Numerical Computation of Invariant Objects with Wavelets

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Outline

1. Motivation
2. Wavelets and Regularity
   - A Primer on Wavelets
   - On the Evaluation of a Wavelet
   - Regularity with Wavelet Coefficients
3. Numerical Computation of Invariant Objects with Wavelets
   - Using the Fast Wavelet Transform

Motivation

The aim of this Thesis was to introduce techniques and algorithms to compute approximations of (geometrically) complicated dynamical invariant objects by means of wavelets. From such approximation, we want to predict – understand changes in the geometry or derive dynamical properties of such objects (among other questions).

Along this talk, we will be focused on skew products on the cylinder of the form

$$\mathfrak{F}_{\sigma,\varepsilon}\begin{pmatrix}\theta_n\\ x_n\end{pmatrix} = \begin{cases}\theta_{n+1} & = R_\omega(\theta_n) = \theta_n + \omega \pmod 1,\\ x_{n+1} & = F_{\sigma,\varepsilon}(\theta_n, x_n),\end{cases} \tag{1}$$

where $\varepsilon, \sigma \in \mathbb{R}^+$, $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} = [0,1)$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $x \equiv 0$ invariant.

On The Use Of Wavelets

The complicated objects that we want to compute will be invariant maps $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$ of the above skew product:

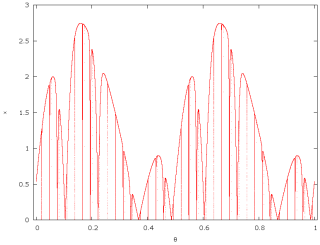
$$\varphi(R_\omega(\theta)) = F_{\sigma,\varepsilon}(\theta, \varphi(\theta)).$$

A standard approach to compute these objects is the use of Fourier approximations:

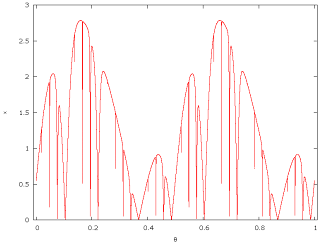
$$\varphi \sim a_0 + \sum_{n=1}^{N} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

On The Use Of Wavelets

The word complicated means, loosely speaking, "*highly non-piecewise continuous*". Thus, the regularity and periodicity of the Fourier basis make this approach too costly.



The graph of the map φ for
$F_{\sigma,0} = 2\sigma \tanh(x_n) \cdot |\cos(2\pi\theta_n)|$.



The graph of the map φ for
$F_{\sigma,\varepsilon} = 2\sigma \tanh(x_n)(\varepsilon + |\cos(2\pi\theta_n)|)$.

Trying to overcome the above questions and problems, we have used other orthonormal basis: the *wavelets*.

An Sketch of the Performed Strategies

So, given a wavelet ψ, we want to compute approximations like:

$$\varphi \sim a_0 + \sum_{j=0}^{J} \sum_{n=0}^{2^j-1} d_{-j,n} \psi_{-j,n}^{\mathrm{PER}}(\theta) = d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta),$$

where ψ^{PER} is the 1-periodic version of ψ, $N = 2^{J+1}$ and $\ell = 2^j + n$.

How We Have Computed The Wavelet Coefficients?

1. Using the Fast Wavelet Transform (where $N = 2^{30}$).
2. Solving, by means of Newton's Method, the Invariance Equation using either the Haar and a Daubechies wavelet

$$\mathbf{F}_{\sigma,\varepsilon}(\cdot) := d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(R_\omega(\theta)) - F_{\sigma,\varepsilon}\left(\theta, d_0 + \sum_{\ell=1}^{N-1} d_\ell \psi_\ell^{\mathrm{PER}}(\theta)\right)$$

Thus, at each Newton step, is mandatory find a fast solution (and a feasible computation of $2^{26} \times 2^{26}$ matrices Ψ and Ψ_R) of:

$$\left(\Psi_R - \frac{\partial F_{\sigma,\varepsilon}}{\partial x}\Psi\right) \mathrm{X_n} = -\mathbf{F}_{\sigma,\varepsilon}(\mathrm{D_n^{PER}}).$$

Wavelets are otained from a function called *mother wavelet*, $\psi(x)$, by the following formula:

$$\psi_{j,n}(x) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{x-2^j n}{2^j}\right).$$

The mother wavelet $\psi(x)$ verifies a relation with a "*list of coefficients*" called *low pass filter* h_n. Imposing conditions on h_n, in [Mal] it is shown that:

Mallat and Meyer Theorem

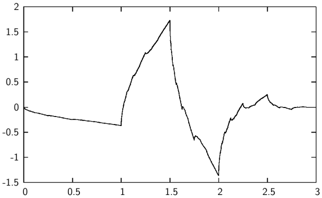
The wavelets $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}\times\mathbb{Z}}$ are an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$.

