Global flow of the parabolic restricted three body problem

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Outline

N-Body Problem

- N-body problem: N point masses m_i, $i = 1, \ldots N$ moving under their gravitational atractions

$$\boxed{m_i \ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}}$$

- 2-body problem: integrable problem. The masses move in Keplerian orbits: elliptic, parabolic or hyperbolic, around their center of mass.

- Restricted Three-Body problem: Two main bodies (primaries) moving in a keplerian orbit + massless particle moving under the gravitational attraction of the primaries, without affecting them.

N-Body Problems

Main tools of the dynamical systems

Hamiltonian formulation:

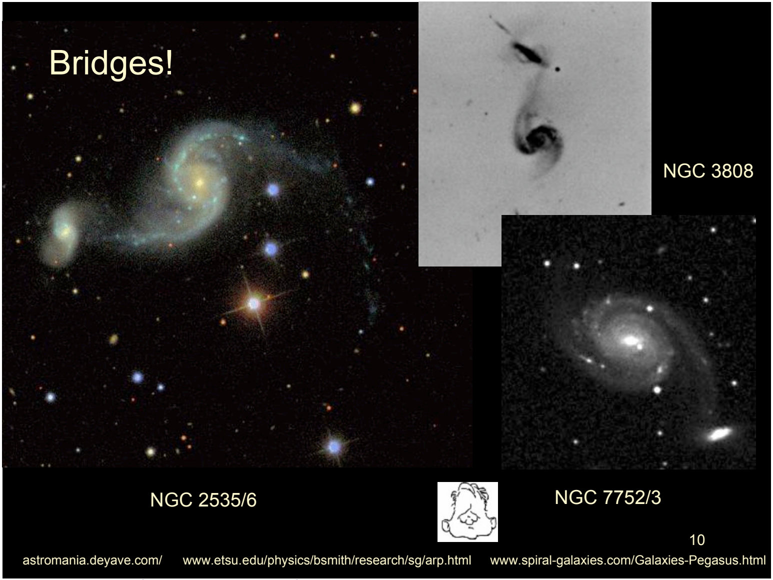
$$\boxed{\dot{\mathbf{z}} = J \cdot \nabla H(\mathbf{z})}$$

- Invariant objects: equilibrium points, periodic and quasi-periodic orbits
- Stability of the invariant objects
- Invariant manifolds:

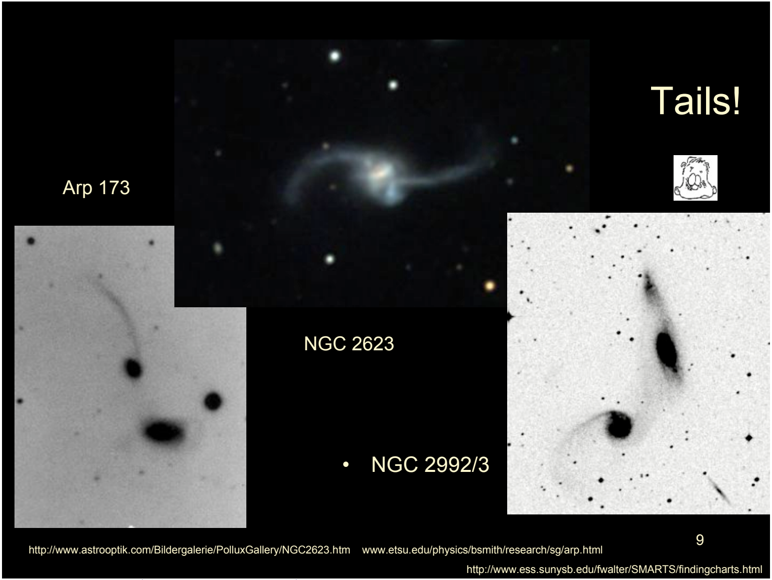
$$W^u(\Gamma) = \{\mathbf{z}(t); \mathbf{z}(t) \underset{t \nearrow -\infty}{\longrightarrow} \Gamma\}$$

$$W^s(\Gamma) = \{\mathbf{z}(t); \mathbf{z}(t) \underset{t \searrow +\infty}{\longrightarrow} \Gamma\}$$

