Dynamics of a particle in some cases of the N-body problem

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- Two-body problem (no secrets)
- N-body problem (too difficult)
- R3BP (still difficult and 200 years of history)
- Add complications to the R3BP (non spherity, radiation pressure ..., ribbon,...)
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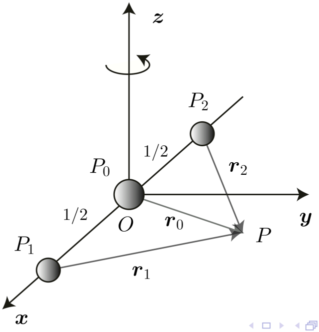
- we consider the Restricted collinear four-body problem with radiation pressure
- 3 primaries:
  (P_0, m_0, q_0) and two identical bodies (P_1, m, q), (P_2, m, q)
- planar motion of a massless P in a synodic reference frame

$$m_0 = \beta m, \qquad \omega^2 = \Delta = 2(1 + 4\beta)$$



Radiation coefficients

$$b = \frac{F_r}{F_g} \rightarrow \quad q_i = 1 - b_i$$

- $b_i = 0$ classical problem
- $b_i \in (0, 1)$ reduction of gravitational forces by radiation
- $b_i \geq 1$ radiation has overhelmed gravitational forces by radiation

So, $q_i \in (-\infty, 1] \quad (i = 0, 1, 2)$

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Equations of motion

$$\ddot{x} - 2\,\dot{y} = \frac{\partial U}{\partial x}, \qquad \ddot{y} + 2\,\dot{x} = \frac{\partial U}{\partial y},$$

where the effective potential U is

$$U(x,y) = \frac{1}{2}\,(x^2 + y^2) + \frac{1}{\Delta}\left(\frac{\beta\,q_0}{r_0} + \frac{q_1}{r_1} + \frac{q_1}{r_2}\right),$$

in which

$$\begin{aligned} r_0 &= \sqrt{x^2 + y^2}, \\ r_1 &= \sqrt{(x - 1/2)^2 + y^2}, \\ r_2 &= \sqrt{(x + 1/2)^2 + y^2}. \end{aligned}$$

Equilibrium points

$$U_x = x - \frac{1}{\Delta}\left[\frac{\beta\, q_0}{r_0^3}x + \frac{q_1}{r_1^3}\left(x - \tfrac{1}{2}\right) + \frac{q_1}{r_2^3}\left(x + \tfrac{1}{2}\right)\right] \quad = 0,$$

$$U_y = y\left[1 - \frac{1}{\Delta}\left(\frac{\beta\, q_0}{r_0^3} + \frac{q_1}{r_1^3} + \frac{q_1}{r_2^3}\right)\right] \quad = 0.$$

Two types of solutions:

- triangular points when $y \neq 0$
- collinear points when $y = 0$

Triangular points $(y \neq 0)$

Are defined by the value of y verifying the equation

$$\Delta = \left[\frac{\beta\, q_0}{y^3} + \frac{2\, q_1}{(y^2 + 1/4)^{3/2}} \right].$$

Proposition

The number of triangular equilibria is

a) *2, when $q_0 > 0$,*

b) *0, when $q_0 < 0$ and $q_1 < 0$,*

c) *if $q_0 < 0$ and $q_1 > 0$, for each value of β there exists a function $\Psi_\beta(q_0)$ such that the number of equilibria is*

- **c.1)** *0, when $q_1 < \Psi_\beta(q_0)$,*
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Collinear points ($y = 0$)

The second equation is always satisfied. Then,

$$x - \frac{1}{\Delta}\left[\frac{\beta\, q_0}{r_0^3}x + \frac{q_1}{r_1^3}\left(x - \frac{1}{2}\right) + \frac{q_1}{r_2^3}\left(x + \frac{1}{2}\right)\right] = 0,$$

where now $r_0 = |x|, \quad r_1 = |x - 1/2|, \quad r_2 = |x + 1/2|.$

There are four intervals in which the collinear equilibria can appear:

- the outer positive interval $O = (1/2, \infty)$
- the inner positive interval $I = (0, 1/2)$
- the inner negative interval $I_n = (-1/2, 0)$
- the outer negative interval $O_n = (-\infty, -1/2)$

Outer collinear equilibria

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The number of positive outer collinear equilibria, regardless of the value of β, is

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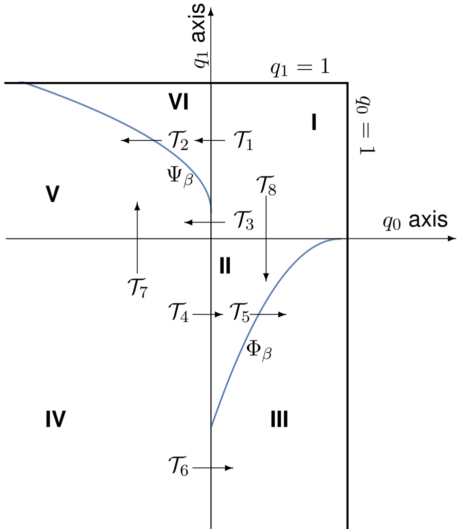
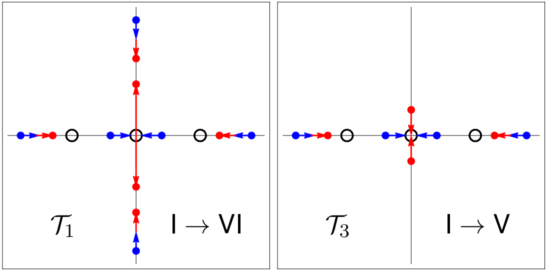
Region	For any β		Equilibria
			$(T; N_O, N_I, P_I, P_O)$
I	$q_0 \in [0,1]$	$q_1 \in [0,1]$	(2; 1,1,1,1)
II	$q_0 \in [0,1]$	$q_1 \in (\Phi_\beta(q_0), 0)$	(2; 0,2,2,0)
Φ_β curve	$q_0 \in [0,1]$	$q_1 = \Phi_\beta(q_0)$	(2; 0,1,1,0)
III	$q_0 \in [0,1]$	$q_1 < \Phi_\beta(q_0)$	(2; 0,0,0,0)
IV	$q_0 < 0$	$q_1 < 0$	(0; 0,1,1,0)
V	$q_0 < 0$	$q_1 \in (0, \Psi_\beta(q_0))$	(0; 1,0,0,1)
Ψ_β curve	$q_0 < 0$	$q_1 = \Psi_\beta(q_0)$	(2; 1,0,0,1)
VI	$q_0 < 0$	$q_1 > \Psi_\beta(q_0)$	(4; 1,0,0,1)

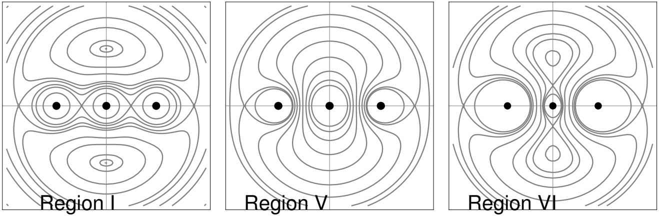
Six regions and two bifurcation curves



Each $\mathcal{T}_i$ represents a transition between adjacent regions.

