Analytical tools to study the criticality at the outer boundary of potential centers

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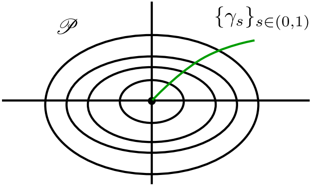
Outline

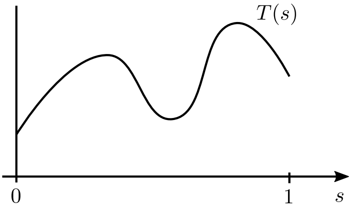
Definitions

Consider planar differential systems

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \end{cases}$$

where f and g are analytic functions on $U \subset \mathbb{R}^2$, with a center at $p \in U$. We denote the *period annulus* of the center by $\mathscr{P}$. If $\{\gamma_s\}_{s\in(0,1)}$ is a parametrization of the set of periodic orbits in $\mathscr{P}$, the *period function* can be written as the map $s \mapsto T(s) := \{\text{period of } \gamma_s\}$.

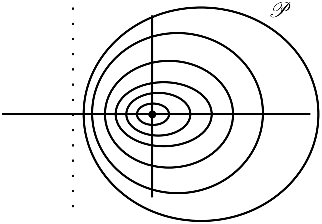


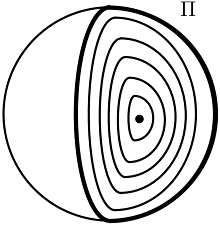


Definitions

Definition 1.1. For a given $\hat{s} \in (0,1)$ we say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center if $\hat{s}$ is an isolated zero of $T'(s)$ of multiplicity k. □

The period annulus $\mathscr{P}$ may be unbounded. We compactify $\mathbb{R}^2$ into $\mathbb{RP}^2$ in order to define properly $\partial\mathscr{P}$ as a compact subset of $\mathbb{RP}^2$.





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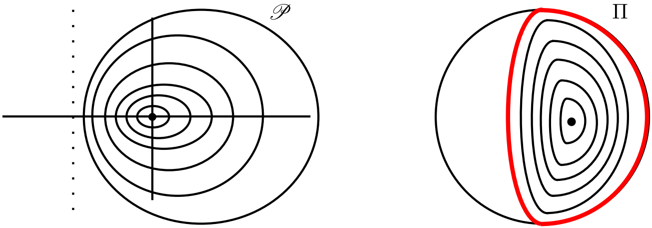

We denote the *outer boundary* of $\mathscr{P}$ by $\Pi := \partial\mathscr{P} \setminus \{p\}$.

Definitions

Let us assume $X_\mu = f_\mu(x,y)\partial_x + g_\mu(x,y)\partial_y$ be a continuous family of analytic centers at p_μ, with $\mu \in \Lambda \subset \mathbb{R}^d$.

Aim

To decompose the parameter space $\Lambda = \cup\Lambda_i$ in such a way that if $\mu_1, \mu_2 \in \Lambda_i$ then T_{μ_1} and T_{μ_2} are qualitatively the same.

P. Mardešić, D. Marín, J. Villadelprat,
The period function of reversible quadratic centers, J. Differential Equations 134, (1997) 216–268.

Three places where a critical periodic orbit may bifurcate from:

- Bifurcation from p_μ.
- Bifurcation from the "interior" of $\mathscr{P}_\mu$.
- Bifurcation from Π_μ.

Outline

Bifurcation from the outer boundary

Definition 2.1. Consider a continuous family $\{X_\mu\}_{\mu\in\Lambda}$ of planar analytic vector fields with a center and fix some $\hat{\mu}\in\Lambda$.

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Bifurcation from the outer boundary

Definition 2.1. Consider a continuous family $\{X_\mu\}_{\mu\in\Lambda}$ of planar analytic vector fields with a center and fix some $\hat{\mu}\in\Lambda$. Suppose that the outer boundary of the period annulus varies continuously at $\hat{\mu}\in\Lambda$. Then, setting

$$N(\delta,\varepsilon)=\sup\left\{\begin{array}{l}\#\text{critical periodic orbits } \gamma \text{ of } X_\mu \\ \text{ with } d_H(\gamma,\Pi_{\hat{\mu}})\leqslant\varepsilon \text{ and } \|\mu-\hat{\mu}\|\leqslant\delta\end{array}\right\}$$

we define $\text{Crit}\left((\Pi_{\hat{\mu}},X_{\hat{\mu}}),X_\mu\right):=\inf_{\delta,\varepsilon}N(\delta,\varepsilon)$ to be the *criticality* of $(\Pi_{\hat{\mu}},X_{\hat{\mu}})$ with respect to the deformation X_μ. □

Outline

Some notation

Consider potential systems

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = V_\mu'(x), \end{cases}$$

where V_μ is analytic and $V_\mu(0) = V_\mu'(0) = 0$ and $V_\mu''(0) > 0$.