Galactic encounters: bridges and tails



Galactic encounters: bridges and tails



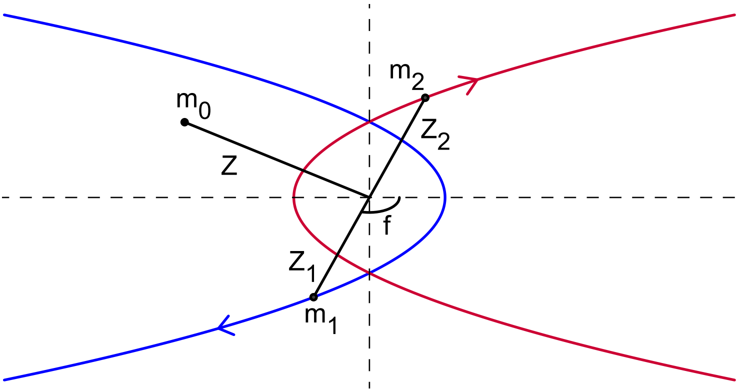
Motivations and Aims

- Close approach of two galaxies: it causes significant modification of the mass distribution or disc structure. One particle that initially stays in one galaxy (or around one star), after the close encounter, it can jump to the other galaxy or escape.

- To study the mechanisms that explain that a particle remains or not around each galaxy, considering a very simple model: the planar parabolic restricted three-body problem.

Outline

The Planar Parabolic Restricted Three-Body Problem



Equations (I)

Parabolic problem:

$$\frac{d^2\mathbf{Z}}{dt^2} = -(1-\mu)\frac{\mathbf{Z}-\mathbf{Z}_1}{|\mathbf{Z}-\mathbf{Z}_1|^3} - \mu\frac{\mathbf{Z}-\mathbf{Z}_2}{|\mathbf{Z}-\mathbf{Z}_2|^3},$$

$\mathbf{Z}_2 = -\mathbf{Z}_1 = \frac{1}{2}(\sigma^2-1, 2\sigma)$, and $\sigma = \tan(f/2)$

- Change to a synodic frame (primaries at fixed positions) + change of time:

$$\mathbf{z}_1 = (-\frac{1}{2}, 0), \qquad \mathbf{z}_2 = (\frac{1}{2}, 0),$$

$$\frac{dt}{ds} = \sqrt{2}\, r^{3/2}.$$

- Compatification to extend the flow when the primaries are at infinity $(t, s \to \pm\infty)$:

$$\sin(\theta) = \tanh(s).$$

Equations (II)

Global system

$$\begin{cases} \theta' &= \cos\theta, \\ \mathbf{z}' &= \mathbf{w}, \\ \mathbf{w}' &= -A(\theta)\mathbf{w} + \nabla\Omega(\mathbf{z}) \end{cases}$$

where $' = \dfrac{d}{ds}$ and

$$A(\theta) = \begin{pmatrix} \sin\theta & 4\cos\theta \\ -4\cos\theta & \sin\theta \end{pmatrix},$$

$$\Omega(\mathbf{z}) = x^2 + y^2 + 2\frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + 2\frac{\mu}{\sqrt{(x-\mu+1)^2+y^2}}.$$

Upper and Lower boundary problems

Global system

$$\left\{ \begin{array}{rcl} \theta' & = & \cos\theta, \\ \mathbf{z}' & = & \mathbf{w}, \\ \mathbf{w}' & = & -A(\theta)\mathbf{w} + \nabla\Omega(\mathbf{z}) \end{array} \right. \quad \xrightarrow[\theta=\pm\pi/2]{} \quad \left\{ \begin{array}{rcl} \mathbf{z}' & = & \mathbf{w}, \\ \mathbf{w}' & = & \mp\mathbf{w} + \nabla\Omega(\mathbf{z}) \end{array} \right.$$

Boundary problems

dim 5

dim 4

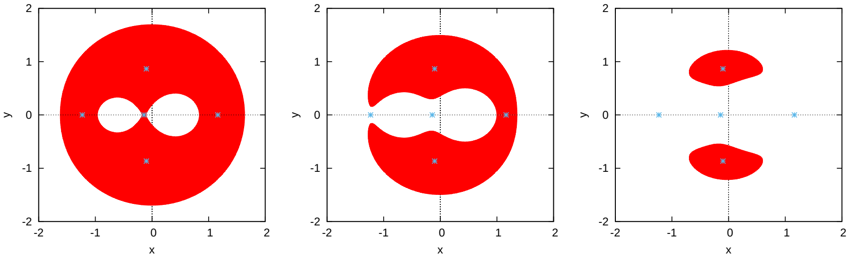
Main properties (I)

- Jacobi function: semi gradient property (no periodic orbits)

$$C = 2\Omega(\mathbf{z}) - |\mathbf{w}|^2, \qquad \frac{dC}{ds} = 2\sin\theta |\,\mathbf{w}|^2$$

- Hill's regions: $\{2\Omega(\mathbf{z}) - C \geq 0\}$ $\quad \to$ C-criterium

$$-\pi/2 \quad \longrightarrow \quad \theta \quad \longrightarrow \quad 0$$



$$\pi/2 \quad \longleftarrow \quad \theta \quad \longleftarrow \quad 0$$

Main properties (II)