Transitions between adjacent regions (a)





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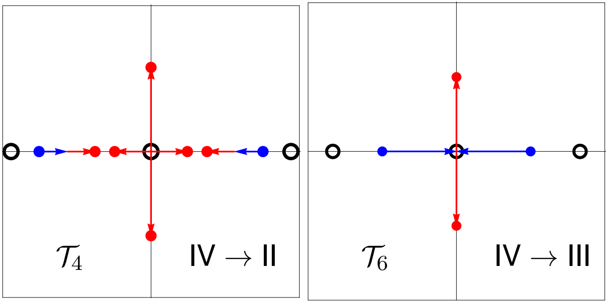
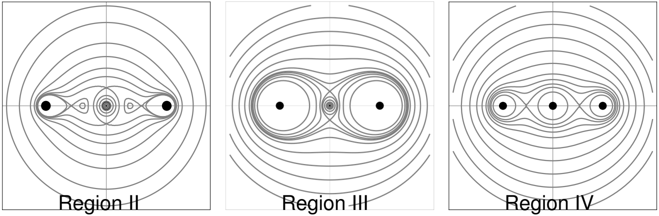
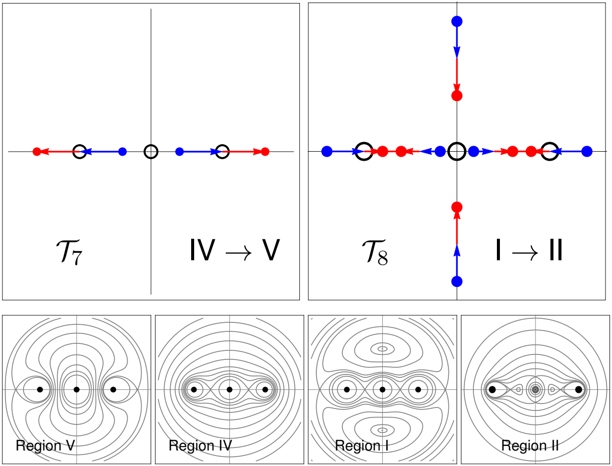




Transitions between adjacent regions (c)



Transition $\mathcal{T}_2$

Transition from region VI to region V.
Through a saddle-node bifurcation.

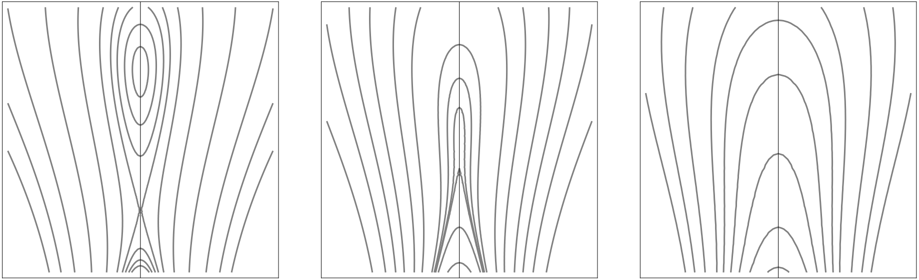
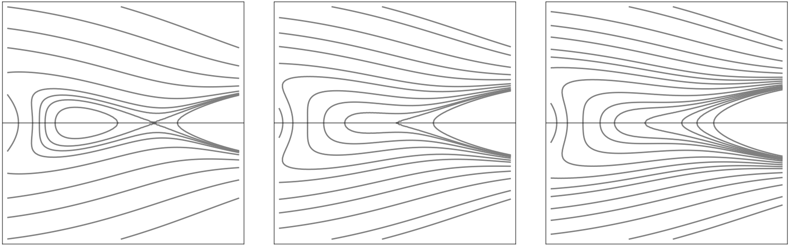


Figure: Transition $\mathcal{T}_2$: saddle-node bifurcation of triangular equilibrium on curve $q_1 = \Psi_\beta(q_0)$. Left: two equilibria in region VI. Center: one cusp equilibrium in bifurcation line. Right: no equilibrium in region V

Transition $\mathcal{T}_5$ from region II to region III

It appears when the value of q_0 crosses the bifurcation line.
The two inner collinear equilibria approach and collide into a cup equilibrium point when $q_0 = \Phi_\beta(q_1)$. After that, the equilibrium disappears.



Linear stability of the equilibrium points

It is characterized by the roots of the characteristic equation:

$$\lambda^4 + a\,\lambda^2 + b = 0,$$

where coefficients

$$a = 4 \,-\, U_{xx} \,-\, U_{yy}, \quad b = U_{xx}\,U_{yy} - U_{xy}^2$$

depend on the three parameters β, q_0, q_1 and the coordinates, $x(\beta, q_0, q_1)$ and $y(\beta, q_0, q_1)$, of the equilibrium points

Stability of triangular equilibria

Coefficients of the characteristic equation, in this case, are

$$\begin{aligned} a &= 4-(1+\frac{T_x}{\Delta_T})-(1+\frac{T_y}{\Delta_T}) = 2-\frac{T_x+T_y}{\Delta_T} = 1, \\ b &= (1+\frac{T_x}{\Delta_T})(1+\frac{T_y}{\Delta_T}) = 2+\frac{T_xT_y}{\Delta_T^2}. \end{aligned}$$

Proposition

In region I there is a β-parameter family of bifurcation lines, $q_1 = \sigma_\beta(q_0)$, such as, for a given β, the line σ_β separates region Ia, $q_1 > \sigma_\beta(q_0)$, from region Ib, $q_1 < \sigma_\beta(q_0)$.
In region Ia, the equilibrium is unstable, while in region Ib is stable.

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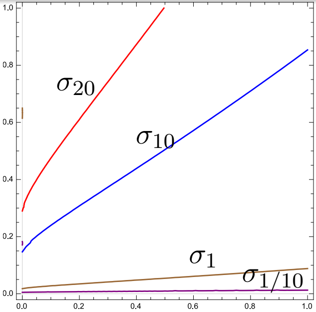
In region Ia, the triangular equilibrium is unstable, while in region Ib is stable.



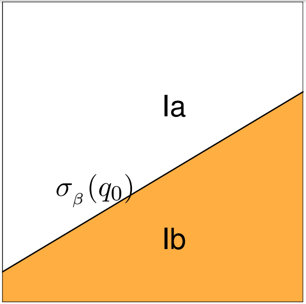


Figure: Left: Bifurcation lines for $\beta = 1/10, 1, 10, 20$. Right: Zones Ia and Ib The equilibria are stable on the colored region Ib.

Proposition

Triangular equilibrium points in regions II and III are unstable.

Proposition

Inside region VI there exists a β-parameter family of bifurcation lines, $q_1 = \psi_\beta(q_0)$, such as, for a given β, the line ψ_β separates region VIa $=\{q_1 \mid \psi_\beta(q_0) < q_1 < \psi_\beta(q_0)\}$, from region VIb = $\{q_1 \mid \psi_\beta(q_0) < q_1\}$. In region VIa the triangular equilibrium farthest to the origin is stable, while in VIb it is unstable; in the rest of region VI it is always unstable.

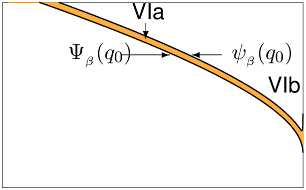


Figure: The equilibrium point is stable on the colored region VIa.

Proposition

Triangular equilibrium points in regions II and III are unstable.

Proposition

Inside region VI there exists a β-parameter family of bifurcation lines, $q_1 = \psi_\beta(q_0)$, such as, for a given β, the line ψ_β separates region VIa $=\{q_1 \mid \psi_\beta(q_0) < q_1 < \psi_\beta(q_0)\}$, from region VIb $= \{q_1 \mid \psi_\beta(q_0) < q_1\}$. In region VIa the triangular equilibrium farthest to the origin is stable, while in VIb it is unstable; in the rest of region VI it is always unstable.

