
  [HirschP69][Fenichel71][Mañé78]
- There are spectral characterizations of hyperbolicity.
  [Mather68][HirschPS77][Swanson83]

Introduction

Continuation of invariant tori, up to breakdown

In this talk:

- Continuation of invariant tori with respect to parameters.
- Two cases:
  - Continuation regardless the internal dynamics;
  - Continuation fixing the internal dynamics to be a rotation with a prescribed (Diophantine) frequency: need of adjustment of parameters.
- Different mechanisms of breakdown.
- Quantitative laws. (Empirically conjectured scaling properties.)
- Open questions

Normally hyperbolic invariant tori

Invariant tori

Invariance equations

- Let us consider a family of maps

$$F_{a,\varepsilon} : \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{T}^d \times \mathbb{R}^n,$$

inducing a family of dynamical systems in $\mathcal{A} = \mathbb{T}^d \times \mathbb{R}^n$, where:
  - $a \in \mathbb{R}^d$ is an adjusting parameter,
  - $\varepsilon \in \mathbb{R}$ is a perturbation parameter.
- For a, ε fixed, a solution (K, f) of the equation

$$F_{a,\varepsilon}(K(\theta)) = K(f(\theta)),$$

where $K : \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{R}^n$ and $f : \mathbb{T}^d \to \mathbb{T}^d$, gives an invariant torus $\mathcal{K} = \{K(\theta) \mid \theta \in \mathbb{T}^d\}$ for $F_{a,\varepsilon}$, whose dynamics is given by the map f.
- Let $\omega \in \mathbb{R}^d$ be a (Diophantine) frequency vector. For ε fixed, a solution (K, a) of the equation

$$F_{a,\varepsilon}(K(\theta)) = K(\theta + \omega),$$

where $K : \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{R}^n$ and $a \in \mathbb{R}^d$, gives an invariant torus $\mathcal{K}$ for $F_{a,\varepsilon}$ whose dynamics is the rotation by ω, i.e. $f = R_\omega$.

Invariant tori

Linearization

The linearization around the torus $K : \mathbb{T}^d \to \mathbb{T}^d \times \mathbb{R}^n$,

$$M(\theta) = \mathrm{D}F_{a,\varepsilon}(K(\theta)),$$

induces:

- a linear skew product $(M, f) : \mathbb{R}^{d+n} \times \mathbb{T}^d \to \mathbb{R}^{d+n} \times \mathbb{T}^d$,

$$\begin{cases} \bar{v} = M(\theta) v \\ \bar{\theta} = f(\theta) \end{cases} ;$$

- a transfer operator $\mathcal{M}$ acting on bounded sections $\mathcal{v} : \mathbb{T}^d \to \mathbb{C}^{d+n}$ by

$$\mathcal{M}\mathcal{v}(\theta) = M(f^{-1}(\theta))\mathcal{v}(f^{-1}(\theta)) \ .$$

The functional analysis properties (6) *are closely related to the dynamical properties of* (6).

Mather, Sacker, Sell, Palmer, Hirsch, Pugh, Shub, Mañé, Chicone, Swanson, Johnson, Latushkin, Stëpin, de la Llave, ...

Invariant tori

Spectral Theory

- If the set of aperiodic orbits of f is dense, then the spectrum of $\mathcal{M}$ is a finite union of spectral annuli.
- Each spectral set of $\mathcal{M}$ induces an invariant subbundle of (M, f), characterized by rates of growth.
- If the internal dynamics is an ergodic rotation, then the spectrum acting on C^r sections is the same spectrum as acting on bounded sections, and then it is also rotationally invariant. (with Rafael de la Llave)

Normal hyperbolicity

A functional analytic definition

The invariant manifold $\mathcal{K}$ is normally hyperbolic if the spectrum of the transfer operator has three spectral components:

- the central component, which corresponds to the tangent bundle;
- the stable component, inside the unit circle, producing the stable bundle;
- the unstable component, outside the unit circle, producing the unstable bundle.

If there is not unstable component, then the torus is a (uniform) attractor, and if there is not stable component, then the torus is a (uniform) repeller.

Normal hyperbolicity

A dynamical definition

The invariant manifold $\mathcal{K}$ is normally hyperbolic if the bundle $\mathrm{T}_{\mathcal{K}}\mathcal{A}$ splits into three continuous invariant subbundles

$$\mathrm{T}_{\mathcal{K}}\mathcal{A} = \mathrm{N}^s\mathcal{K} \oplus \mathrm{T}\mathcal{K} \oplus \mathrm{N}^u\mathcal{K}$$

that are characterized by the spectral gap conditions

$$0 < \rho_s < \rho_{L,-} \leq 1 \leq \rho_{L,+} < \rho_u,$$

and a constant $C > 0$, such that the following uniform rates of growth hold:

$$(v,\theta) \in \mathrm{N}^s\mathcal{K} \Leftrightarrow \forall k > 0, |(M,f)^k(v,\theta)| \leq C\rho_s^k|(v,\theta)|,$$
$$(v,\theta) \in \mathrm{N}^u\mathcal{K} \Leftrightarrow \forall k < 0, |(M,f)^k(v,\theta)| \leq C\rho_u^k|(v,\theta)|,$$
$$(v,\theta) \in \mathrm{T}\mathcal{K} \;\Leftrightarrow \begin{cases} \forall k < 0, |(M,f)^k(v,\theta)| \leq C\rho_{L,-}^k|(v,\theta)| \\ \forall k > 0, |(M,f)^k(v,\theta)| \leq C\rho_{L,+}^k|(v,\theta)|. \end{cases}$$

If $0 < \rho_s < \rho_{L,-}^r \leq 1 \leq \rho_{L,+}^r < \rho_u$ the invariant manifold is of class C^r.

Normal hyperbolicity

Observables

In order to monitor the normal hyperbolicity properties, we may use:

- for regularity properties: C^r norms, Sobolev norms;
- for hyperbolicity properties: Lyapunov multipliers;
- for splitting and uniformity properties: angles between bundles.

The observables are very related.

They let us to distinguish different types of bifurcations and breakdowns.

Continuation across resonances

A 2D-fattened Arnold family

The equations of the model

We consider the following 2D-fattened Arnold family in $\mathbb{T} \times \mathbb{R}$:

$$\begin{cases} \bar{x} = x + a + \frac{\varepsilon}{2\pi}(\sin(2\pi x) + y) & \pmod 1 \\ \bar{y} = b(\sin(2\pi x) + y) \end{cases}$$

where

- b is the dissipative parameter ($b = 0.3$);
- a is an adjusting parameter;
- ε is the perturbation parameter ($\varepsilon = 0.5$).

[Broer, Simó, Tatjer 98]

We will continue with respect to parameter a an attracting invariant torus.