Equilibrium points at the boundaries (as in the RTBP):

- Collinear: $L_i^{\pm} = (x_i(\mu), 0, 0, 0, \pm\pi/2)$, $i = 1, 2, 3$
- Triangular: $L_i^{\pm} = (\mu - \frac{1}{2}, \pm\sqrt{3}/2, 0, 0, \pm\pi/2)$, $i = 4, 5$

Stability:

	$L^{+}_{1,2,3}$	$L^{+}_{4,5}$
$\dim(W^u)$	1	2
$\dim(W^s)$	4	3

	$L^{-}_{1,2,3}$	$L^{-}_{4,5}$
$\dim(W^u)$	4	3
$\dim(W^s)$	1	2

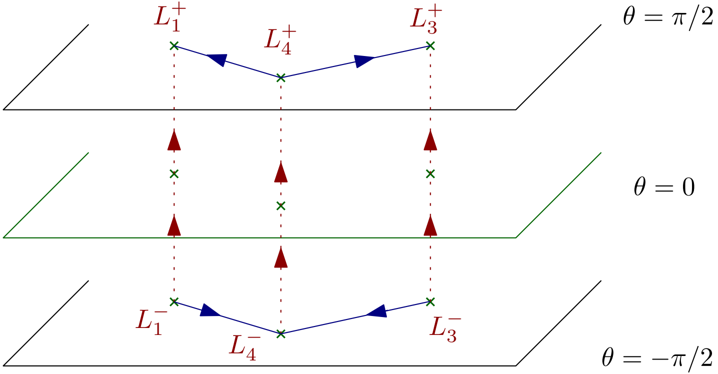
Main properties (II)

- Equilibrium points at the boundaries: $L_i^{\pm}$, $i = 1, ..., 5$ for $\theta = \pm\pi/2$ and $\mu = 1/2$

	(x_i, y_i)	$C(L_i^{\pm}) = C_i$
$L_1^{\pm}$	$(-1.198406145, 0)$	6.91359245
$L_2^{\pm}$	$(0, 0)$	8
$L_3^{\pm}$	$(1.198406145, 0)$	6.91359245
$L_{4,5}^{\pm}$	$(0, \pm\sqrt{3}/2)$	5.5

Main properties (III)

- Homothetic solutions and connections



Dynamics of the problem

In order to describe the dynamics of the parabolic problem, we will focus on two aspects:

- the final evolutions in the synodical system when time tends to infinity,
- the richness in the intermediate stages due to
  - existence of invariant manifolds associated with the homothetic solutions
  - heteroclinic connections that allow the existence of orbits with passages close to collinear and/or equilateral configurations.

Final evolutions

Proposition (Final evolutions)

Let $\gamma(s) = (\theta(s), \mathbf{z}(s), \mathbf{w}(s))$, $s \in [0, \infty)$, be a solution of the global system. Then, either:

- it is a collision orbit,
- $\lim_{s\to\infty} |\mathbf{z}(s)| = \infty$ (escape orbit)
- its ω-limit is an equilibrium point.

Final evolutions

Definition

Let $\mathbf{Z}(t)$ be a solution of the parabolic problem. We say that

- it is a capture orbit around the primary of mass m_i, for $i = 1$ or 2, if $\limsup_{t\to\infty} |\mathbf{Z}(t) - \mathbf{Z}_i(t)| \leq K$, for some constant K;
- it is an escape orbit if $\limsup_{t\to\infty} |\mathbf{Z}(t)| = \infty$ and $\limsup_{t\to\infty} |\mathbf{Z}(t) - \mathbf{Z}_i(t)| = \infty$ for $i = 1$ and 2.

Remark: the definition is given in the inertial frame: $|\mathbf{Z} - \mathbf{Z}_i| = r|\mathbf{z} - \mathbf{z}_i|$

- capture orbit $\rightarrow$ $|\mathbf{z} - \mathbf{z}_i| \rightarrow 0$ (collision orbit)
- $\liminf_{s\to\infty} |\mathbf{z}(s)| \geq K$ $\rightarrow$ escape orbit

