Monomial multisummability through Borel-Laplace transforms. Applications to singularly perturbed differential equations and Pfaffian systems

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The notion of *monomial summability* was introduced in the paper:

Canalis-Durand M.,Mozo-Fernández J., Schäfke R.: *Monomial summability and doubly singular differential equations.* J. Differential Equations, vol. 233, (2007) 485-511.,

in order to study the formal solutions of the *doubly singular equation*

$$\varepsilon^q x^{p+1} \frac{dy}{dx} = F(x, \varepsilon, y).$$

The method combines the variables x and ε in the new one $t = x^p \varepsilon^q$, corresponding to the source of divergence of the solutions.

Formal setting

We work in the $\mathbb{C}-$algebra $\mathbb{C}[[x,\varepsilon]]$ of formal power series in two variables with complex coefficients.

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The series $\hat{f}=\sum a_{n,m}x^n\varepsilon^m$ is $s-$Gevrey in the monomial $x^p\varepsilon^q$ if and only if there are positive constants C, A satisfying

$$|a_{n,m}|\leq CA^{n+m}\min\{n!^{s/p}, m!^{s/q}\},$$

for all $n,m\in\mathbb{N}$.

Analytic setting

A *sector in the monomial* $x^p\varepsilon^q$ is a set defined as

$$\begin{aligned}\Pi_{p,q}(a,b,r) &= S_{p,q}(d,b-a,r)\\ &= \left\{(x,\varepsilon)\in\mathbb{C}^2 \mid 0<|x|^p,|\varepsilon|^q<r,\ a<\arg(x^p\varepsilon^q)<b\right\},\end{aligned}$$

where $a,b\in\mathbb{R}$ with $a<b$ and $r>0$. The number r is called the *radius*, $b-a$ the *opening* and $d=(b+a)/2$ the *bisecting direction* of the sector, respectively.

Figure: $\Pi_{p,q}(\pi/2, 3\pi/2, r)$ for $p = 2,\ q = 3$.

Asymptotic expansions in a monomial

Definition

Let $f \in \mathcal{O}(\Pi_{p,q})$, $\Pi_{p,q} = \Pi_{p,q}(a,b,r)$ and $\hat{f} \in \mathcal{C}$ with $\hat{T}_{p,q}\hat{f} = \sum f_n t^n \in \mathcal{E}_{r'}^{(p,q)}[[t]]$ for some $0 < r' \leq r$.

We say that f *has* $\hat{f}$ *as asymptotic expansion in* $x^p \varepsilon^q$ *on* $\Pi_{p,q}$ ($f \sim^{(p,q)} \hat{f}$ on $\Pi_{p,q}$) if for every subsector $\widetilde{\Pi}_{p,q}$ and $N \in \mathbb{N}$ there is a positive constant C_N such that for $(x,\varepsilon) \in \widetilde{\Pi}_{p,q}$ we have:

$$\left| f(x,\varepsilon) - \sum_{n=0}^{N-1} f_n(x,\varepsilon)(x^p \varepsilon^q)^n \right| \leq C_N |x^p \varepsilon^q|^N. \tag{1}$$

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The asymptotic expansion is said to be of $s-$*Gevrey type* ($f \sim_s^{(p,q)} \hat{f}$ on $\Pi_{p,q}$) if it is possible to choose $C_N = CA^N N!^s$ for some C, A independent of N. In this case $\hat{f} \in \mathbb{C}[[x,\varepsilon]]_s^{(p,q)}$.

Monomial summability

Definition

Let $k>0$ and $\hat{f}\in\mathcal{C}$ be given. We say that $\hat{f}$ is $k-$*summable in the monomial* $x^p\varepsilon^q$ *in the direction* d if there is a sector $\Pi_{p,q}(a,b,r)$ bisected by d with opening $b-a>\pi/k$ and $f\in\mathcal{O}(\Pi_{p,q}(a,b,r))$ with $f\sim_{1/k}^{(p,q)}\hat{f}$ on $\Pi_{p,q}(a,b,r)$.

We simply say that $\hat{f}$ is $k-$*summable in the monomial* $x^p\varepsilon^q$ if it is $k-$summable in the monomial $x^p\varepsilon^q$ in every direction d, with finitely many exceptions mod. 2π.