- $\mathcal{I}_\mu = (x_\ell(\mu), x_r(\mu))$ denotes the projection of the period annulus $\mathscr{P}_\mu$ on the x-axis.
- $H_\mu(x,y) = \frac{1}{2}y^2 + V_\mu(x)$.
- The energy level of Π_μ is $+\infty$. That is, $V_\mu(\mathcal{I}_\mu) = [0, +\infty)$.
- $g_\mu(x) := x\sqrt{V_\mu(x)/x^2}$.

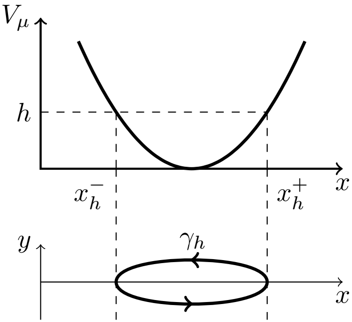

Figure: Interpretation of the periodic orbit γ_h.

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So,

$$T'_\mu(h) = \frac{1}{\sqrt{2}h}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_\mu^{-1})''(\sqrt{h}\sin\theta)\sqrt{h}\sin\theta d\theta.$$

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Complications:

- The function $T_\mu(h)$ is not analytic in $h = +\infty$.

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Complications:

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Idea:

- Show that $T'_\mu(h)$ has sign for $h \approx +\infty$.
- IMPORTANT: The uniformity on the parameters is required.

Outline

The bifurcation problem at the outer boundary turns into give a uniform upper-bound of the number of zeroes of $T'_\mu(h)$ near $h \approx +\infty$.

Definition 2.2. Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on I. The ordered set $(f_0, f_1, \ldots, f_{n-1})$ is an *extended complete Chebyshev system* (ECT-system) on I if, for all $k = 1, 2, \ldots, n$, any nontrivial linear combination

$$a_0 f_0(x) + a_1 f_1(x) + \cdots a_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on I counted with multiplicities. □

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We consider the operator $\mathscr{F} : \mathcal{C}^\omega([0, +\infty)) \to \mathcal{C}^\omega([0, +\infty))$ defined by

$$\mathscr{F}[f](x) := \int_0^{\frac{\pi}{2}} f(x \sin\theta) d\theta.$$

The main idea is to find some analytic real functions $\phi_\mu^1, \ldots, \phi_\mu^n$ satisfying that there exist $\varepsilon, M > 0$ such that if $\|\mu - \hat{\mu}\| < \varepsilon$ then $(\phi_\mu^1, \ldots, \phi_\mu^n, \mathscr{F}[f_\mu])$ form an ECT-system on the interval $(M, +\infty)$.

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Lemma

Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on I. $(f_0, f_1, \ldots, f_{n-1})$ in an ECT-system on I if and only if for each $k \in \{1, \ldots, n\}$, $W[f_0, f_1, \ldots, f_{k-1}](x) \neq 0$ for all $x \in I$.

We want to find continuous functions $\nu_1, \ldots, \nu_n$ pairwise distinct at $\mu = \hat{\mu}$ satisfying that there exist $\varepsilon, M > 0$ such that if $\|\mu - \hat{\mu}\| < \varepsilon$ then the function

$$x \mapsto W[x^{\nu_1(\mu)}, \ldots, x^{\nu_n(\mu)}, \mathscr{F}[f_\mu](x)]$$

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This is based in assuming that $\mathscr{F}[f_\mu](x)$ has an asymptotic development

$$\mathscr{F}[f_\mu](x) = a_1(\mu)x^{\nu_1(\mu)} + a_2(\mu)x^{\nu_2(\mu)} + \dots + a_n(\mu)x^{\nu_n(\mu)} + \dots$$

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and the division-derivation algorithm.