[Mal] Mallat, Stéphane, *A wavelet tour of signal processing*, Academic Press Inc., San Diego, CA, 1998, xxiv+577.

Fixing And Translating The Wavelet

We will be focused on the Daubechies wavelets family. Each Daubechies wavelet minimize its support, $[1-p, p]$, constrained to the maximal number of vanishing moments, p:

$$\int_{1-p}^{p} x^k \psi(x)\, dx = 0 \text{ for } 0 \le k < p.$$

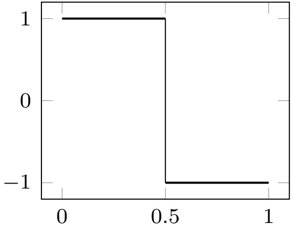


Daubechies wavelet (compact support)

No closed formula

$$h[n] = \begin{cases} 0.48296291314\ldots & \text{if } n = 0, \\ 0.83651630373\ldots & \text{if } n = 1, \\ 0.22414386804\ldots & \text{if } n = 2, \\ -0.12940952255\ldots & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Other Daubechies Example



Haar wavelet (compact support)

$$\psi(x) := \mathbf{1}_{[0,\frac{1}{2})}(x) - \mathbf{1}_{[\frac{1}{2},1)}(x)$$

where $\mathbf{1}_{[a,b)}(x) = \begin{cases} 1 & \text{if } x \in [a,b), \\ 0 & \text{otherwise.} \end{cases}$

$$h[n] = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is the unique Daubechies wavelet with an explicit formula.

Haar against the Invariance Equation

Using the Haar wavelet basis, by performing the Gauss Method formally on the aforesaid (*preconditioned*) linear system we have given an explicit recurrence that solves a $2^{28} \times 2^{28}$ linear system in less than 10 seconds.

Fixing And Translating The Wavelet

Since our framework is $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, we transform a $\mathbb{R}$-function into a $\mathbb{S}^1$-function by setting $\psi_{j,n}^{\mathrm{PER}}$ as follows:

$$\psi_{j,n}^{\mathrm{PER}}(\theta) = \sum_{\iota\in\mathbb{Z}} \psi_{j,n}\overbrace{(\theta+\iota)}^{x\in\mathbb{R}\ :\ \mathrm{frac}(x)=\theta} = 2^{-j/2}\sum_{\iota\in\mathbb{Z}} \psi\left(\frac{\overbrace{(\theta+\iota)}^{x} - 2^j n}{2^j}\right).$$

$\psi_{j,n}^{\mathrm{PER}}$ are 1-periodic functions belonging to $\mathscr{L}^1(\mathbb{S}^1)$ and they are an orthonormal basis of $\mathscr{L}^2(\mathbb{S}^1)$ (see [HeWe]).

Hence, once ψ is given, we are (almost) ready to compute

$$\varphi \sim a_0 + \sum_{j=0}^{N}\sum_{n=0}^{2^j-1} d_{-j,n}\psi_{-j,n}^{\mathrm{PER}}(\theta).$$

[HeWe] Hernández, Eugenio and Weiss, Guido, *A first course on wavelets*, CRC Press, Boca Raton, FL, 1996, xx+489.

Adapted Daubechies-Lagarias Algorithm On The Circle

Definition (The key ingredient of Daubechies-Lagarias algorithm)

$$\mathrm{dyad}(\{2x\}) := d_1 d_2 \dots d_n \dots,$$

$$\mathbf{v}(x) = \mathbf{v}(\{x\}) := \frac{1}{2p-1} \left(\lim_{n\to\infty} \mathbf{M}_{d_1} \cdot \mathbf{M}_{d_2} \cdots \mathbf{M}_{d_n} \right) \mathbf{1}_{2p-1}$$

$$= \frac{1}{2p-1} \lim_{n\to\infty} \mathbf{M}_{d_1} \cdot \mathbf{M}_{d_2} \cdots \mathbf{M}_{d_n} \cdot \mathbf{1}_{2p-1},$$

where $\mathrm{dyad}(x)$ denotes the dyadic (binary) expansion of x, $\{\cdot\}$ denotes the fractional part function (i.e. $\{x\} = x - \lfloor x \rfloor$), and $\mathbf{1}_{2p-1}$ denotes the $2p-1$ dimensional column vector whose entries are all equal to 1.

