Local uniformization of codimension one foliations

Miguel Fernández Duque

Joint work with F. Cano and C. A. Roche

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DDays, Salou

Resolution of singularities: varieties

Given a variety M, a resolution of singularities of M is another variety $\tilde{M}$ with a "well-behaved" morphism

$$\tilde{M} \longrightarrow M$$

such that all points of $\tilde{M}$ are non-singular.

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Algebraic geometry setting:

- $M \subset \mathbb{P}^N$ a projective variety;
- $\tilde{M} \longrightarrow M$ a proper morphism, in particular a finite composition of blow-ups with non-singular centers.

Known results in characteristic zero

- Curves: Newton, Noether and others;
- Surfaces: Walker (1935, *Ann. of Math.*);
- 3-folds: Zariski (1944, *Ann. of Math.*);
- Arbitrary dimension: Hironaka (1964, *Ann. of Math.*).

Resolution of singularities: other problems

Let $\mathcal{P}$ be a well-defined "regularity property". For any variety M consider

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- "Well-defined" implies that $\mathrm{Reg}_{\mathcal{P}}(M)$ is a non-empty open subset of M.
- Problem: determine $\tilde{M} \to M$ such that $\mathrm{Reg}_{\mathcal{P}}(\tilde{M}) = \tilde{M}$.

Examples of such problems

- monomialization of morphisms;
- reduction of singularities of vector fields;
- foliations of codimension greater than one;
- monomialization of Liouvillian first integrals;
- subordinated and simultaneous resolutions;
- constructiveness of the resolution process;
- etc.

Known results about foliations

- Surfaces: Seidenberg (1968, *Ann. of Math.*);
- Vector fields in 3-folds: Panazzolo (2006, *Acta Mathematica*), McQuillan & Panazzolo (2013, *Ann. of Math.*);
- Codimension one foliations in 3-folds: Cano (2004, *Ann. of Math.*);
- $\dim \geq 4$: still an open problem.

- The Local Uniformization Problem can be thought as a "local version" of the Resolution of Singularities Problem;
- It consists in solving the singularities "following" a closed subvariety (the center of a valuation);
- In fact, in this way we solve the singularities in a open dense subset of M.;
- In addition, we obtain finitely many local uniformizations in such a way that they "cover" M.

Zariski's Local Uniformization

Zariski's proof of resolution of singularities for 3-folds has two different parts:

- He obtains local uniformization for algebraic varieties in arbitrary dimension (1940 *Ann. of Math.*);
- He also develops a patching method for local uniformizations to obtain resolution of singularities in dimension 3 (1944, *Ann. of Math.*).

More examples:

- Cossart & Piltant (2009, *Journal of Algebra*): 3-folds in positive characteristic.
- Cano, Roche & Spivakovsky (2014, *RACSAM*): Vector fields in dimension 3.

Main result of my PhD. Thesis

There exists Local Uniformization for codimension one foliations in arbitrary dimension.