Definition 2.4. Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions in $[0, +\infty)$. Given $\hat{\mu} \in \Lambda$ we say that $\{f_\mu\}$ is *continuously quantifiable* in $\hat{\mu}$ at $+\infty$ by $\alpha(\mu)$ with limit ℓ if $\lim_{(x,\mu)\to(+\infty,\hat{\mu})} \frac{f_\mu(x)}{x^{\alpha(\mu)}} = \ell$ and $\ell \neq 0$. Let us denote it by $f_\mu(x) \sim_\infty x^{\alpha(\mu)}$ in $\hat{\mu}$. □

Definition 2.5. For $f \in \mathcal{C}^\omega([0, +\infty))$ we call $M_n[f] := \int_0^{+\infty} x^{2n-2} f(x)dx$ the n-th *momentum* of f, whenever it is well defined. □

Case $n = 0$ (there are no functions ν_i)

Theorem A

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(a) *If* $\alpha(\hat{\mu}) > -1$, *then* $\mathscr{F}[f_\mu](x) \sim_\infty x^{\alpha(\mu)}$ *in* $\hat{\mu}$.

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(b) *If $\alpha(\hat{\mu}) < -1$, let us take $n \in \mathbb{N}$ such that $\alpha(\hat{\mu}) + 2n \in [-1, 1)$.*

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- $(b1)$ *If* $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \ldots \equiv M_{j-1}[f_\mu] \equiv 0$ *and* $M_j[f_{\hat{\mu}}] \neq 0$ *for some* $1 \leqslant j \leqslant n$, *then*

$$\mathscr{F}[f_\mu](x) \sim_\infty x^{1-2j} \quad \text{in } \hat{\mu}.$$

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- $(b2)$ *If $M_1[f_\mu] \equiv M_2[f_\mu] \equiv \cdots \equiv M_n[f_\mu] \equiv 0$ and $\alpha(\hat{\mu}) + 2n \notin \{-1, 0\}$, then*

$$\mathscr{F}[f_\mu](x) \sim_\infty x^{\alpha(\mu)} \quad \text{in } \hat{\mu}.$$

Case $n=0$ (there are no functions ν_i)

Theorem A

Consider a continuous family $\{f_\mu\}_{\mu\in\Lambda}$ of analytic functions on $[0,+\infty)$. Suppose that $f_\mu(x)\sim_\infty x^{\alpha(\mu)}$ in $\hat{\mu}$. The following assertions hold:

- (a) *If $\alpha(\hat{\mu})>-1$, then $\mathscr{F}[f_\mu](x)\sim_\infty x^{\alpha(\mu)}$ in $\hat{\mu}$.*
- (b) *If $\alpha(\hat{\mu})<-1$, let us take $n\in\mathbb{N}$ such that $\alpha(\hat{\mu})+2n\in[-1,1)$. In this case:*
  - $(b1)$ *If $M_1[f_\mu]\equiv M_2[f_\mu]\equiv\ldots\equiv M_{j-1}[f_\mu]\equiv 0$ and $M_j[f_{\hat{\mu}}]\neq 0$ for some $1\leqslant j\leqslant n$, then*
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$$\mathscr{F}[f_\mu](x)\sim_\infty x^{\alpha(\mu)} \quad \text{in } \hat{\mu}.$$

So we have $\mathscr{F}[f_\mu](x)=x^{\nu(\mu)}(\Delta(\mu)+r_\mu(x))$ for $x\approx+\infty$, where $r_\mu(x)$ is a reminder that tends uniformly to zero as x tends to infinity.

Case $n > 0$. Given $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{R}$ we define the differential operator $\mathscr{L}_{\boldsymbol{\nu}_n} : \mathscr{C}^\omega\big((0,+\infty)\big) \longrightarrow \mathscr{C}^\omega\big((0,+\infty)\big)$ given by

$$\mathscr{L}_{\boldsymbol{\nu}_n}[f](x) := \frac{W\left[x^{\nu_1}, x^{\nu_2}, \ldots, x^{\nu_n}, f(x)\right]}{x^{\sum_{i=1}^{n}(\nu_i - i)}},$$

where $\boldsymbol{\nu}_n = (\nu_1, \ldots, \nu_n)$ and $\mathscr{L}_{\boldsymbol{\nu}_0} = id$.

Idea: to apply Theorem A with the function $\mathscr{L}_{\boldsymbol{\nu}_n}[f](x)$.