C-criterium

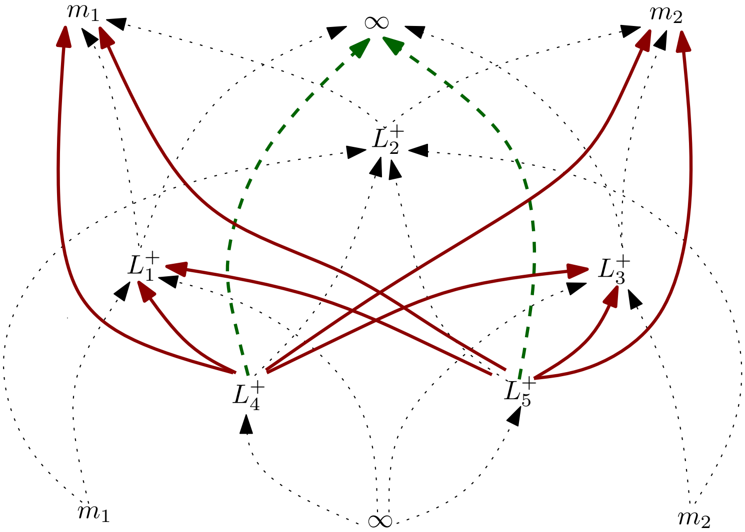
Proposition

Let $\mathbf{q} \in Int(D)$ with $\theta \geq 0$, and $\gamma(s) = (\theta(s), \mathbf{z}(s), \mathbf{w}(s))$, $s \in [0, \infty)$, the solution of the global system through $\mathbf{q}$. Then,

(i) if for some time s_0 the value of the Jacobi function $C(\gamma(s_0)) > C_2$ and $\mathbf{z}(s)$ is located in one of the bounded components of the Hill's region, then it is a collision orbit;

(ii) if for some time s_0 the value of the Jacobi function $C(\gamma(s_0)) > C_3$ and $\mathbf{z}(s)$ is located in the unbounded component of the Hill's region, then it is an escape orbit.

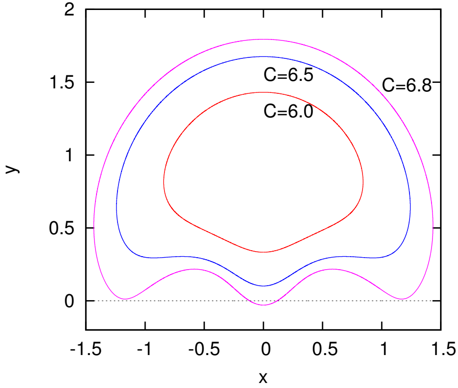
Connections in the the upper boundary problem



Heteroclinic $L_4^+ \quad \rightarrow \quad L_3^+$

Invariant manifold $W^u(L_4^+)$ (dim=2) and its intersections with

$$\Sigma_{C^*} = \{(\mathbf{z}, \mathbf{w}) \mid C(\mathbf{z}, \mathbf{w}) = C^*\}$$



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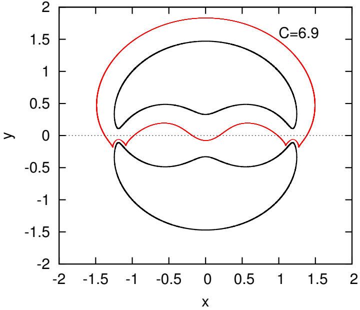

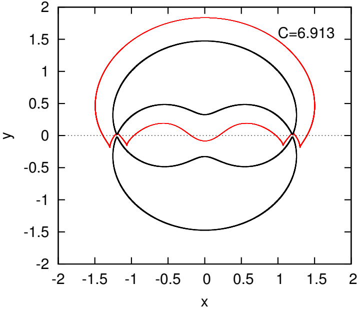

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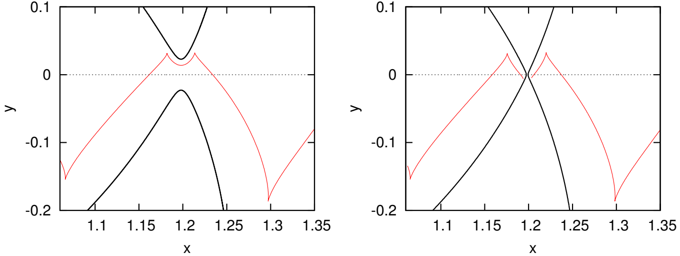


Outline

Explorations

Role of the invariant manifolds in the sets of connecting orbits between primaries

- Equal masses ($\mu = 0.5$):
  Barrabés, Cors, Ollé *Dynamics of the parabolic restricted three-body problem* Communications in Nonlinear Science and Numerical Simulation, 29: 400–415, 2015