(with Marta Canadell)

A 2D-fattened Arnold family

Continuation of an attracting invariant curve

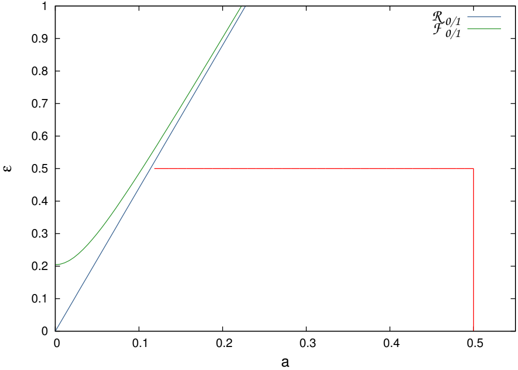
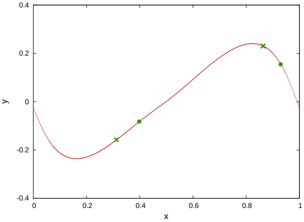


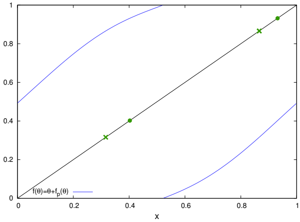
Figure: Continuation path (in red), and regions $\mathcal{R}_{0/1}$ and $\mathcal{F}_{0/1}$ for $b = 0.3$ in the 2D-FAF.

A 2D-fattened Arnold family ($\varepsilon = 0.5$)

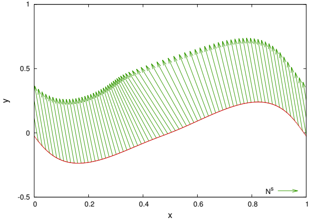
$a = 0.4904$: 1/2 resonance



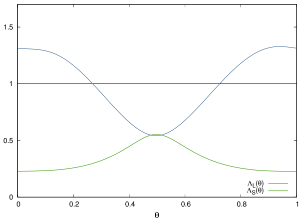
(a) Invariant curve



(b) Internal dynamics



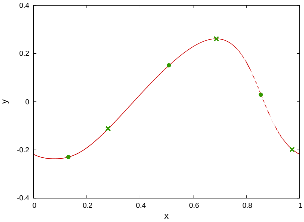
(c) Stable bundle



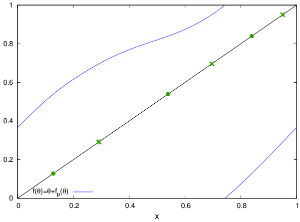
(d) Linearized dynamics

A 2D-fattened Arnold family ($\varepsilon = 0.5$)

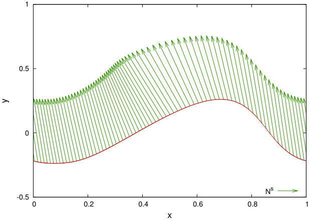
$a = 0.3377$: 1/3 resonance



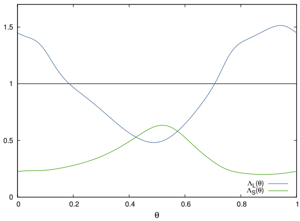
(a) Invariant curve



(b) Internal dynamics



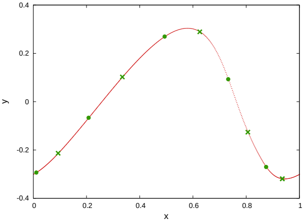
(c) Stable bundle



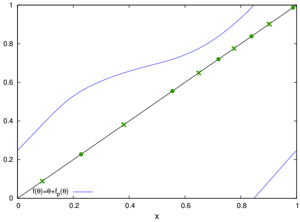
(d) Linearized dynamics

A 2D-fattened Arnold family ($\varepsilon = 0.5$)

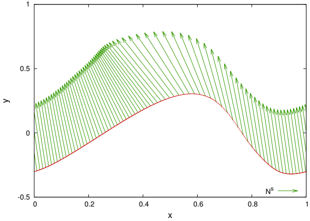
$a = 0.2142$: 1/5 resonance



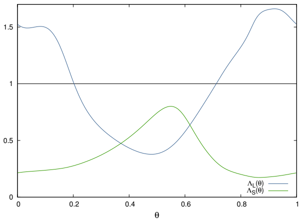
(a) Invariant curve



(b) Internal dynamics



(c) Stable bundle



(d) Linearized dynamics

A 2D-fattened Arnold family

Devil staircase

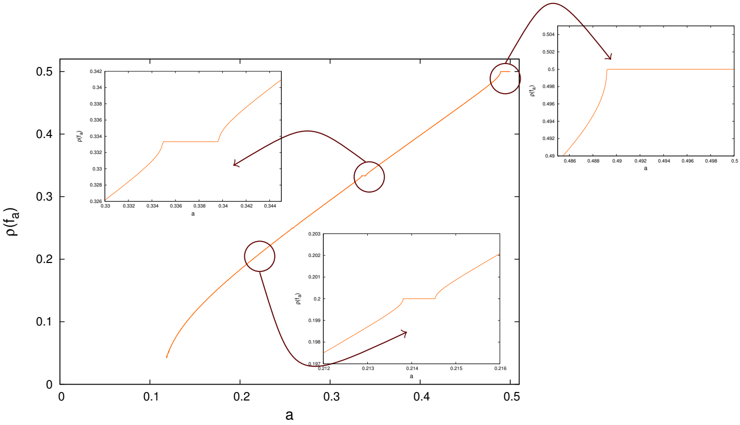


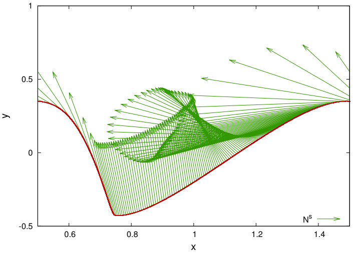
Figure: Rotation number for the internal dynamics of 2D-FAF, for $b = 0.3$ and $\varepsilon = 0.5$ fixed. Magnifications near the resonances $\rho(f_a) = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}$.

A 2D-fattened Arnold family

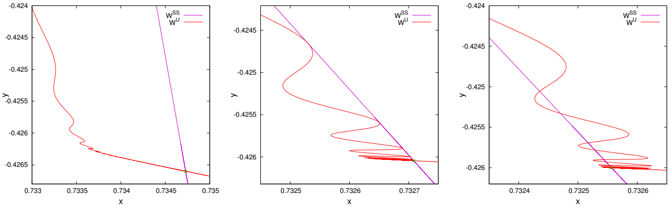
Breakdown: homoclinic bifurcation

A cascade of phenomena occurs after the last computed torus, leading to its destruction [AfraimovichS83,DiazLR01,BroerST98,GonchenkoSV13]:

- a saddle bifurcation producing a saddle and an attracting node;
- a quadratic tangency of the strong stable manifold of the node with the unstable manifold of the saddle.



$$a_{\mathrm{last}} \simeq 0.1162 > a_{\mathrm{sn}} \simeq 0.1137 > a_{\mathrm{ct}} > a_{\mathrm{qt}} \simeq 0.1130 > a_{\mathrm{nf}} \simeq 0.1037$$



Continuation inside a resonance

A 3D-fattened Arnold family

The equations of the model

We consider the following 3D-fattened Arnold family in $\mathbb{T} \times \mathbb{R}^2$:

$$\begin{cases} \bar{x} = x + a + \dfrac{\varepsilon}{2\pi}(\sin(2\pi x) + y + z/2) \qquad \pmod 1 \\ \bar{y} = b(\sin(2\pi x) + y) \\ \bar{z} = c(\sin(2\pi x) + y + z) \end{cases}$$

where

- b, c are fixed parameters ($b = 0.3$, $c = 2.4$, hence $bc = 0.72$);
- a is an adjusting parameter;
- ε is the perturbation parameter ($\varepsilon = 0.5$).