Daubechies-Lagarias algorithm for the mother wavelet ψ^{PER}

From $\psi(x) := \mathbf{u}(x)^\top \mathbf{v}(x)$, we have shown

$$\psi^{\mathrm{PER}}(\theta) = \sum_{m\in\mathbb{Z}} \psi(\theta+m) = \left(\sum_{m\in\mathbb{Z}} \mathbf{u}(\theta+m) \right)^\top \mathbf{v}(\theta).$$

Computing Regularities With Wavelet Coefficients

Theorem (uses of the wavelet coefficients)

Let $s \in \mathbb{R} \setminus \{0\}$ and let ψ be a mother Daubechies wavelet with more than $\max(s, 5/2 - s)$ vanishing moments. Then $f \in \mathscr{B}^s_{\infty,\infty}$ if and only if there exists $C > 0$ such that for all $j \leq 0$

$$\sup_{n\in\mathbb{Z}} |\langle f, \psi^{\mathrm{PER}}_{j,n} \rangle| \leq C 2^{\tau j} \qquad \text{with} \qquad \tau = \begin{cases} s + \frac{1}{2} & \text{if } s > 0, \\ s - \frac{1}{2} & \text{if } s < 0, \end{cases}$$

In the case of Haar, [Tri02], there is an analogous result.

- **[Coh]** Cohen, Albert, *Numerical analysis of wavelet methods*, North-Holland, 2003, xviii+336.
- **[Tri01]** Triebel, Hans, *Theory of function spaces. III*, Birkhäuser Verlag, Basel, 2006, xii+426.
- **[Tri02]** Triebel, Hans, *Bases in function spaces, sampling, discrepancy, numerical integration*, European Mathematical Society, Zürich, 2010, x+296.

Outline

The [GOPY]-Keller Model: a Testing Ground

We will use $\omega = \frac{\sqrt{5}-1}{2}$ and the following one-parameter family of skew products (with $x \equiv 0$ invariant) as a testing case for our techniques.

$$\mathfrak{F}_{\sigma,\varepsilon(\sigma)} \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} = \begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = 2\sigma \tanh(x_n)(\overbrace{\varepsilon}^{\varepsilon(\sigma)} + |\cos(2\pi\theta_n)|), \end{cases} \tag{2}$$

where

$$\varepsilon(\sigma) = \begin{cases} (\sigma - 1.5)^2 & \text{when } 1.5 < \sigma \leq 2, \\ 0 & \text{when } 1 < \sigma \leq 1.5. \end{cases}$$

This testing ground is similar to the [GOPY] model.

[GOPY] Grebogi, Celso *et al.*, *Strange attractors that are not chaotic*, Phys. D 13 1984 1–2 261–268.

Keller'S Theorem (Shortened and Rephrased)

There exists an upper semi-continuous map $\varphi\colon \mathbb{S}^1 \longrightarrow [0,\infty)$ whose graph is invariant under the Model (2). Moreover,

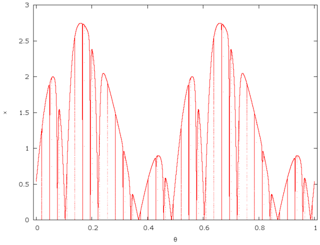
1. if $\sigma > 1$ then
$$\varphi \in \mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1) \text{ being } \begin{cases} s = 0 & \text{if } \varepsilon = 0, \text{ (there is an SNA)} \\ s > 0 & \text{if } \varepsilon > 0. \end{cases}$$
2. if $\sigma \neq 1$ then $|x_n - \varphi(\theta_n)| \to 0$ exponentially fast for almost every θ and every $x > 0$.

[Kel] Keller, Gerhard, *A note on strange non-chaotic attractors*, Fund. Math. 151 1996 2 139–148.

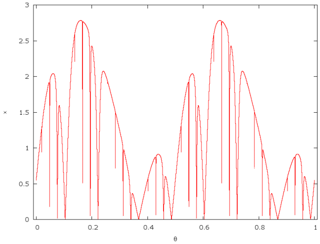
[AMR] LL. Alsedà, J.M. Mondelo and D. Romero, *A numerical estimate of the regularity of a family of Strange Non-Chaotic Attractors*, submitted.

The Regularity Using Wavelet Coefficients (informal)

The regularity spaces $\mathscr{B}^s_{\infty,\infty}$ provide a natural framework for the approximations of $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$ that one gets. For example, for our *generic* toy model can we "distinguish" these two instances of φ's using the "computed" wavelet coefficients?



Pinched case (i.e. $\varepsilon = 0$).