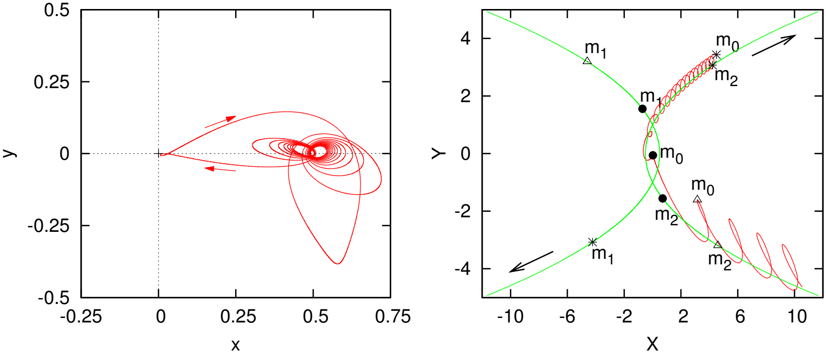
- Different masses ($\mu < 0.5$):

Connecting orbits with passages to collinear or triangular configurations

Connection of type $m_i - L_k - m_j$:

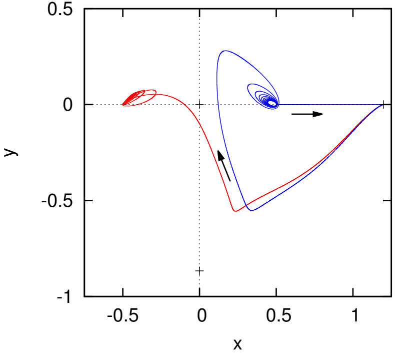
- collision orbit with m_i backwards in time
- collision orbit with m_j forwards in time
- along its trajectory it has a close passage to L_k

Connecting orbits: examples ($\mu = 0.5$)



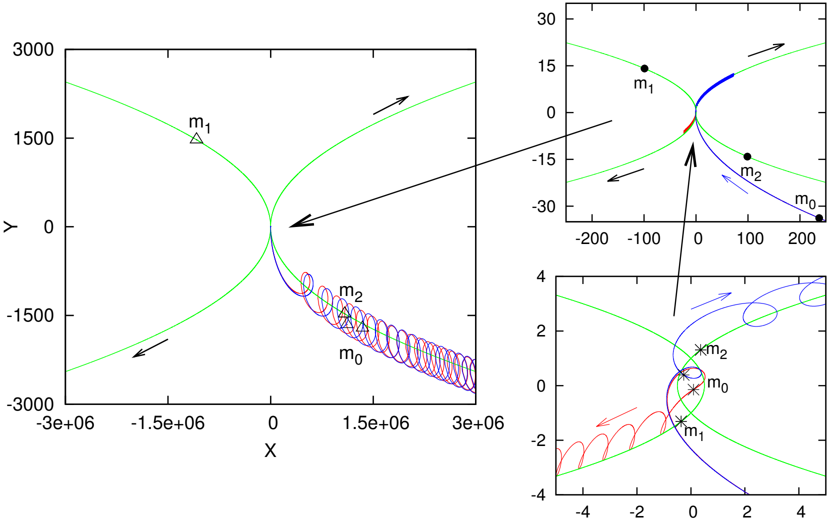
$$m_2 - L_2 - m_2$$

Connecting orbits: examples ($\mu = 0.5$)



$$m_2 - L_3 - L_2 - m_2/m_1$$

Connecting orbits: examples ($\mu = 0.5$)



Symmetric connecting orbits ($\mu = 0.5$)

- Connection $m_i - m_i$: crosses the section $\theta = 0$ such that $y = x' = 0$

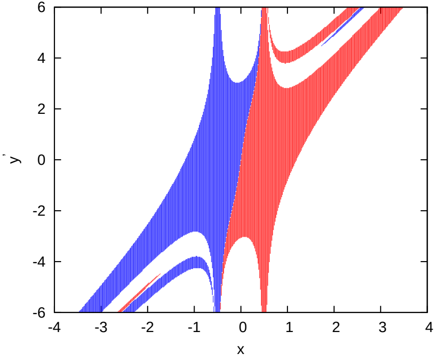
  I.C. $(x_0, 0, 0, y'_0)$

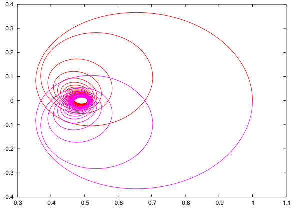
- Connection $m_i - m_j$: crosses the section $\theta = 0$ such that $x = y' = 0$