[Broer, Osinga, Vegter 97], [Broer, Hagen, Vegter 07]

We fix $a = 0.1/(2\pi)$, and continue with respect to parameter ε a saddle invariant torus.

(with Marta Canadell)

A 3D-fattened Arnold family

Continuations of saddle invariant tori

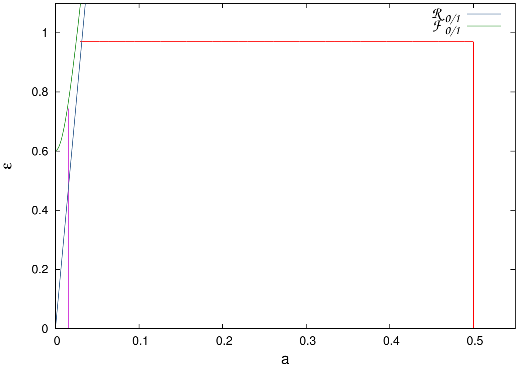
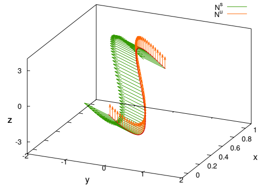


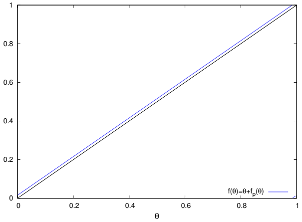
Figure: Continuation paths (in red and purple) and regions $\mathcal{R}_{0/1}$ and $\mathcal{F}_{0/1}$ for $b = 0.3$ and $c = 2.4$ in the 3D-FAF. In the following, we focus in the purple line.

A 3D-fattened Arnold family ($a = 0.1/(2\pi)$)

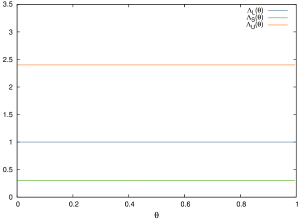
$\varepsilon = 0$: initial torus



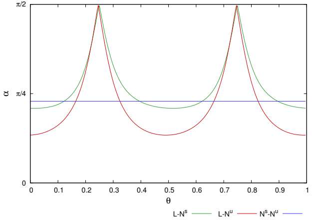
(a) Invariant curve and bundles



(b) Internal dynamics



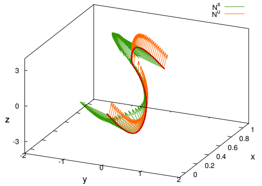
(c) Angles between bundles



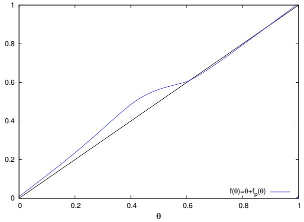
(d) Linearized dynamics

A 3D-fattened Arnold family ($a = 0.1/(2\pi)$)

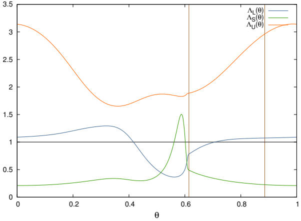
$\varepsilon = 0.7440768923$: last computed torus



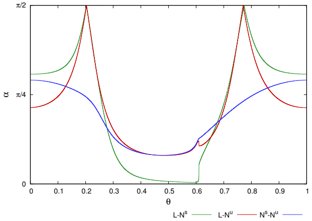
(a) Invariant curve and bundles



(b) Internal dynamics



(c) Angles between bundles



(d) Linearized dynamics

A 3D-fattened Arnold family ($a = 0.1/(2\pi)$)

Open questions

Which is the mechanism of breakdown of the saddle invariant curve?

Some ideas:

- Since the invariant curve contains two saddle fixed points of stability indices 2 and 1, which is the role of the corresponding invariant manifolds?
- Global phenomena ... homoclinic bifurcations?

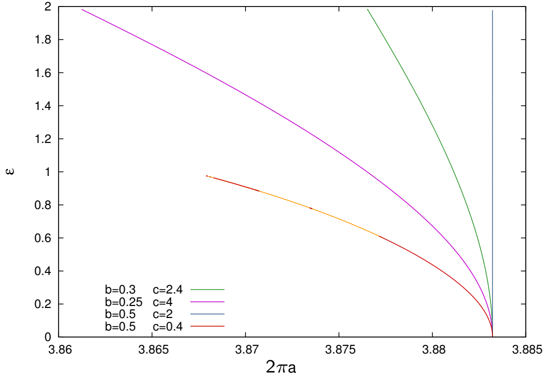
Breakdown of normally hyperbolic quasiperiodic invariant tori

Continuations with fixed frequency

... in the 3D-fattened Arnold family

We look for invariant tori with fixed frequency $\omega = \frac{\sqrt{5}-1}{2}$ for some 3D-fattened Arnold families:

- dissipative: $b = 0.3, c = 2.4$;
- **conservative**: $b = 0.25, c = 4$;
- reversible: $b = 0.5, c = 2$;
- **attracting**: $b = 0.5, c = 0.4$.