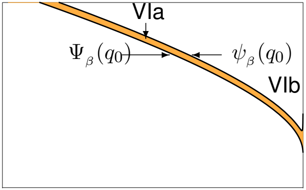


Figure: The equilibrium point is stable on the colored region VIa.

Stability of collinear equilibria (a)

Two kind of positive collinear equilibria: the outer equilibria, in $O = (1/2, \infty)$, and the inner ones, in $I = (0, 1/2)$.

Proposition

Outer collinear equilibria in regions I, V and VI are unstable.

Proposition

Inner collinear equilibria in regions I and IV are unstable.

Stability of collinear equilibria (a)

Two kind of positive collinear equilibria: the outer equilibria, in $O = (1/2, \infty)$, and the inner ones, in $I = (0, 1/2)$.

Proposition

Outer collinear equilibria in regions I, V and VI are unstable.

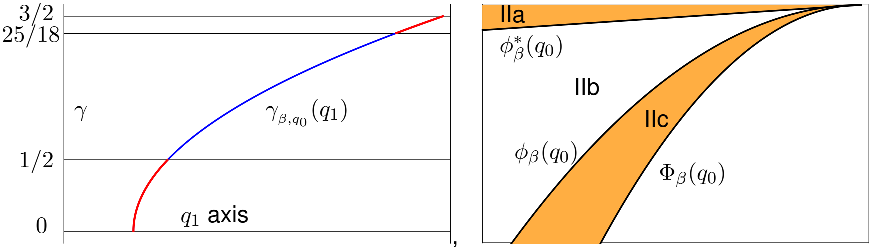
Proposition

Inner collinear equilibria in regions I and IV are unstable.

Stability of collinear equilibria (b)

Region II: there are two β*-parameter families of bifurcation lines* $q_1 = \phi_\beta(q_0)$ *and* $q_1 = \phi^*_\beta(q_0)$ *that separates region im*
IIb $= \{q_1 \,|\, \phi_\beta(q_0) < q_1 < \phi^*_\beta(q_0)\}$,
IIa $= \{q_1 \,|\, q_1 > \phi^*_\beta(q_0)\}$ *IIc* $= \{q_1 \,|\, \Phi_\beta(q_0) < q_1 < \phi_\beta(q_0)\}$.

Inner collinear equil.	unstable	stable
nearest to the origin	in IIb	in IIa and IIc
farthest to the origin	in II	

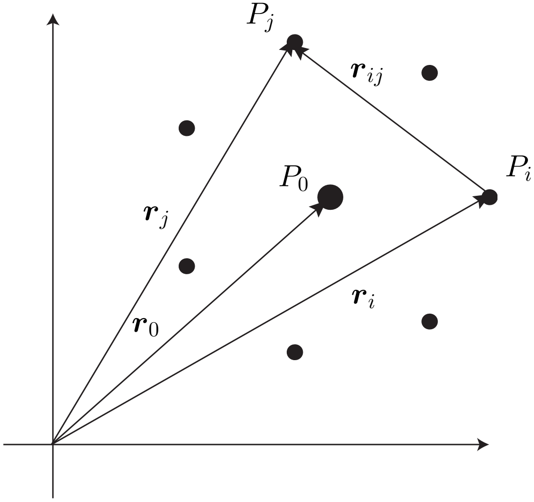


Another problem

Periodic solutions and their parametric evolution in the planar case of the $(n+1)$ ring problem with oblateness

Problem:

- ⋆ n equal bodies of mass m at the vertices of a regular n- gon
- ⋆ a central body of mass $m_0 = \beta m$ at the center of the n- gon
- ⋆ the n- gon is rotating on its plane
- ⋆ an infinitesimal mass orbiting around the bodies



An old problem (Maxwell, 1859), Saturn ring

Recent interest:

Astrodynamics, Dynamical systems, . . .

– Scheeres (1992)

– Kalvouridis (1999, 2000, . . .)

– Arribas and Elipe (2004, 2005)

– Pinotsis (2005)

– – –

New problem:

Consider the central body spheroid or radiating source

Consequences:

- ⋆ A new parameter ϵ
- ⋆ New equilibria
- ⋆ New bifurcations
- ⋆ Dynamics much richer

Equations of motion

$$\ddot{x} - 2\dot{y} = -\frac{\partial U}{\partial x}, \qquad \ddot{y} + 2\dot{x} = -\frac{\partial U}{\partial y},$$

effective potential

$$U(x,y) = -\frac{1}{2}(x^2+y^2) - \frac{1}{\Delta}\left(\beta\left(\frac{1}{r_0} + \frac{\epsilon}{r_0^2}\right) + \sum_{i=1}^{n}\frac{1}{r_i}\right),$$

Introduce the angle $\theta = \pi/n$, then

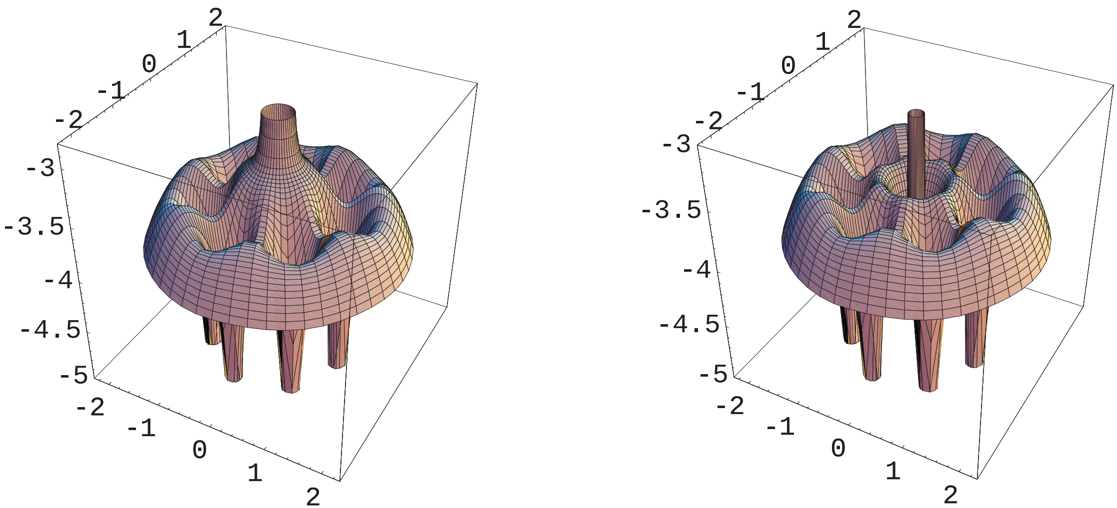
$$x_i = \frac{1}{M}\cos 2(i-1)\theta, \qquad y_i = \frac{1}{M}\sin 2(i-1)\theta,$$

with $M = 2\sin\theta$, and $\Delta = M(\Lambda + \beta M^2)$ a constant

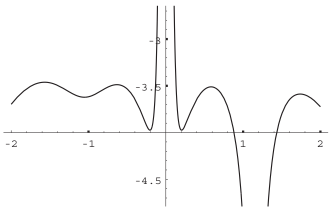
$$\Lambda = \sin^2\theta \sum_{i=2}^{n}\frac{1}{\sin(i-1)\theta}.$$