- $\mathbb{C}\{x,\varepsilon\}_{1/k,d}^{(p,q)}$: $k-$summable series in $x^p\varepsilon^q$ in the direction d,
- $\mathbb{C}\{x,\varepsilon\}_{1/k}^{(p,q)}$: $k-$summable series in $x^p\varepsilon^q$.

Monomial summability and blow-ups

Consider the charts of the classical blow-up of the origin of $\mathbb{C}^2$, given by

$$\pi_1(x,\varepsilon)=(x\varepsilon,\varepsilon),\qquad \pi_2(x,\varepsilon)=(x,x\varepsilon).$$

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Proposition

Let $\hat{f}\in\mathbb{C}\{x,\varepsilon\}_{1/k,d}^{(p,q)}$ *with sum* f. *Then* $\hat{f}\circ\pi_1\in\mathbb{C}\{x,\varepsilon\}_{1/k,d}^{(p,p+q)}$, $\hat{f}\circ\pi_2\in\mathbb{C}\{x,\varepsilon\}_{1/k,d}^{(p+q,q)}$ *and have sums* $f\circ\pi_1$, $f\circ\pi_2$, *respectively.*

Tauberian properties for monomial summability

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If $\hat{f} \in \mathbb{C}\{x,\varepsilon\}_{1/k}^{(p,q)}$ *has no singular directions then* $\hat{f} \in \mathbb{C}\{x,\varepsilon\}$.

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Theorem

Let $k, l > 0$ *be positive real numbers and let* $x^p\varepsilon^q$ *and* $x^{p'}\varepsilon^{q'}$ *be two monomials. Then* $\mathbb{C}\{x,\varepsilon\}_{1/k}^{(p,q)} \cap \mathbb{C}\{x,\varepsilon\}_{1/l}^{(p',q')} = \mathbb{C}\{x,\varepsilon\}$*, except in the case* $p/p' = q/q' = l/k$ *where* $\mathbb{C}\{x,\varepsilon\}_{1/k}^{(p,q)} = \mathbb{C}\{x,\varepsilon\}_{1/l}^{(p',q')}$.

Borel transform

Definition

Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$. The $k-$*Borel transform associated to the monomial* $x^p\varepsilon^q$ *with weight* (s_1, s_2) of a function f is defined by the formula

$$\mathcal{B}_{k,(s_1,s_2)}^{(p,q)}(f)(\xi,\upsilon) = \frac{(\xi^p\upsilon^q)^{-k}}{2\pi i}\int_\gamma f(\xi u^{-s_1/pk}, \upsilon u^{-s_2/qk})e^u du,$$

where γ denotes a Hankel path.

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where γ denotes a Hankel path.

The formula is adapted from the papers:

Balser W.: *Summability of power series in several variables, with applications to singular perturbation problems and partial differential equations*. Ann. Fac. Sci. Toulouse Math, vol. XIV, n°4 (2005) 593-608.

Balser W., Mozo-Fernández J.: *Multisummability of Formal Solutions of Singular Perturbation Problems*. J. Differential Equations, vol. 183, (2002) 526-545.

Laplace transform

Definition

Let $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $|\alpha| < \pi/2$. The $k-$*Laplace transform associated to the monomial* $x^p\varepsilon^q$ *with weight* (s_1, s_2) *in direction* α of a function f is defined by the formula

$$\mathcal{L}^{(p,q)}_{k,\alpha,(s_1,s_2)}(f)(x,\varepsilon) = (x^p\varepsilon^q)^k \int_0^{e^{i\alpha}\infty} f(xu^{s_1/pk}, \varepsilon u^{s_2/qk})e^{-u}du.$$

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We assume that f has an exponential growth of the form

$$|f(\xi, \upsilon)| \leq C e^{B\max\{|\xi|^{pk/s_1}, |\upsilon|^{qk/s_2}\}}. \tag{2}$$

Monomial Borel-Laplace summation methods

Definition

Let $\hat{f}$ be a $1/k-$Gevrey series in $x^p\varepsilon^q$ and set $\hat{\varphi}_{s_1,s_2} = \hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}((x^p\varepsilon^q)^k\hat{f})$.