(with Marta Canadell)

Continuation of a saddle torus with respect to ε

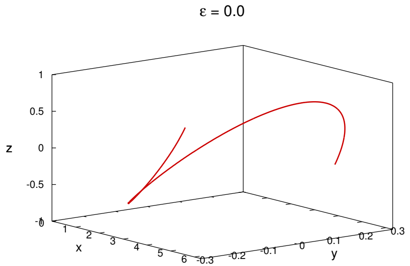
Conservative case: $b = 0.25, c = 4$

ε	$2\pi a$	λ_S	λ_U	α_{LS}	α_{LU}	α_{SU}
0.000000	3.883222077	0.2500000000	4.000000000	1.23529e+00	9.67700e-01	7.53151e-01
1.000000	3.876507483	0.2470073613	4.048462340	5.84550e-01	7.24296e-01	4.74488e-01
1.900000	3.862725196	0.2360025603	4.237242166	4.06820e-02	4.85572e-01	4.55743e-01
1.980000	3.861290122	0.2341969934	4.269909640	1.58441e-03	4.64434e-01	4.60560e-01
1.983000	3.861235891	0.2341236575	4.271247128	1.48398e-04	4.63643e-01	4.60725e-01
1.983200	3.861232275	0.2341187477	4.271336704	5.26130e-05	4.63591e-01	4.60736e-01
1.983210	3.861232094	0.2341185021	4.271341184	4.78291e-05	4.63588e-01	4.60736e-01
1.983211	3.861232076	0.2341184775	4.271341632	4.73511e-05	4.63588e-01	4.60736e-01

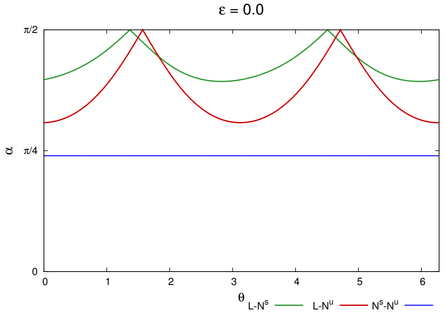
Table: Continuation of a **saddle torus**: continuation parameter, adjusting parameter, eigenvalues of the linearized equation and minimum angle between bundles.

Continuation of a saddle torus with respect to ε

$\varepsilon = 0.0$



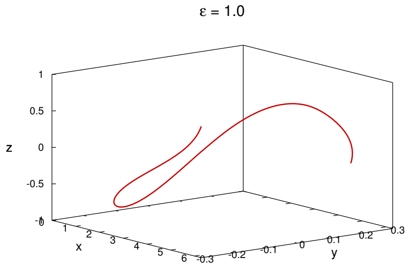
invariant torus



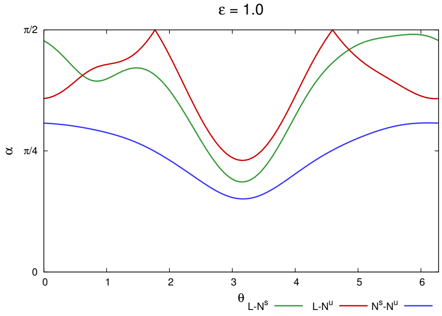
angles between fibers of the bundles

Continuation of a saddle torus with respect to ε

$\varepsilon = 1.0$



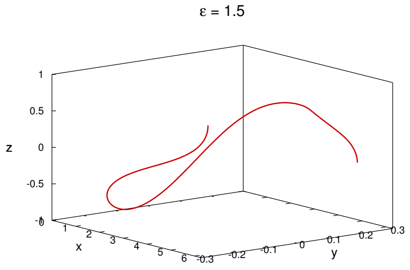
invariant torus



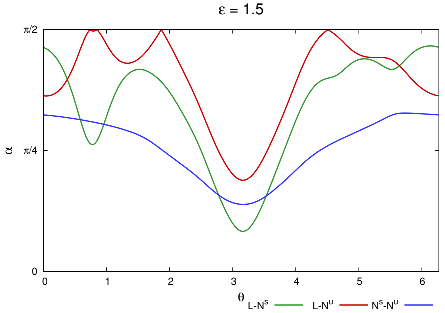
angles between fibers of the bundles

Continuation of a saddle torus with respect to ε

$\varepsilon = 1.5$



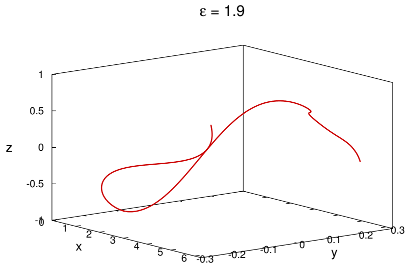
invariant torus



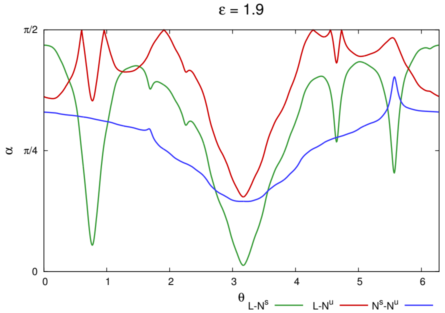
angles between fibers of the bundles

Continuation of a saddle torus with respect to ε

$\varepsilon = 1.9$



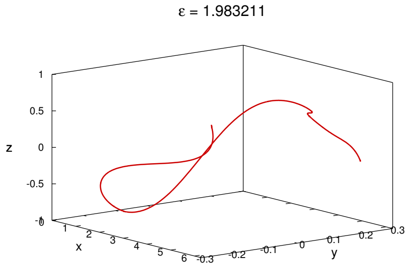
invariant torus



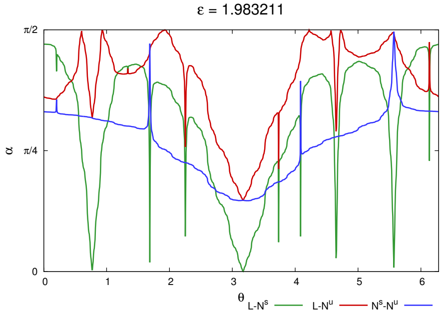
angles between fibers of the bundles

Continuation of a saddle torus with respect to ε

$\varepsilon = 1.983211$



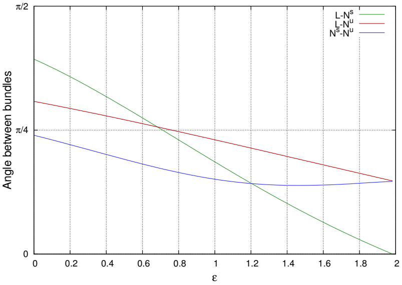
invariant torus

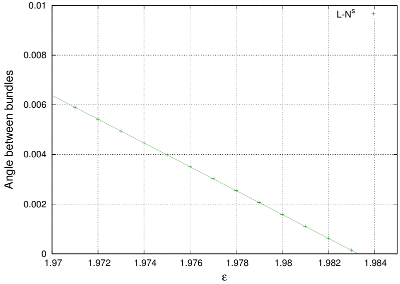


angles between fibers of the bundles

Bundle collision between tangent and stable bundles

Angles between bundles





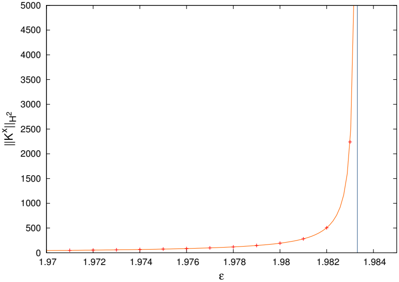
Minimum angles between bundles $\quad \alpha_{LS}(\varepsilon) \simeq 0.95101870 - 0.47951222\,\varepsilon.$

Estimated critical value of breakdown: $\varepsilon_{c,LS} \simeq \mathbf{1.98330}439.$