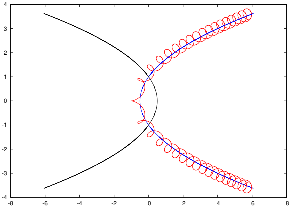
  I.C. $(0, y_0, x'_0, 0)$

Symmetric connecting orbits ($\mu = 0.5$)

$$m_i - m_i$$

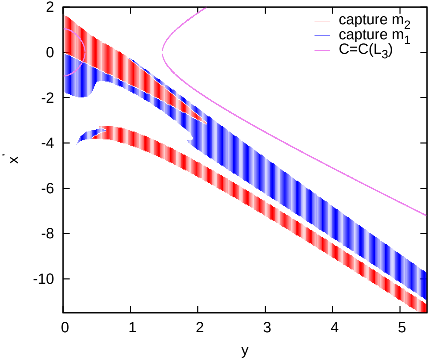


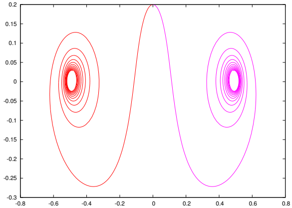


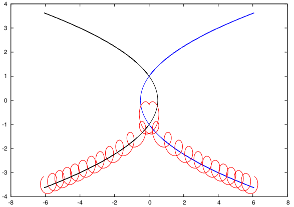


Symmetric connecting orbits ($\mu = 0.5$)

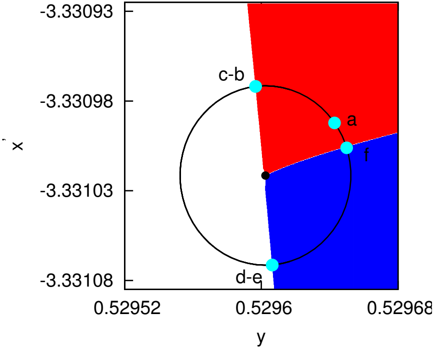
$$m_i - m_j$$

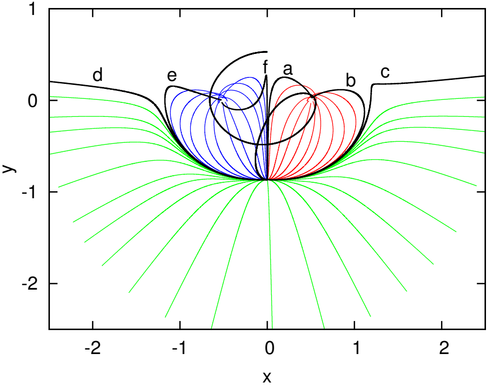




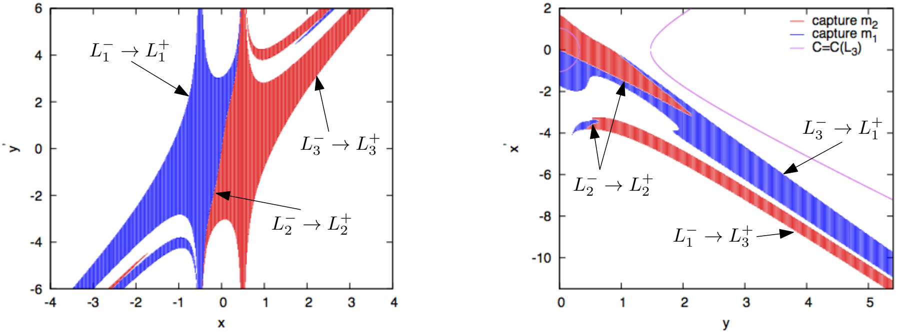


Symmetric connecting orbits ($\mu = 0.5$)

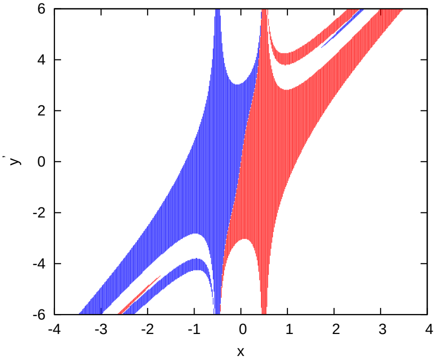




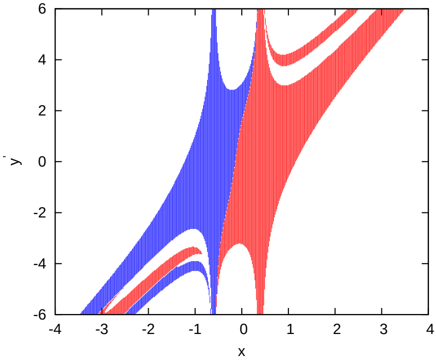
Symmetric connecting orbits ($\mu = 0.5$)