The parameter $\epsilon \lesseqgtr 0$,

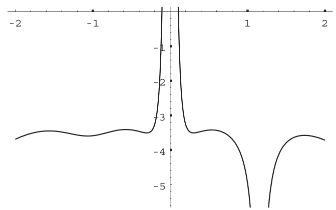
Jacobian first integral $C = 2U + (\dot{x}^2 + \dot{y}^2)$


2
1
0
-1
-2
-3
-3.5
-4
-4.5
-5
-2
-1
0
1
2
2
1
0
-1
-2
-3
-3.5
-4
-4.5
-5
-2
-1
0
1
2

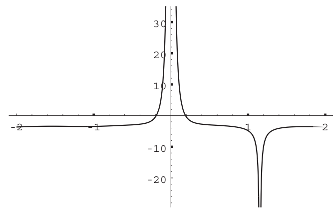
ε=–0.1



ε=–0.14



ε=–0.3



Periodic orbits

$\beta = 2$, $n = 7$, $\epsilon = -0.1, -0.14$ and -0.3;

Grid search in the plane C—x_0

Periodic orbits with x- symmetry, then, $y_0 = 0$, $\dot{x}_0 = 0$

$$C = (\dot{x}_0^2 + \dot{y}_0^2) + 2U(x_0, y_0) \Longrightarrow C, x_0, y_0 = 0, \dot{x}_0 = 0, \quad \dot{y}_0$$

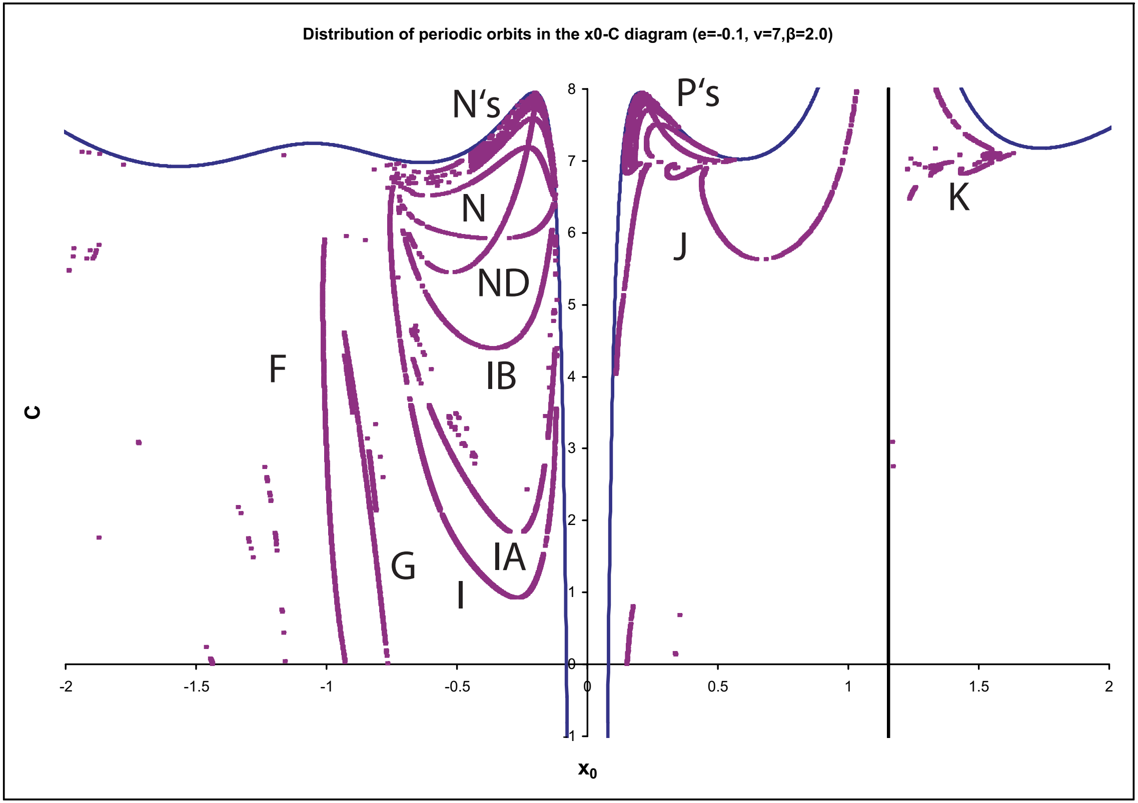
Integrate until the orbit crosses the x- axis $(T/2)$;