Loss of regularity

Sobolev norms

ε	H^2	$\mathrm{N_F}$
0.000000	0.00000e+00	64
1.000000	3.93198e-01	64
1.900000	9.25609e+00	1024
1.980000	2.74229e+02	16384
1.983000	3.16805e+03	131072
1.983200	9.23013e+03	524288
1.983210	1.01840e+04	1048576
1.983211	1.02901e+04	1048576



H^2 seminorm, and number of modes

$$H^2(\varepsilon) \simeq \frac{0.52106152}{(1.98331250-\varepsilon)^{1.03657369}}$$

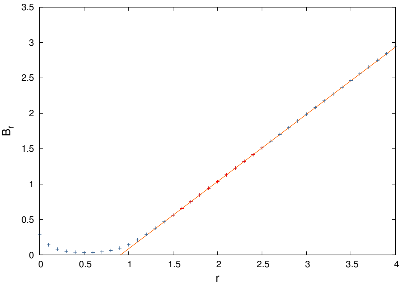
Estimated critical value of breakdown: $\varepsilon_{c,H^2} \simeq \mathbf{1.98331}250$.

Critical exponent: $B_2 \simeq 1.03657369$.

Sobolev regularity at breakdown

Linear behaviour of critical exponents

r	ε_{c,H^r}	B_r
1.0	1.9834618324	0.1454271549
1.2	1.9833384372	0.2908069846
1.4	1.9833176436	0.4695009290
1.6	1.9833139955	0.6572301080
1.8	1.9833129682	0.8467031525
2.0	1.9833125000	1.0365736863
2.2	1.9833122392	1.2266035739
2.4	1.9833120811	1.4167235069
2.6	1.9833119793	1.6068996870
2.8	1.9833119103	1.7971125472
3.0	1.9833118614	1.9873500514



Critical value ε_c and critical exponent, B_r $\qquad B_r \simeq -0.86158681 + 0.94921635\ r$

Estimated critical Sobolev regularity: $r_c \simeq 0.90768225$.

Continuation of an attracting torus with respect to ε

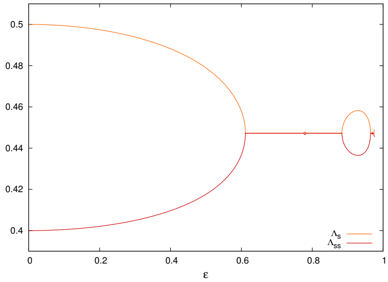
Dissipative case: $b = 0.5, c = 0.4$

ε	$2\pi a$	λ_2	λ_3	α_{LN}	index $\times 2$
0.0000000000	3.8832220775	0.4000000000	0.5000000000	1.17932e+00	0
0.6000000000	3.8773582471	0.4367722458	0.4579030644	2.80952e-01	0
0.6112956166	3.8771457591	0.4472071096	0.4472200814	2.68560e-01	0
0.6500000000	3.8763877719	$\lvert\cdot\rvert \simeq 0.4472135955$		2.28944e-01	-
0.7000000000	3.8753378014			1.83379e-01	
0.7780000000	3.8735326319	0.4467502443	0.4476774260	1.20776e-01	13
0.7810000000	3.8734589788	0.4468415827	0.4475859170	1.20060e-01	13
0.8000000000	3.8729849911	$\lvert\cdot\rvert \simeq 0.4472135955$		1.06723e-01	-
0.8700000000	3.8711212578			6.15360e-02	
0.9000000000	3.8702618736	-0.4390977148	-0.4554795000	4.38126e-02	-8
0.9400000000	3.8690546139	-0.4369038657	-0.4577666206	2.14837e-02	-8
0.9650000000	3.8682620624	$\lvert\cdot\rvert \simeq 0.4472135955$		8.21239e-03	-
0.9660000000	3.8682297243			7.69179e-03	
0.9664195313	3.8682161425	0.4471829177	0.4472442872	7.47361e-03	-152
0.9665195313	3.8682129039	0.4471254316	0.4473017886	7.42163e-03	-152
0.9670000000	3.8681973361	$\lvert\cdot\rvert \simeq 0.4472135955$		7.17195e-03	-
0.9680000000	3.8681648979			6.65290e-03	
0.9700597656	3.8680979246	0.4469783005	0.4474490260	5.58617e-03	81
0.9702000000	3.8680933571	$\lvert\cdot\rvert \simeq 0.4472135955$		5.51366e-03	-
0.9725000000	3.8680183043			4.32655e-03	
0.9738288651	3.8679748195	-0.4471093093	-0.4473179175	3.64248e-03	-63
0.9748679777	3.8679407539	-0.4458282085	-0.4486032989	3.10852e-03	-63
0.9761123718	3.8678998859	-0.4455276098	-0.4489059727	2.47015e-03	-63
0.9787153549	3.8678141438	$\lvert\cdot\rvert \simeq 0.4472135955$		1.13849e-03	-

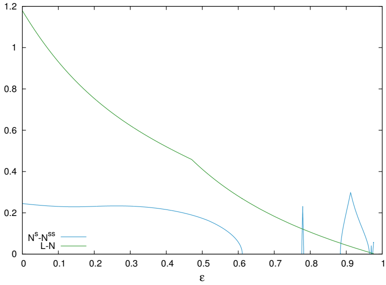
Table: Continuation of an attracting torus: continuation parameter, adjusting parameter, eigenvalues of the linearized equation, angle between bundles and index of the bundles.

Continuation of an attracting torus with respect to ε

Lyapunov multipliers and angles between bundles



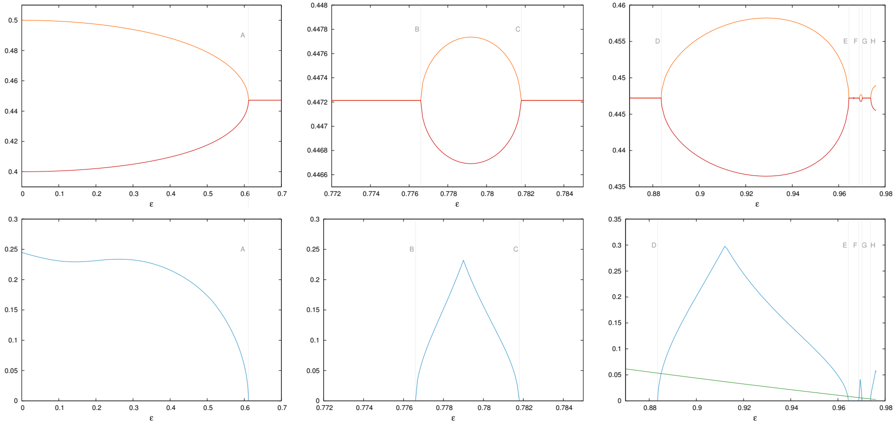
Lyapunov multipliers



Minimum angle between bundles

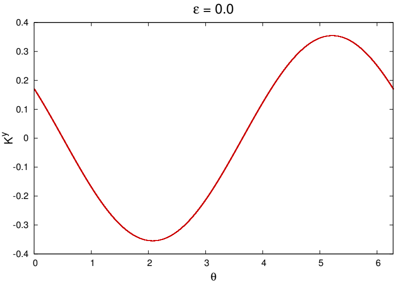
Continuation of an attracting torus with respect to ε