Evolution of sets of symmetric connecting orbits

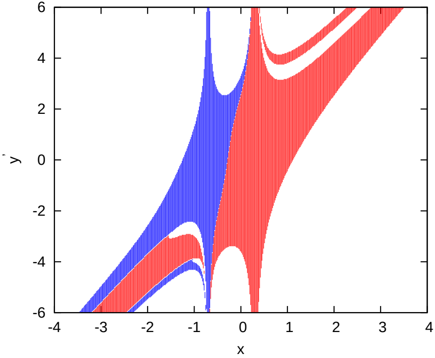


$\mu = 0.5$

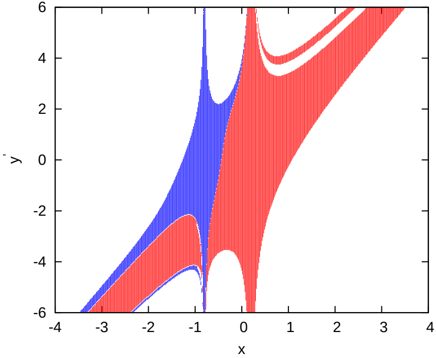


$\mu = 0.4$

Evolution of sets of symmetric connecting orbits

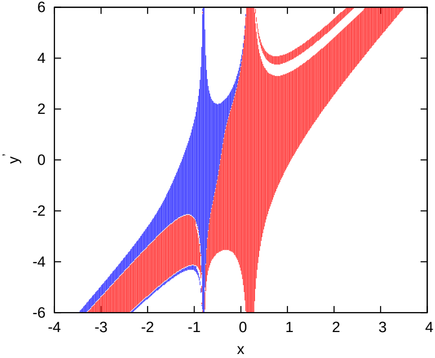


$\mu = 0.3$

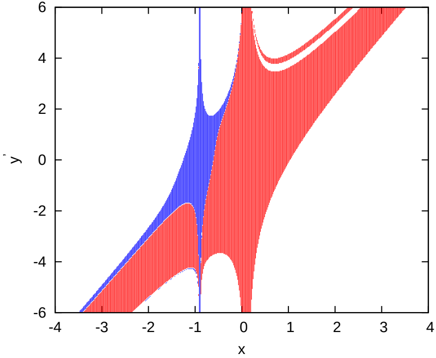


$\mu = 0.2$

Evolution of sets of symmetric connecting orbits



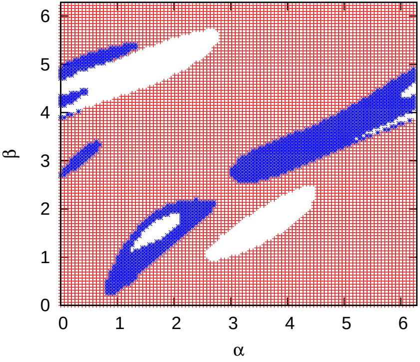
$\mu = 0.2$



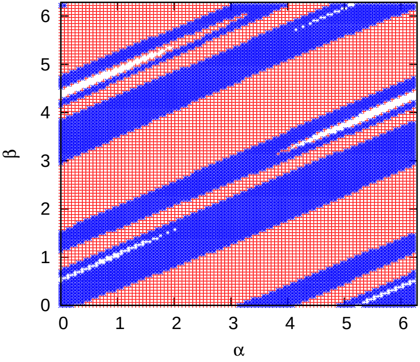
$\mu = 0.1$

Bridges and Tails?

We consider a bunch of initial conditions around m_1 for $\theta = -\pi/4$ and a value $C \geq C_2 = 8$ (for $\mu = 1/2$). For this value of C, we fix a radius, r (distance to m_1) and move $\alpha \in [0, 2\pi]$. Since, velocity module is given by position and Jacobi function C, we move $\beta \in [0, 2\pi]$ (velocity direction).