We will say that $\hat{f}$ is $k-(s_1,s_2)-$*Borel summable in the monomial* $x^p\varepsilon^q$ *in direction* d if:

1. $\hat{\varphi}_{s_1,s_2}$ can be analytically continued, say as φ_{s_1,s_2}, to a monomial sector of the form $S_{p,q}(d, 2\epsilon, +\infty)$,
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In this case the $k-(s_1,s_2)-$*Borel sum of* $\hat{f}$ in direction d is defined as

$$f(x,\varepsilon) = \frac{1}{(x^p\varepsilon^q)^k}\mathcal{L}^{(p,q)}_{k,(s_1,s_2)}(\varphi_{s_1,s_2})(x,\varepsilon).$$

Monomial summability and Borel-Laplace method

Theorem

Let $\hat{f} \in \mathbb{C}[[x, \varepsilon]]_{1/k}^{(p,q)}$ be a $1/k-$Gevrey series in the monomial $x^p\varepsilon^q$. Then it is equivalent:

1. $\hat{f} \in \mathbb{C}\{x, \varepsilon\}_{1/k,d}^{(p,q)}$,
2. *There are $s_1, s_2 > 0$ with $s_1 + s_2 = 1$ such that $\hat{f}$ is $k - (s_1, s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d.*
3. *For all $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$, $\hat{f}$ is $k - (s_1, s_2)-$Borel summable in the monomial $x^p\varepsilon^q$ in direction d.*

In all cases the corresponding sums coincide.

Applications

Doubly singular equations

Theorem

Consider the singularly perturbed differential equation

$$\varepsilon^q x^{p+1} \frac{dy}{dx} = F(x, \varepsilon, y),$$

where $y \in \mathbb{C}^l$, $p, q \in \mathbb{N}^*$, F *analytic in a neighborhood of* $(0, 0, \mathbf{0})$ *and* $F(0, 0, \mathbf{0}) = \mathbf{0}$.

If $\partial F/\partial y(0, 0, \mathbf{0})$ *is invertible then the previous equation has a unique formal solution* $\hat{y}$. *Furthermore it is* $1-$*summable in* $x^p \varepsilon^q$.

Monomial summability of a family of PDEs

Consider the problem

$$x^p \varepsilon^q \left(\frac{s_1}{p} x \frac{\partial y}{\partial x} + \frac{s_2}{q} \varepsilon \frac{\partial y}{\partial \varepsilon} \right) = C(x, \varepsilon) y(x, \varepsilon) + \gamma(x, \varepsilon), \qquad (3)$$

where $p, q \in \mathbb{N}^*$, $s_1, s_2 > 0$ satisfy $s_1 + s_2 = 1$ and $C \in \mathrm{Mat}(l \times l, \mathbb{C}\{x, \varepsilon\})$, $\gamma \in \mathbb{C}\{x, \varepsilon\}^l$.

Theorem

If $C(0,0)$ is invertible then equation (3) has a unique formal solution $\hat{y}$ and it is $1-$summable in $x^p \varepsilon^q$. Its possible singular directions are the directions passing through the eigenvalues of $C(0,0)$.

Pfaffian system with normal crossings

Consider the following the system of PDEs:

$$\begin{cases} \varepsilon^q x^{p+1} \dfrac{\partial y}{\partial x} = f_1(x, \varepsilon, y), & \text{(4a)} \\ x^{p'} \varepsilon^{q'+1} \dfrac{\partial y}{\partial \varepsilon} = f_2(x, \varepsilon, y), & \text{(4b)} \end{cases}$$

where $p, q, p', q' \in \mathbb{N}^*$, $y \in \mathbb{C}^l$, and f_1, f_2 are analytic in a neighborhood of $(0, 0, \mathbf{0})$.

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where $p, q, p', q' \in \mathbb{N}^*$, $y \in \mathbb{C}^l$, and f_1, f_2 are analytic in a neighborhood of $(0, 0, \mathbf{0})$.