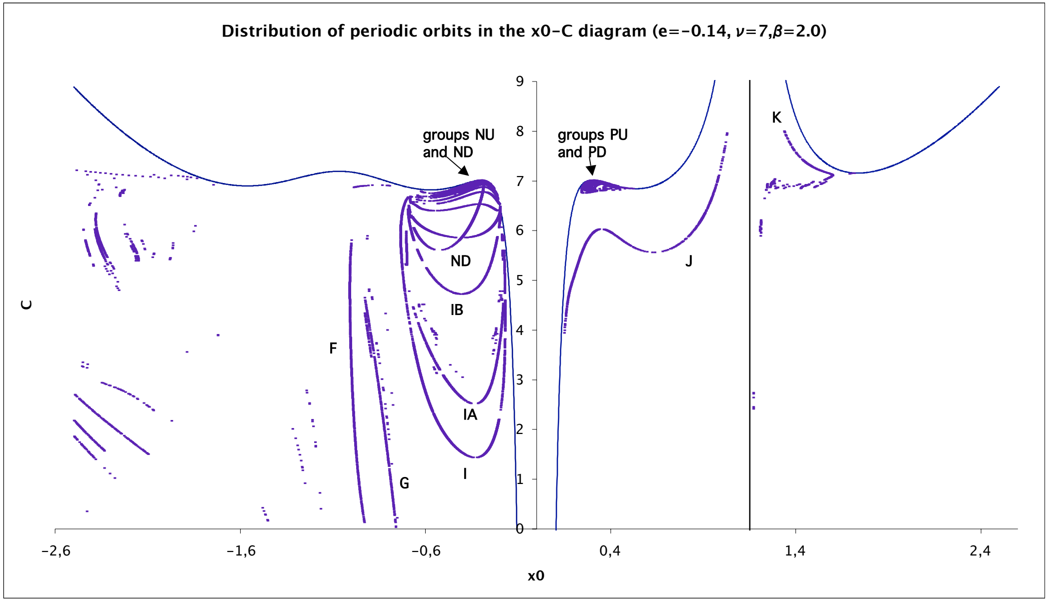
Keep the value $\dot{x}(T/2)$

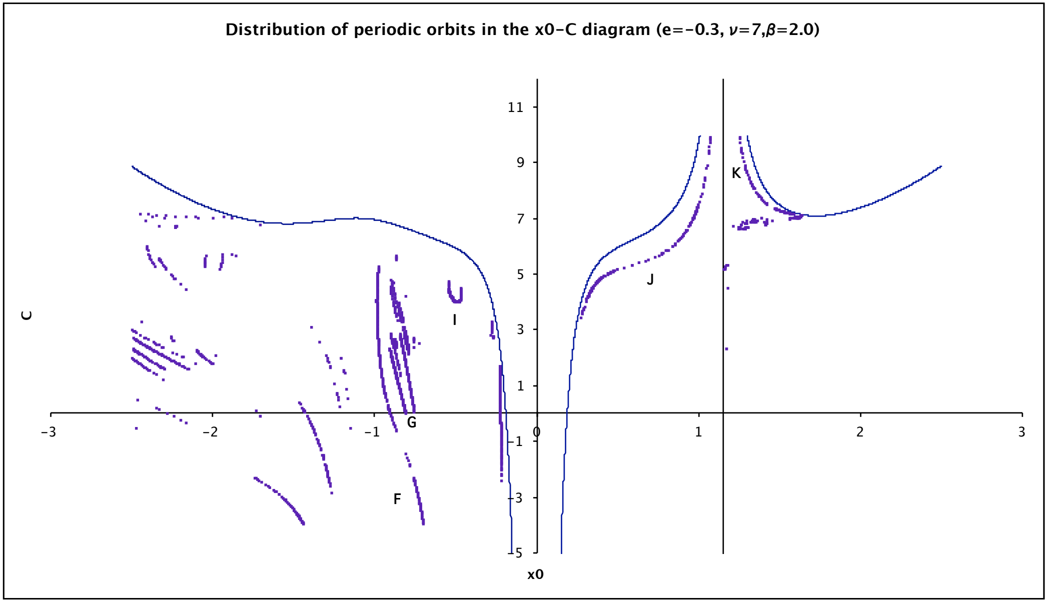
Repeat for $x_0' = x_0 + \delta x_0$

Check the sign $\dot{x}(T/2)\dot{x}'(T/2)$ until negative

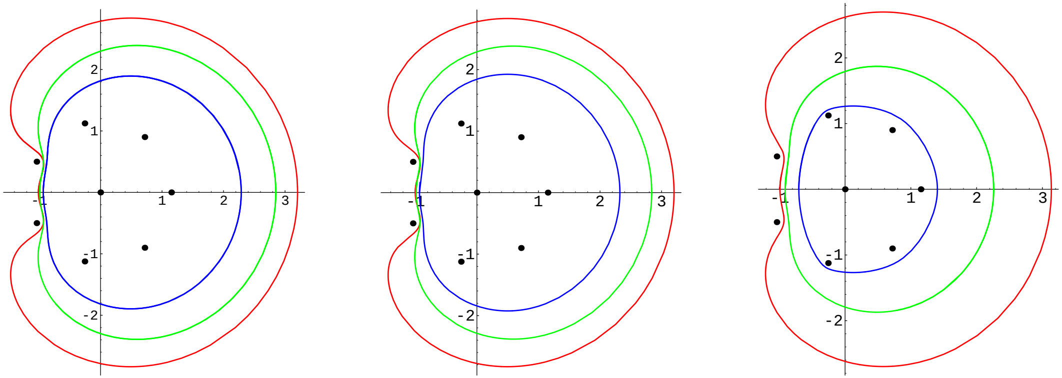
Repeat the procedure for another $C + \delta C$


Distribution of periodic orbits in the x0-C diagram (e=-0.1, v=7,β=2.0)
N's
P's
N
J
K
ND
IB
F
G
IA
I
C
x0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
8
7
6
5
4
3
2
1
0
-1

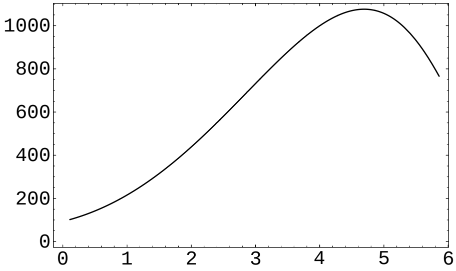

Distribution of periodic orbits in the x0–C diagram (e=−0.14, ν=7,β=2.0)
groups NU
and ND
groups PU
and PD
K
ND
IB
F
IA
G
I
J
C
x0
−2,6
−1,6
−0,6
0,4
1,4
2,4
0
1
2
3
4
5
6
7
8
9


Distribution of periodic orbits in the x0–C diagram (e=−0.3, ν=7,β=2.0)
C
x0
−3
−2
−1
0
1
2
3
11
9
7
5
3
1
−1
−3
−5
F
G
I
J
K