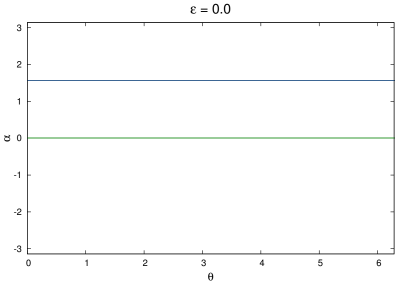
Some node-focus transitions



Continuation of an attracting torus with respect to ε

$\varepsilon = 0.0$

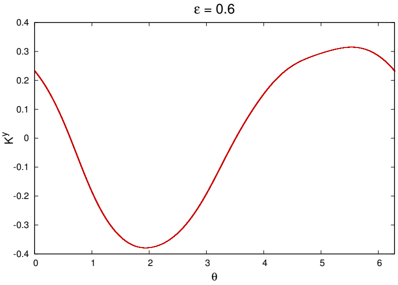


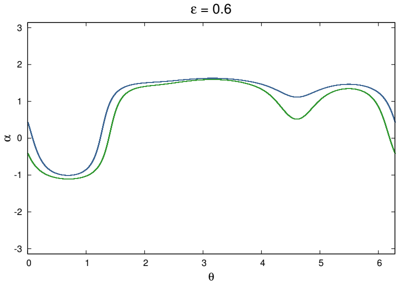


y-component of parameterization   projectivized bundles in the stable bundle

Continuation of an attracting torus with respect to ε

$\varepsilon = 0.6$



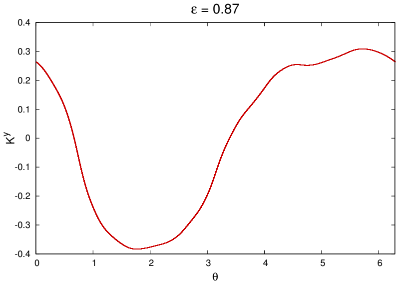


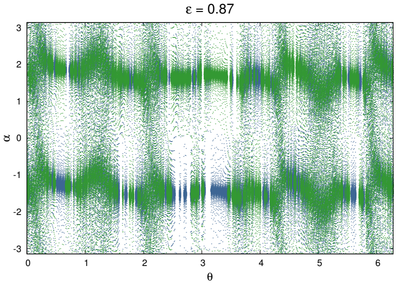
y-component of parameterization

projectivized bundles in the stable bundle

Continuation of an attracting torus with respect to ε

$\varepsilon = 0.87$



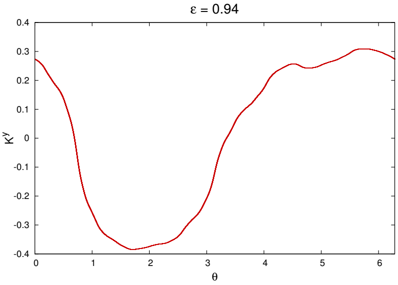


y-component of parameterization

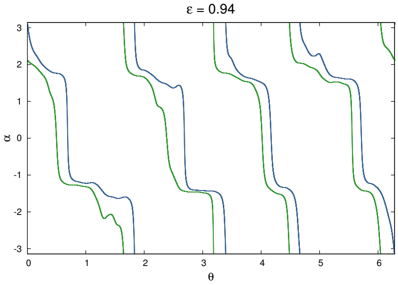
projectivized bundles in the stable bundle

Continuation of an attracting torus with respect to ε

$\varepsilon = 0.94$



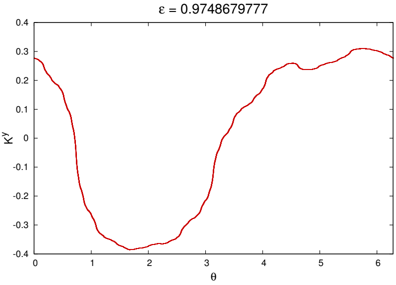
y-component of parameterization



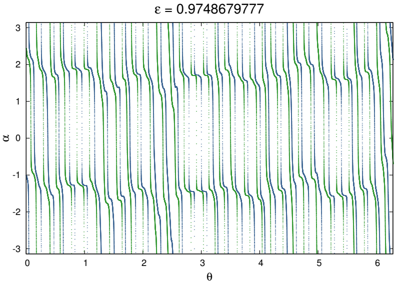
projectivized bundles in the stable bundle

Continuation of an attracting torus with respect to ε

$\varepsilon = 0.9748679777$



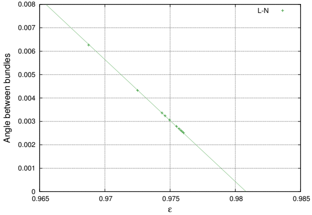
y-component of parameterization

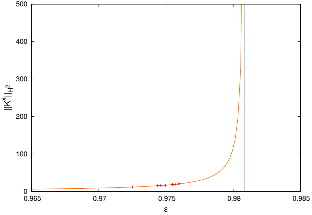


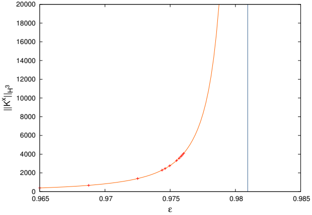
projectivized bundles in the stable bundle

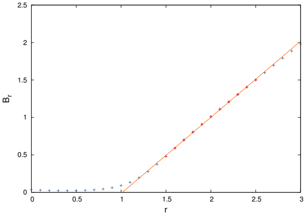
Continuation of an attracting torus with respect to ε

Observables at breakdown









$$\alpha_{L,S} \simeq 1.270754 - 0.641802\ \varepsilon \quad \Longrightarrow \quad \varepsilon_{c_{LS}} \simeq 0.980821$$

$$H^2(\varepsilon) \simeq \frac{0.093014}{(0.980863-\varepsilon)^{1.010053}} \quad \Longrightarrow \quad \varepsilon_{c_{H^2}} \simeq 0.980863$$

$$H^3(\varepsilon) \simeq \frac{0.108759}{(0.980941-\varepsilon)^{1.981441}} \quad \Longrightarrow \quad \varepsilon_{c_{H^3}} \simeq 0.980941$$

$$B_r \simeq -1.031283 + 1.016700\ r \quad \Longrightarrow \quad r_c \simeq 1.014343$$