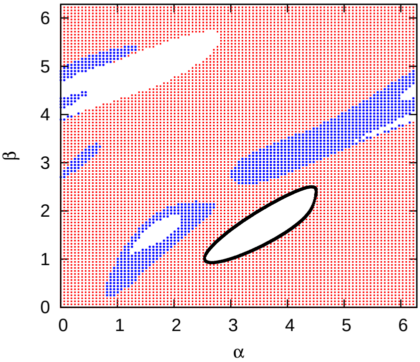


$r = 0.2$

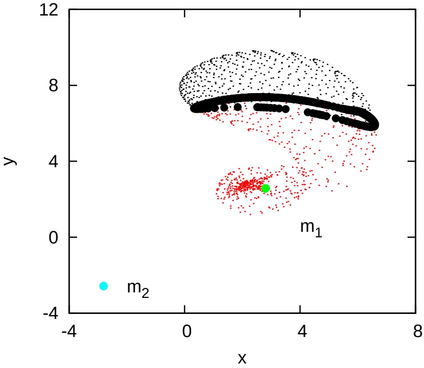


$r = 0.001$

Tails

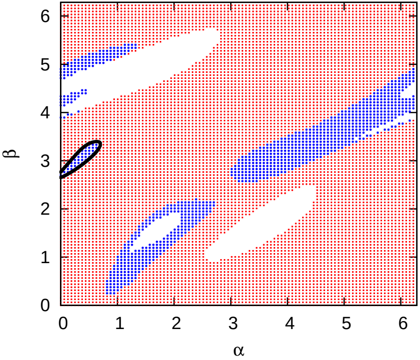


$r = 0.2$

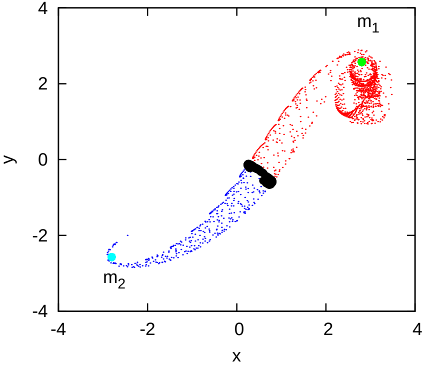


$\theta = 1.2$

Bridges

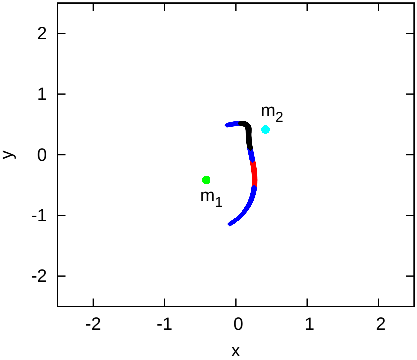


$r = 0.2$

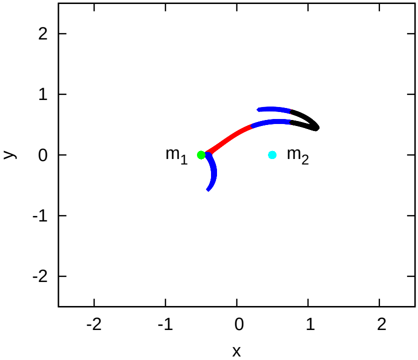


$\theta = 1.2$

A Movie

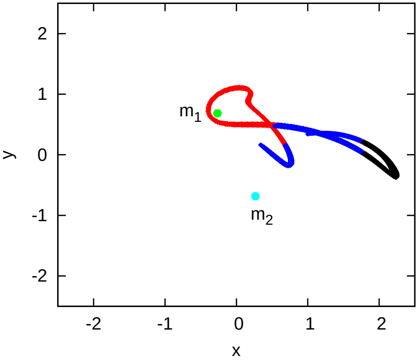


$\theta = -\pi/8$

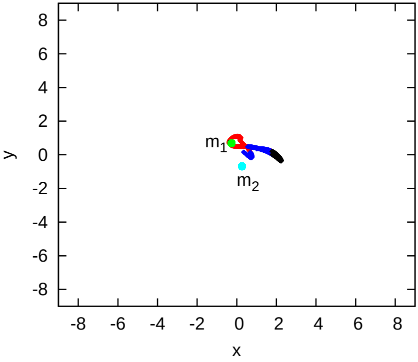


$\theta = 0$

A Movie

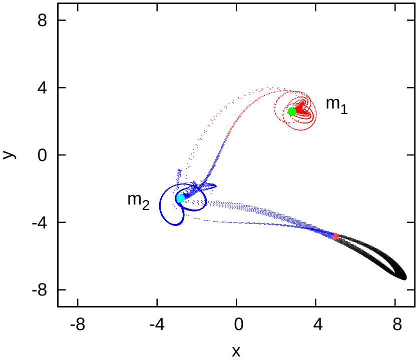


$\theta = \pi/8$

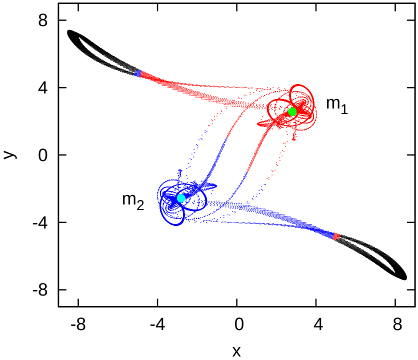


$\theta = \pi/8$

A Movie



$\theta = 1.2$



$\theta = 1.2$

Outline

Introduction

Dynamics of the parabolic problem

Numerical results

Conclusions

Conclusions

- Using the invariant manifolds, the symmetries of the problem and the C-criterium it is possible to construct connecting orbits of different types.

- The regions of the phase space where the test particles remain or not around each galaxy are confined by the invariant manifolds of the collinear equilibrium points.

Further work

- How does the mass parameter of the parabolic problem affects Bridges and tails?
- Explorations varying the inclination (Spatial parabolic problem)
- Hyperbolic problem (make sense)