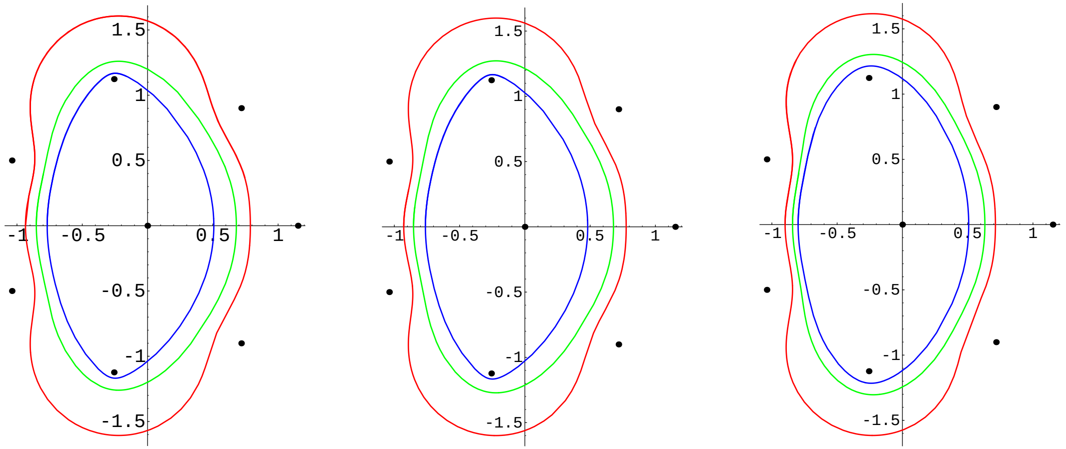
Family F



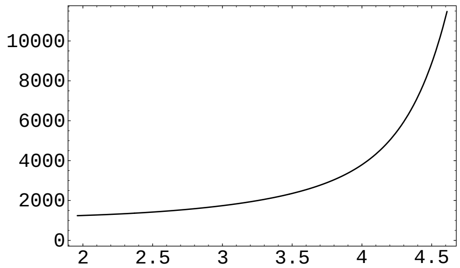
Stability



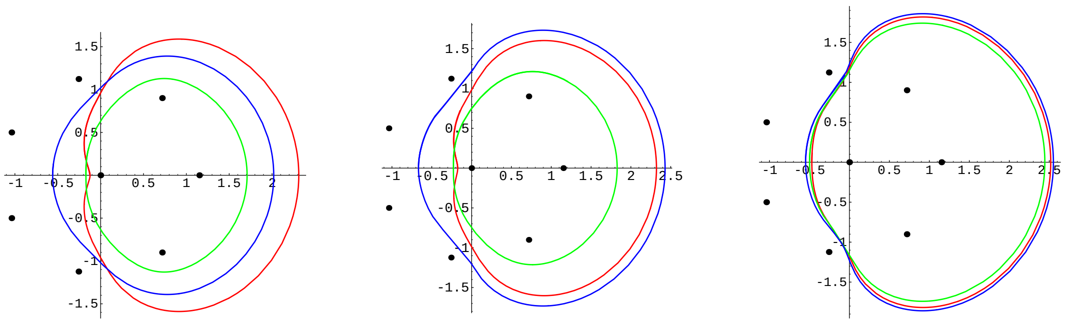
Family G



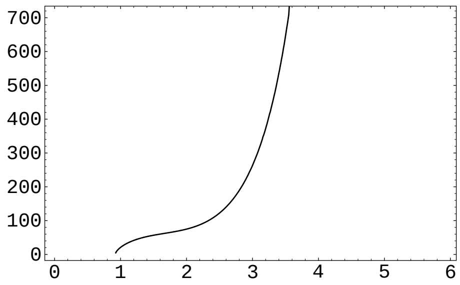
Stability



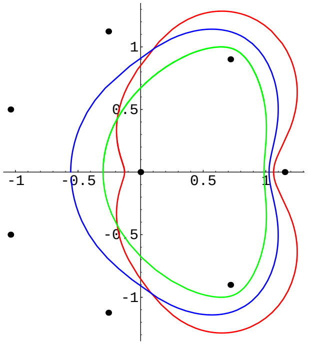
Family I

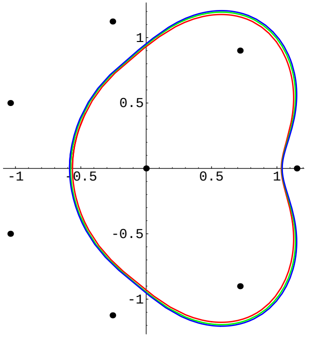


Stability

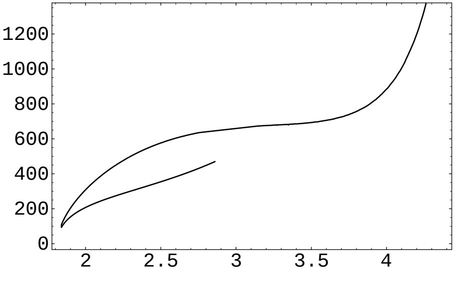


Family IA

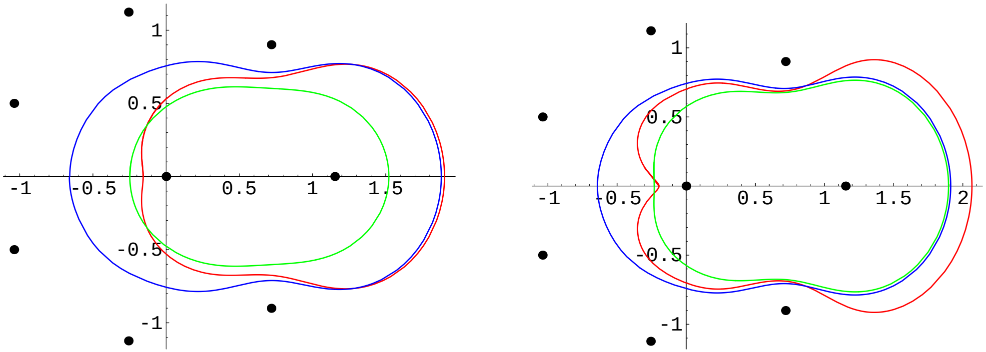




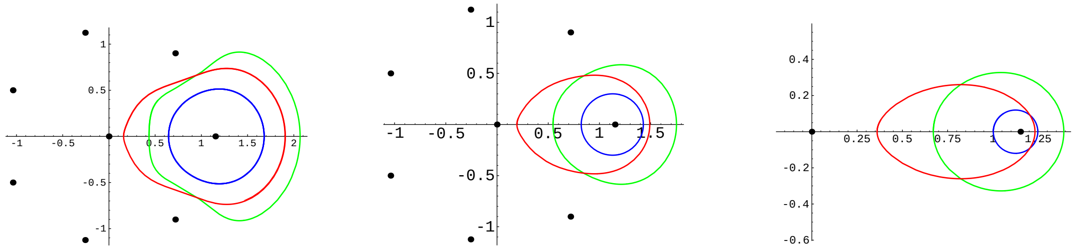
Stability



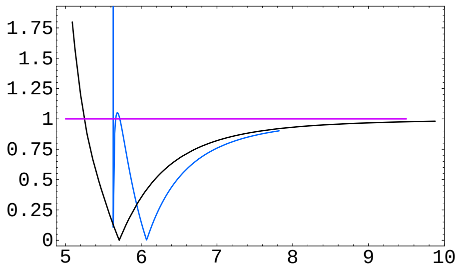
Family IB



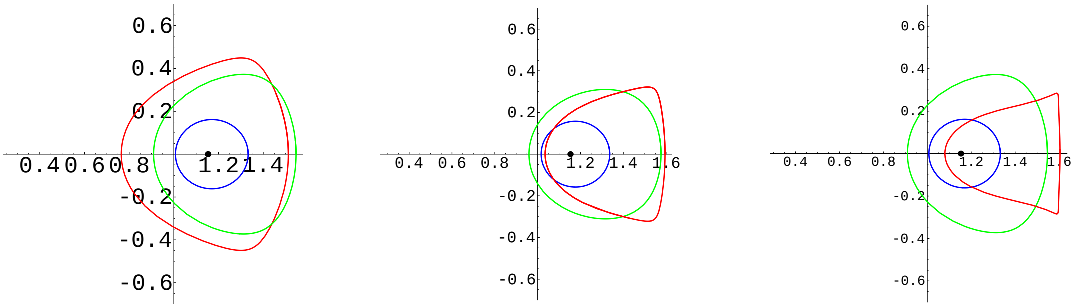
Family J



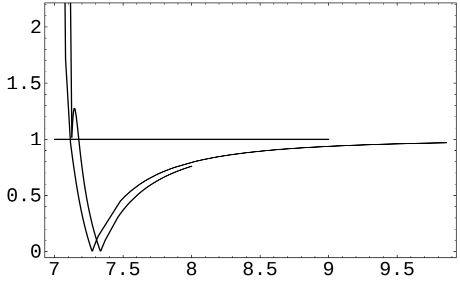
Stability