It is called *completely integrable* if $f_1(x, \varepsilon, \mathbf{0}) = f_2(x, \varepsilon, \mathbf{0}) = \mathbf{0}$ and the functions f_1, f_2 satisfy the following identity on their domains of definition:

$$\frac{\partial}{\partial \varepsilon}\left(\frac{1}{x^{p+1}\varepsilon^q}\right) f_1 + \frac{1}{x^{p+1}\varepsilon^q}\left(\frac{\partial f_1}{\partial \varepsilon} + \frac{\partial f_1}{\partial y}\frac{f_2}{x^{p'}\varepsilon^{q'+1}}\right) =$$

$$\frac{\partial}{\partial x}\left(\frac{1}{x^{p'}\varepsilon^{q'+1}}\right) f_2 + \frac{1}{x^{p'}\varepsilon^{q'+1}}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial y}\frac{f_1}{x^{p+1}\varepsilon^q}\right).$$

If the system is completely integrable, $f_1 = Ay + h.o.t.$ and $f_2 = By + h.o.t.$ then A and B satisfy

$$x^{p'}\varepsilon^{q'}\left(\varepsilon\frac{\partial A}{\partial \varepsilon} - qA\right) - x^{p}\varepsilon^{q}\left(x\frac{\partial B}{\partial x} - p'B\right) + AB - BA = 0.$$

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From this equation we have deduced that:

1. If $p' < p$ or $q' < q$ then $A(0,0)$ is nilpotent.
2. If $p < p'$ or $q < q'$ then $B(0,0)$ is nilpotent.
3. If $p = p'$ and $q = q'$, for every eigenvalue μ of $B(0,0)$ there is an eigenvalue λ of $A(0,0)$ such that $q\lambda = p\mu$. The number λ is an eigenvalue of $A(0,0)$, when restricted to its invariant subspace $E_\mu = \{v \in \mathbb{C}^n | (B(0,0) - \mu I)^k v = 0 \text{ for some } k \in \mathbb{N}\}$.

Convergence of solutions for different monomials

Theorem (Gérard-Sibuya)

Consider the completely integrable Pffafian system (4a), (4b), with $q = p' = 0$*. If* $\frac{\partial f_1}{\partial y}(0, 0, \mathbf{0})$ *and* $\frac{\partial f_2}{\partial y}(0, 0, \mathbf{0})$ *are invertible then the Pfaffian system admits a unique analytic solution* y *at the origin such that* $y(0, 0) = \mathbf{0}$*.*

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Theorem

Consider the system (4a), (4b). Suppose the system has a formal solution $\hat{y}$*. If* $\frac{\partial f_1}{\partial y}(0,0,\mathbf{0})$ *and* $\frac{\partial f_2}{\partial y}(0,0,\mathbf{0})$ *are invertible and* $x^p \varepsilon^q \neq x^{p'} \varepsilon^{q'}$ *then* $\hat{y}$ *is convergent.*

Convergence of solutions for the same monomial

For the same monomial, in the linear case we have

$$\begin{cases} \varepsilon^q x^{p+1}\dfrac{\partial y}{\partial x} = A(x,\varepsilon)y(x,\varepsilon)+a(x,\varepsilon), & \text{(5a)}\\ x^p\varepsilon^{q+1}\dfrac{\partial y}{\partial \varepsilon} = B(x,\varepsilon)y(x,\varepsilon)+b(x,\varepsilon), & \text{(5b)}\end{cases}$$

Corollary

If the system has a formal solution $\hat{y}$ and there are $s_1, s_2 > 0$ such that $s_1 + s_2 = 1$ and $s_1/pA(0,0) + s_2/qB(0,0)$ is invertible, then $\hat{y}$ is $1-$summable in $x^p\varepsilon^q$. Its possible singular directions are the directions passing through the eigenvalues of $s_1/pA(0,0) + s_2/qB(0,0)$.

Towards Monomial Multisummability

Examples of series not $k-$summable in any monomial

Theorem

Let $\hat{f}_j \in \mathbb{C}\{x,\varepsilon\}_{1/k_j}^{(p_j,q_j)} \setminus \mathbb{C}\{x,\varepsilon\}$ *be* k_j-*summable in* $x^{p_j}\varepsilon^{q_j}$, *for* $j = 1, ..., r$, *respectively.*

Then $\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_r$ *is* k_0-*summable in* $x^{p_0}\varepsilon^{q_0}$ *if and only if* $k_0 p_0 = k_j p_j$ *and* $k_0 q_0 = k_j q_j$ *for all* $j = 1, ..., r$.