Bundle collisions

Findings in different contexts

	Continuation of qp NHIT	Breakdown and bundle merging
On skew-products over rotations [Haro, de la Llave 06,07] [Figueras, Haro 12, 16]	⇝ attracting tori saddle tori	Collision of slow and fast stable bundles prior to the breakdown (fractalization route)
	⇝ saddle tori	Collision of stable and unstable bundles producing the breakdown
On conformally symplectic systems [Calleja, Figueras 12]	⇝ attracting tori	Collision of stable and tangent bundles producing the breakdown
On general systems [Canadell, Haro 14, 16]	⇝ attracting tori	Sequence of node-focus transitions before collision of 2D stable and 1D tangent bundles Fractalization route
	⇝ saddle tori	Collision of tangent, stable and unstable bundles producing the breakdown

A normally hyperbolic invariant cylinder

A 4D-Froeschlé map

The equations of the model

We consider the Froeschlé map, in the form:

$$\begin{array}{rcl} F_\varepsilon : \mathbb{T}\times\mathbb{R}\times\mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R}\times\mathbb{T}\times\mathbb{R} \\ \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} & \longrightarrow & \begin{pmatrix} x_1 + y_1 - \frac{\kappa_1}{2\pi}\sin(2\pi x_1) - \frac{\varepsilon}{2\pi}\sin(2\pi(x_1+x_2)) \\ y_1 - \frac{\kappa_1}{2\pi}\sin(2\pi x_1) - \frac{\varepsilon}{2\pi}\sin(2\pi(x_1+x_2)) \\ x_2 + y_2 - \frac{\kappa_2}{2\pi}\sin(2\pi x_2) - \frac{\varepsilon}{2\pi}\sin(2\pi(x_1+x_2)) \\ y_2 - \frac{\kappa_2}{2\pi}\sin(2\pi x_2) - \frac{\varepsilon}{2\pi}\sin(2\pi(x_1+x_2)) \end{pmatrix}, \end{array}$$

where:

- κ_1, κ_2 are the (fixed) parameters of the coupled standard maps;
- ε is the coupling (and perturbing) parameter.

We fix $\kappa_1 = 0.1$, $\kappa_2 = 1.5$, so that the cylinder

$$\mathcal{C}_0 = \{(x_1, y_1, \tfrac{1}{2}, 0) \mid (x_1, y_1) \in \mathbb{T}\times\mathbb{R}\}$$

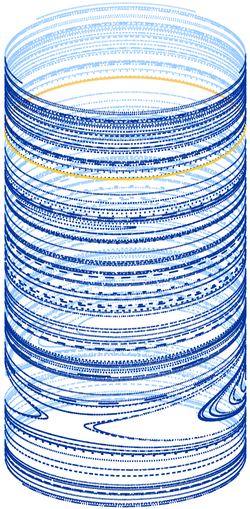
is a normally hyperbolic invariant manifold for F_0.

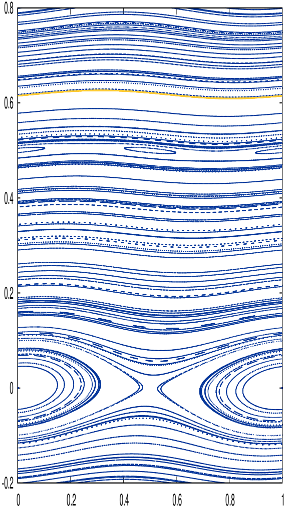
We continue the cylinder $\mathcal{C}_\varepsilon$ with respect to ε.

(with Marta Canadell)