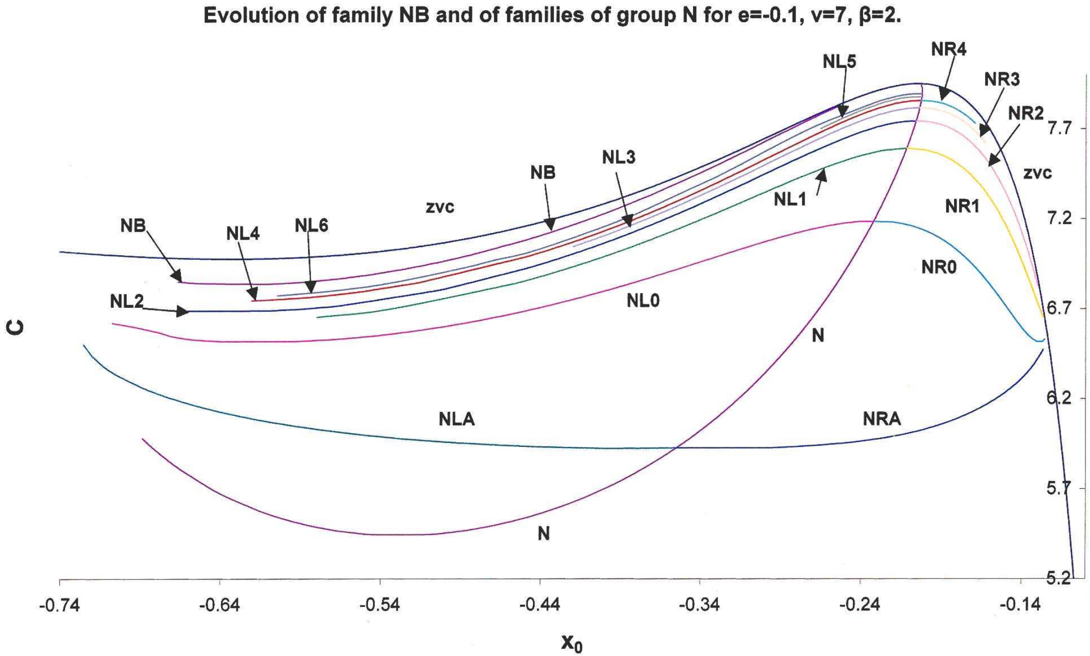
Family K

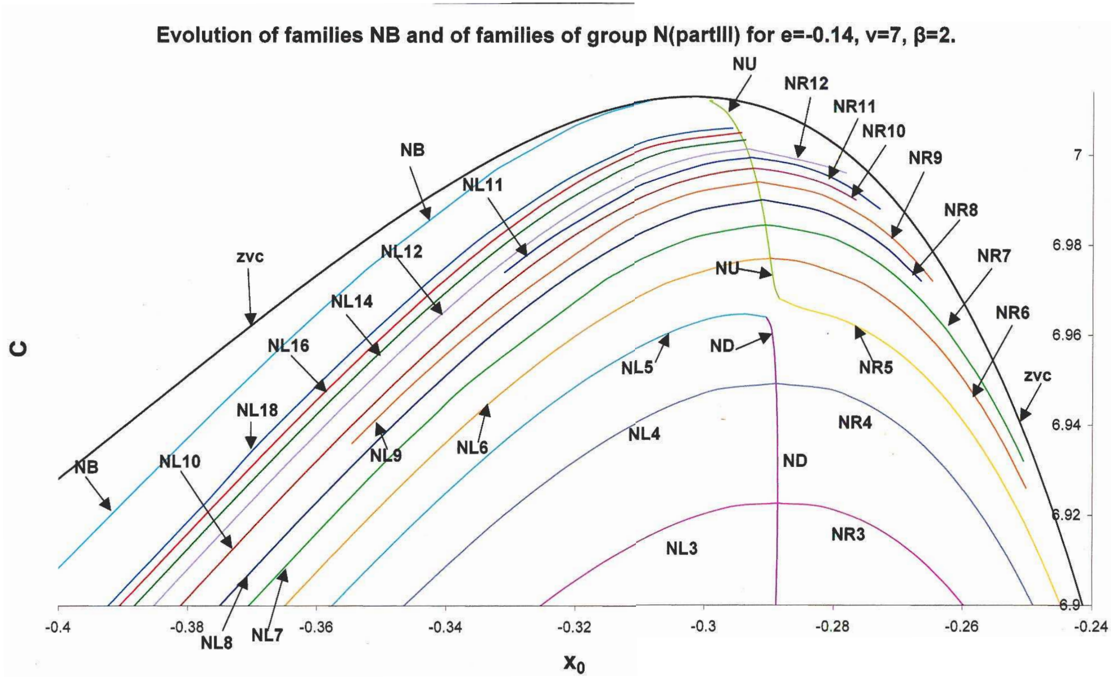


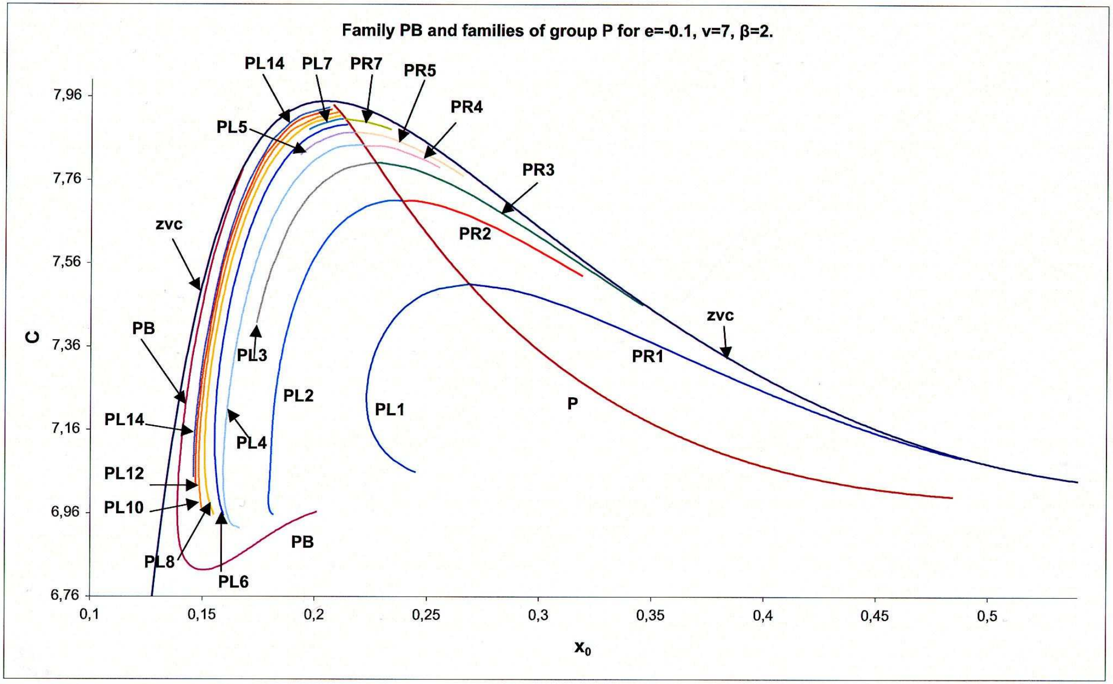
Stability

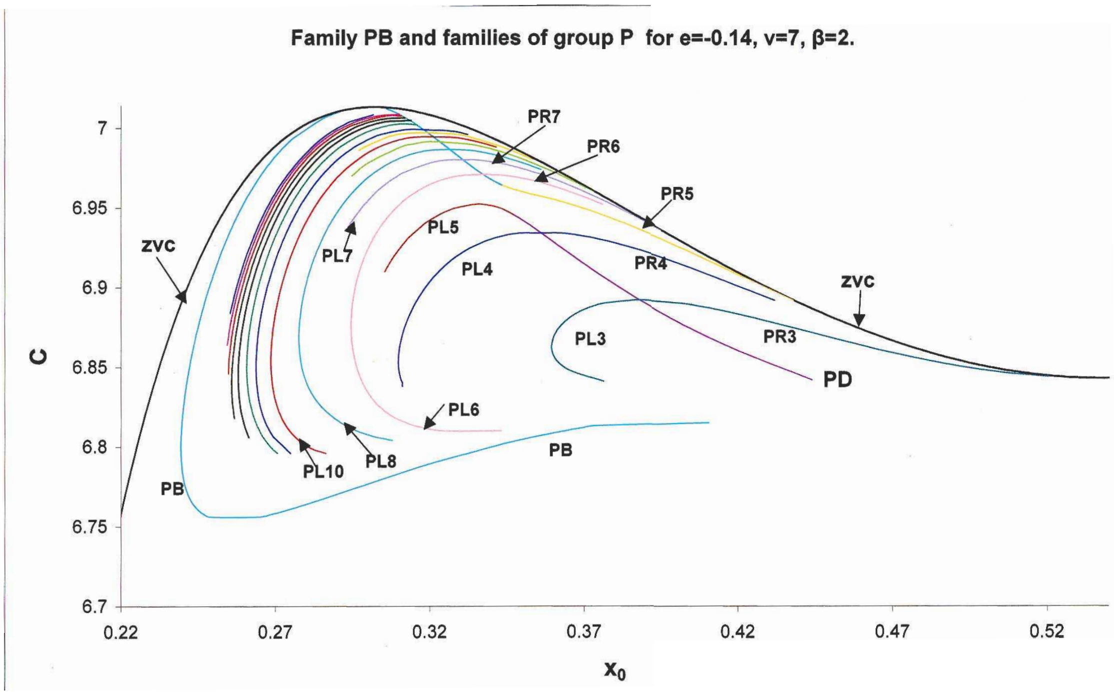


Groups N's and P's

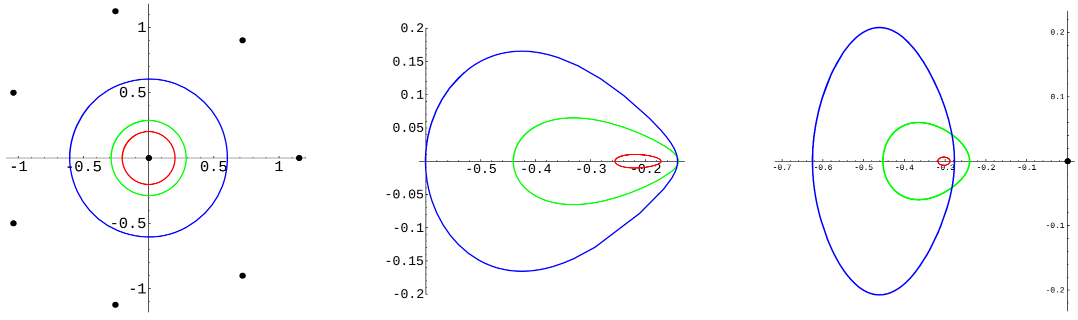



Evolution of families NB and of families of group N(partIII) for e=-0.14, v=7, β=2.
NU
NR12
NR11
NR10
NR9
NR8
NR7
NR6
NR5
NR4
NR3
NB
NL11
NL12
NL14
NL16
NL18
NL10
NL9
NL8
NL7
NL6
NL5
NL4
NL3
ND
zvc
C
x0
-0.4
-0.38
-0.36
-0.34
-0.32
-0.3
-0.28
-0.26
-0.24
7
6.98
6.96
6.94
6.92
6.9

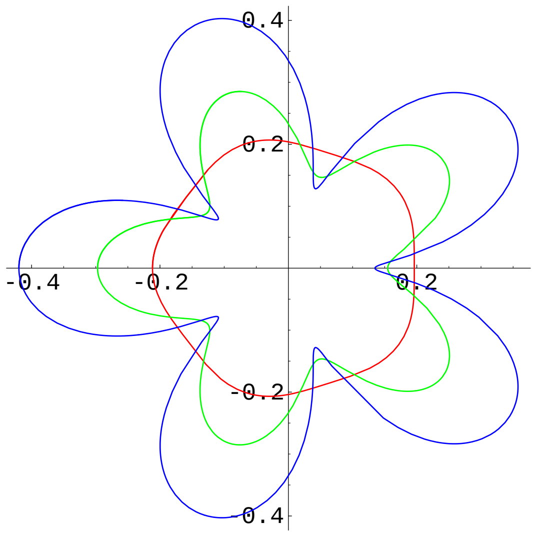

Family PB and families of group P for e=-0.1, v=7, β=2.