Monomial acceleration operators

Following the same idea as in the one variable case, we formally compute the composition of a Borel and Laplace transform for different indexes. Indeed, we see that

$$\mathcal{B}^{(p',q')}_{l,(s_1',s_2')} \circ \mathcal{L}^{(p,q)}_{k,d,(s_1,s_2)}(f)(\xi,\upsilon) = \frac{(\xi^p\upsilon^q)^k}{(\xi^{p'}\upsilon^{q'})^l}\int_0^{e^{i\theta}\infty} f(\xi\tau^{s_1/pk},\upsilon\tau^{s_2/qk})C_{\Lambda l/k}(\tau)d\tau.$$

where the parameters satisfy the relations

$$\Lambda := \frac{s_1}{s_1'}\frac{p'}{p} = \frac{s_2}{s_2'}\frac{q'}{q},$$

$$s_1(p'q-pq') > \frac{p}{l}(qk-q'l).$$

Let $I = (p', q', l, s'_1, s'_2, p, q, k, s_1, s_2)$, with parameters as before. The *acceleration operator associated to* I *in direction* θ is given by

$$\mathfrak{A}_{I,\theta}(f)(\xi, \upsilon) = \frac{(\xi^p \upsilon^q)^k}{(\xi^{p'} \upsilon^{q'})^l} \int_0^{e^{i\theta}\infty} f(\xi\tau^{s_1/pk}, \upsilon\tau^{s_2/qk}) C_{\Lambda l/k}(\tau) d\tau,$$

and it is defined for functions f with exponential growth

$$|f(\xi, \upsilon)| \leq C e^{M \max\{|\xi|^{\kappa_1}, |\upsilon|^{\kappa_2}\}}, \tag{6}$$

$$\frac{1}{\kappa_1} := \frac{s_1}{pk} - \frac{s'_1}{p'l}, \quad \frac{1}{\kappa_2} := \frac{s_2}{qk} - \frac{s'_2}{q'l}.$$

Monomial multisummability for two levels

Definition

We say that $\hat{f}$ is *I-multisummable in the multidirection* (d_1, d_2) if the following conditions are satisfied

1. $\hat{f}$ is $1/k-$Gevrey in the monomial $x^p \varepsilon^q$,
2. $\hat{\mathcal{B}}^{(p,q)}_{k,(s_1,s_2)}(\hat{f})$ can be analytically continued, say φ, to some $S_{p,q}(d_1, \theta_1, +\infty)$, with exponential growth as in (6).
3. $\mathfrak{A}_I(\varphi)$ can be analytically continued, say ψ, to $S_{p,q}(d_1, \theta_1', +\infty) \cap S_{p',q'}(d_2, \theta_2', +\infty)$, with exponential growth as in (2).

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The *$I-$multisum of $\hat{f}$ in the multidirection* (d_1, d_2) is defined as

$$f(x, \varepsilon) = \mathcal{L}^{(p',q')}_{l,(s_1',s_2')}(\psi)(x, \varepsilon),$$

and it is analytic on some $S_{p,q}(d_1, \theta_1'' + \pi/l\Lambda, r) \cap S_{p',q'}(d_2, \theta_2'' + \pi/l, r)$, where $\theta_1'' < \theta_1'$, $\theta_2'' < \theta_2'$ and r is small enough.

The results of this thesis are contained in the following papers:

1. Carrillo, S. A., Mozo-Fernández, J. *An extension of Borel-Laplace methods and monomial summability*. Submitted to publication. Available at **arxiv.org/abs/1609.07893**.
2. Carrillo, S. A., Mozo-Fernández, J. *Tauberian properties for monomial summability with appliactions to Pffafian systems*. Journal of Differential Equations 261 (2016) pp. 7237-7255. ISSN 0022-0396, http://dx.doi.org/10.1016/j.jde.2016.09.017.

Thanks for your attention.