A 4D-Froeschlé map

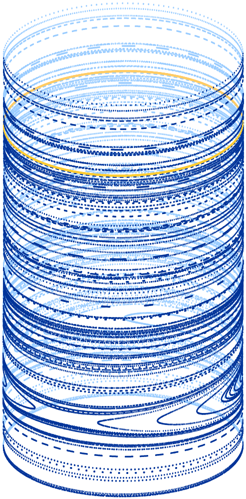
$\varepsilon = 0.00$

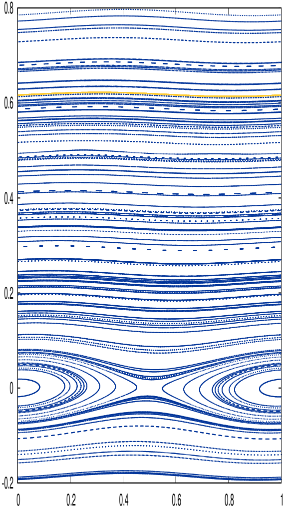




A 4D-Froeschlé map

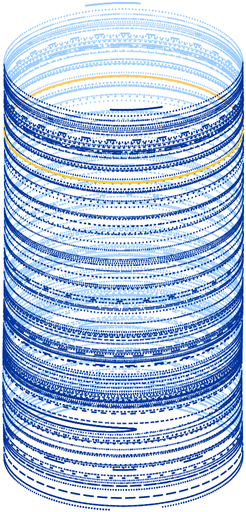
$\varepsilon = 0.05$

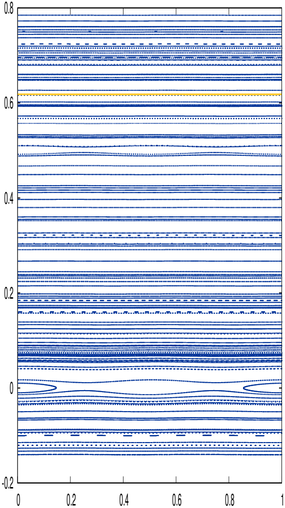




A 4D-Froeschlé map

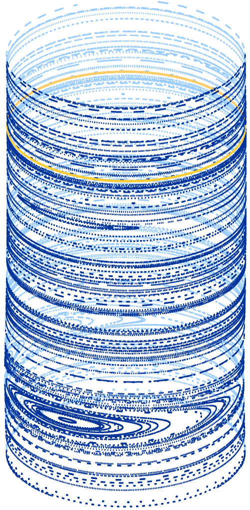
$\varepsilon = 0.10$

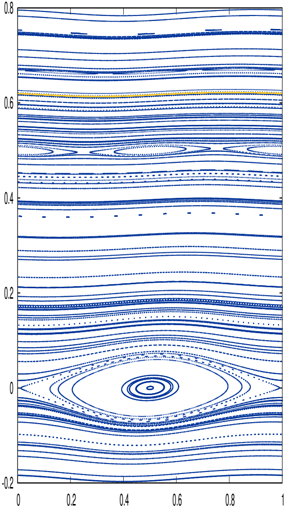




A 4D-Froeschlé map

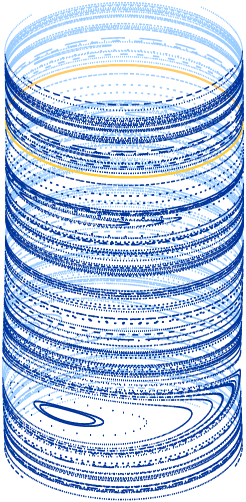
$\varepsilon = 0.15$

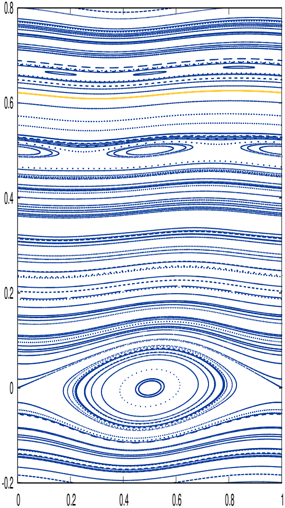




A 4D-Froeschlé map

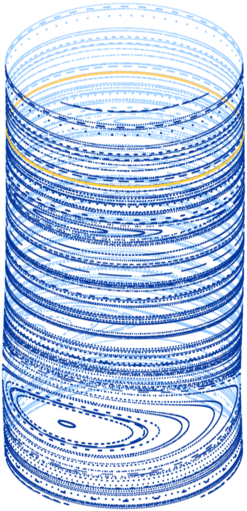
$\varepsilon = 0.20$

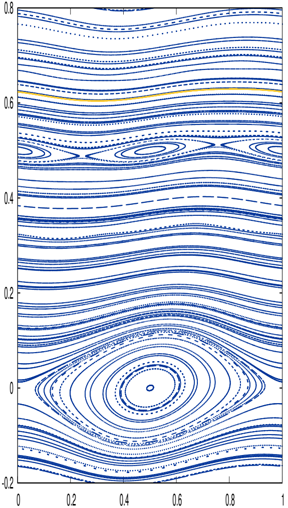




A 4D-Froeschlé map

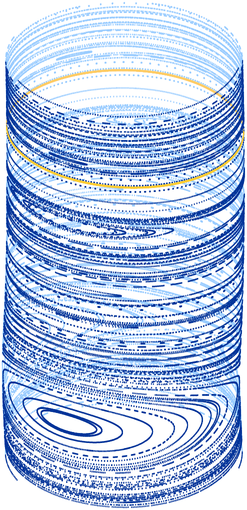
$\varepsilon = 0.25$

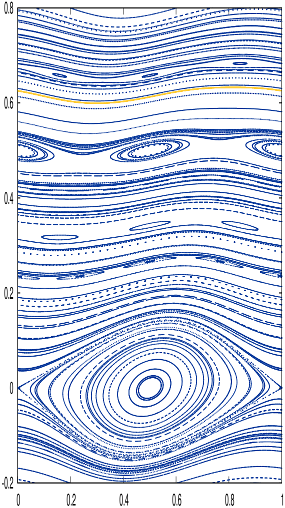




A 4D-Froeschlé map

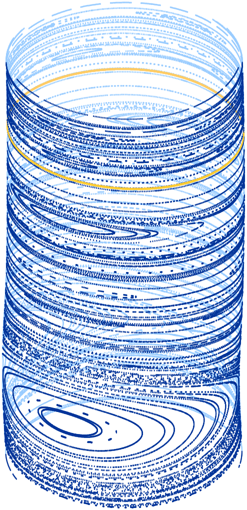
$\varepsilon = 0.30$

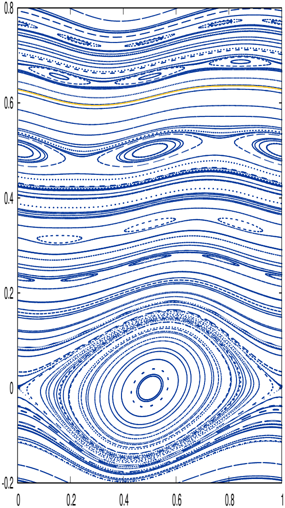




A 4D-Froeschlé map

$\varepsilon = 0.35$





A 4D-Froeschlé map

Dynamics on the invariant cylinder

We observe the typical features of area preserving maps:

- eliptic and hyperbolic periodic orbits;
- rotational invariant curves;
- librational invariant curves (in the islands);
- chaos.

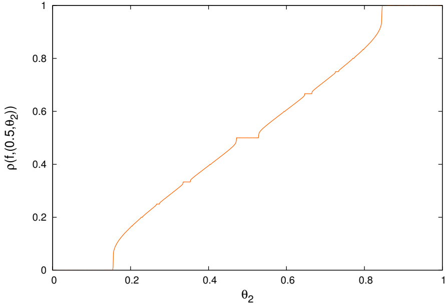


Figure: Rotation number of points in $\mathcal{C}_{0.35}$, parameterized by $(0.5, \theta_2)$.

A 4D-Froeschlé map

Open questions

Which are the mechanisms of breakdown of the invariant cylinder?

Some ideas:

- the internal periodic orbits can bifurcate, and the normal eigenvalues can collide with the internal eigenvalues;
- the internal invariant curves can be destroyed:
  - as a KAM curve (this is not dangerous for the cylinder);
  - with a bundle collision, involving stable, unstable or/and tangent invariant bundles (this would break pieces of the invariant cylinder).
- homoclinic bifurcations?

Conclusions

Conclusions

Reduced to two

- Something is known about mechanisms of breakdown of nhim ...

Conclusions

Reduced to two

- Something is known about mechanisms of breakdown of nhim ...
- But much more work, both numerical and rigorous, has to be done!

Some papers

(with Marta Canadell, Jordi Lluís Figueras and Rafael de la Llave)

- Àlex Haro, Rafael de la Llave. *Manifolds on the verge of a hyperbolicity breakdown.* (Chaos, 2006)

- Àlex Haro, Rafael de la Llave. *A parameterization method for the computation of invariant tori and their whiskers in quasi periodic maps: explorations and mechanisms for the breakdown of hyperbolicity.* (SIADS, 2007)

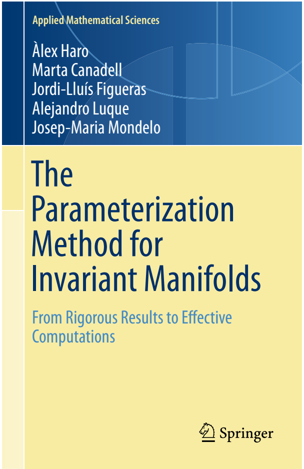
- Jordi-Lluís Figueras, Àlex Haro. *Different scenarios for hyperbolicity breakdown in quasiperiodic area preserving twist maps.* (Chaos, 2015)

- Marta Canadell, Àlex Haro. *Parameterization method for computing quasi-periodic reducible normally hyperbolic invariant tori*, (Advances in Differential Equations and Applications, 2014).

- Jordi Lluís Figueras, Àlex Haro. *A note on the fractalization of saddle invariant curves in quasiperiodic systems*. (DCDS, 2016)

A book

(with Marta Canadell, Jordi-Lluís Figueras, Alejandro Luque and Josep-Maria Mondelo)



... and a prequel
(by Pedro Almodóvar)