PL14
PL7
PR7
PR5
PR4
PL5
PR3
zvc
PR2
PB
PL3
PR1
PL2
PL1
P
PL14
PL4
PL12
PL10
PB
PL8
PL6
C
7,96
7,76
7,56
7,36
7,16
6,96
6,76
0,1
0,15
0,2
0,25
0,3
0,35
0,4
0,45
0,5
x0

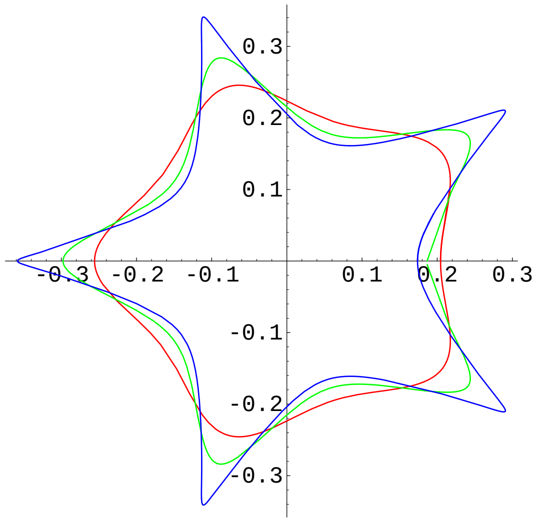

Family PB and families of group P for e=-0.14, v=7, β=2.
PR7
PR6
PR5
PR4
PR3
PD
ZVC
ZVC
PL3
PL4
PL5
PL6
PL7
PL8
PL10
PB
PB
C
7
6.95
6.9
6.85
6.8
6.75
6.7
0.22
0.27
0.32
0.37
0.42
0.47
0.52
x0

Families N and NB

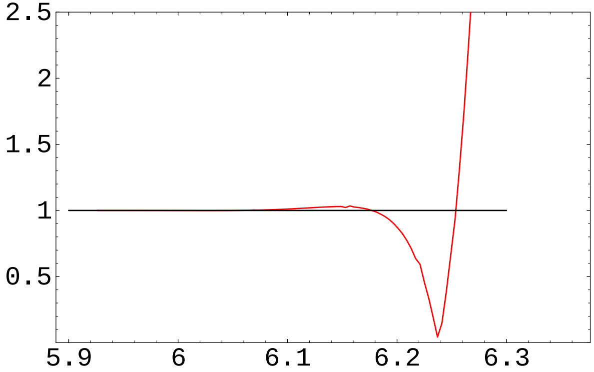


Families NL and PL

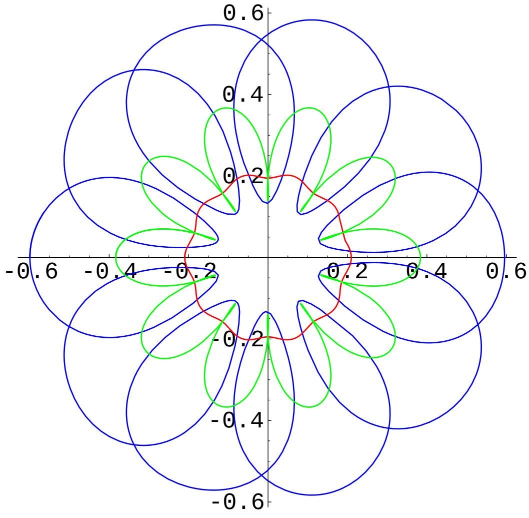


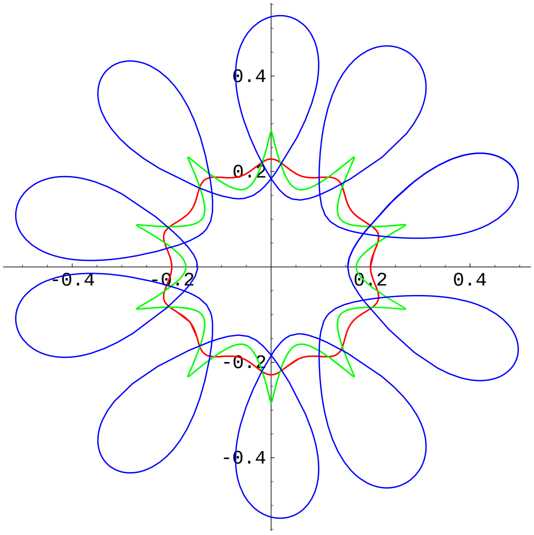


Stability



Families NL and PL (cont.)





Families PL and PR