Key point:

Proposition

$\mathscr{F} \circ \mathscr{L}_{\boldsymbol{\nu}_n} = \mathscr{L}_{\boldsymbol{\nu}_n} \circ \mathscr{F}$

Theorem B

Let $\{f_\mu\}_{\mu\in\Lambda}$ be a continuous family of analytic functions on $[0,+\infty)$. Assume that there exist $n \geqslant 0$ continuous functions $\nu_1, \nu_2, \ldots, \nu_n$ such that $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu](x) \sim_\infty x^{\xi(\mu)}$ in $\hat{\mu}$. The following assertions hold:

(a) *If $\xi(\hat{\mu}) > -1$, then $(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)} \circ \mathscr{F})[f_\mu](x) \sim_\infty x^{\xi(\mu)}$ in $\hat{\mu}$.*

(b) *If $\xi(\hat{\mu}) < -1$, let us take $m \in \mathbb{N}$ such that $\xi(\hat{\mu}) + 2m \in [-1,1)$. In this case:*

$(b1)$ *If $M_1\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big] \equiv \ldots \equiv M_{j-1}\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big] \equiv 0$ and $M_j\big[\mathscr{L}_{\boldsymbol{\nu}_n(\hat{\mu})}[f_{\hat{\mu}}]\big] \neq 0$ for some $1 \leqslant j \leqslant m$, then*

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$(b2)$ *If $M_1\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big] \equiv \ldots \equiv M_m\big[\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu]\big] \equiv 0$ and $\xi(\hat{\mu}) + 2m \notin \{-1, 0\}$, then*

$$(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)} \circ \mathscr{F})[f_\mu](x) \sim_\infty x^{\xi(\mu)} \quad \text{in } \hat{\mu}.$$

Consequently, there exist $\eta : \Lambda \to \mathbb{R}$ such that

$$(\mathscr{L}_{\boldsymbol{\nu}_n(\mu)} \circ \mathscr{F})[f_\mu](x) = x^{\eta(\mu)}(\Delta(\mu) + r_\mu(x))$$

for $x \approx +\infty$, where $r_\mu(x)$ is a reminder that tends to zero uniformly as x tends to infinity. That is,

$$W[x^{\nu_1(\mu)}, \ldots, x^{\nu_n(\mu)}, \mathscr{F}[f_\mu]](x) = x^{\eta(\mu)+\sum_{i=1}^{n}(\nu_i(\mu)-i)}(\Delta(\mu) + r_\mu(x))$$

for $x \approx +\infty$.

Outline

Bounding criticality for potential centers with infinite energy level

Remember that

$$T'_{\mu}(h) = \frac{1}{\sqrt{2}h} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (g_{\mu}^{-1})''(\sqrt{h}\sin\theta)\sqrt{h}\sin\theta d\theta.$$

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Then, setting $f_\mu(x) := \mathcal{P}[z(g_\mu^{-1})''(z)](x)$, where $\mathcal{P}[f](x) := f(x) - f(-x)$, we have

$$\sqrt{2}h^2 T'_\mu(h^2) = \mathscr{F}[f_\mu](h).$$

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Using Theorem B, if $\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[f_\mu](x) \sim_\infty x^{\xi(\mu)}$ (and hypothesis of momenta),

$$\mathscr{L}_{\boldsymbol{\nu}_n(\mu)}[\sqrt{2}h^2 T'_\mu(h^2)] = h^{\eta(\mu)}(\Delta(\mu) + r_\mu(h))$$

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for $h \approx +\infty$, where $r_\mu(h)$ is a reminder that tends uniformly to zero as $h \to +\infty$. Then,

$$W[h^{\nu_1(\mu)}, \ldots, h^{\nu_n(\mu)}, \sqrt{2}h^2 T'_\mu(h^2)] = h^{\eta(\mu) + \sum_{i=1}^n (\nu_i(\mu) - i)}(\Delta(\mu) + r_\mu(h))$$

for $h \approx +\infty$. Consequently, $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \leqslant n$.

References

F. Mañosas, D. Rojas, J. Villadelprat,
The criticality of centers of potential systems at the outer boundary., J. Differential Equations 260, (2016) 4918–4972.

F. Mañosas, D. Rojas, J. Villadelprat,
Study of the period function of a two-parameter family of centers., Preprint, (2016).

F. Mañosas, D. Rojas, J. Villadelprat,
Analytic tools to bound the criticality at the outer boundary of the period annulus., J. Dynamics and Differential Equations, DOI 10.1007/s10884-016-9559-x, (2016).

D. Rojas,
Analytical tools to study the criticality at the outer boundary of potential centers., Ph.D. Dissertation, (2016).

Thank you very much for your attention!