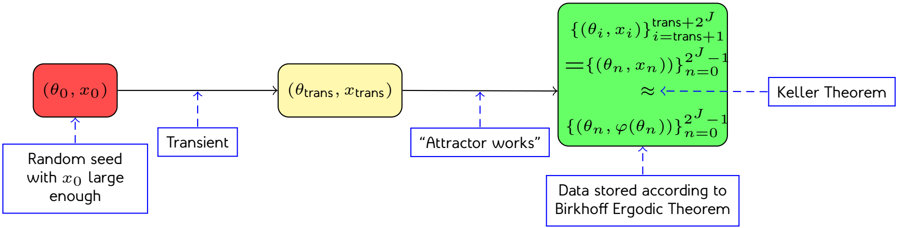
Quasi-Pinched case (i.e. $\varepsilon \simeq 0$).

Using the FWT To Compute Wavelet Coefficients

To compute an estimate of the regularity of the attractor (Graph(φ)), with $J = 30$ for the FWT, we will perform the following steps:

Remark: The Fast Wavelet Transform must be understood as an iterative change of basis from the coefficients $a_{-J}[n]$ for $0 \leq n \leq 2^J - 1$ to the wavelet coefficients $d_{-j}[n]$ for $j \in \{0, 1, \ldots J\}$ and $n \in \{0, 1, \ldots, 2^j - 1\}$. To perform it, the low pass filter h_n is needed.

Step 0: Obtain an approximate mesh of the attractor of size 2^J *storing it according to Birkhoff Ergodic Theorem*:



Using the FWT To Compute Wavelet Coefficients

Step 1: Sort the data $\{(\theta_n, x_n)\}_{n=0}^{2^J-1}$ with respect to the first component to a sequence $\{(\widetilde{\theta}_n, z_n)\}_{n=0}^{2^J-1}$ so that $\widetilde{\theta}_{n_1} < \widetilde{\theta}_{n_2}$ if and only if $n_1 < n_2$.

Remark: Since the data was stored according to the Birkhoff Ergodic Theorem this can be done most efficiently just with an insertion sort algorithm.

Then, the sequence $\{z_n\}_{n=0}^{2^J-1}$ are the $x-$ coordinates of an approximate *equispaced mesh* of the function $\varphi \circ \xi$ where ξ is a $\mathcal{C}^2-$diffeomorphism that sends $\widetilde{\theta}_n$ to $\frac{n}{2^J}$ for $n = 0, 1, \ldots, 2^J - 1$. With the help of a result from [Trio3] one can prove that the function $\varphi \circ \xi$ has the same regularity as φ. So, we can work with this equispaced data instead of with the original one.

[Trio3] Triebel, Hans, *Theory of function spaces. II*, Birkhäuser Verlag, Basel, 1992, viii+370.

Using the FWT To Compute Wavelet Coefficients

Step 2: Calculate $a_{-J}[n]$ for $0 \leq n \leq 2^J - 1$ to initialize the FWT. From the proof of Keller Theorem, Frazier formula and the Dominated Convergence Theorem it follows that

$$a_{-J}[n] \approx \langle \varphi \circ \xi, \phi_{-J,n} \rangle \approx (\varphi \circ \xi)(\tfrac{n}{2^J}) \approx z_n$$

Step 3 Use the FWT (with the above initialization) to compute the coefficients a_0 and

$$d_{-j}[n] = \langle \varphi \circ \xi, \psi_{j,n} \rangle$$

for $j \in \{0, 1, \ldots J\}$ and $n \in \{0, 1, \ldots, 2^j - 1\}$.

Using the FWT To Compute Wavelet Coefficients

Step 4 For $0 \leq j \leq J$, calculate

$$s_j = \log_2 \left(\sup_{0 \leq n \leq 2^j - 1} |d_{-j}[n]| \right).$$

Step 5 Make a linear regression to estimate the slope τ of the graph of the pairs (j, s_j) with $j = 0, -1, -2, \ldots, -J$. Afterwards, use the *regularity* theorem to get s provided that the wavelet used had more than $\max(s, \frac{5}{2} - s)$ vanishing moments. This algorithm gives an effective way of computing wavelet coefficients and regularities in a generic way.

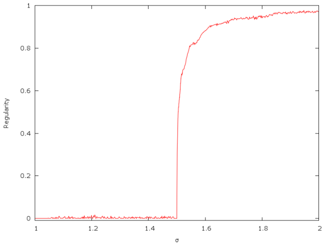
Remark

Steps 4 and **5** justify why we need a massive computation of wavelets coefficients. Indeed,

$$J \text{ samples} \Leftrightarrow 2^{J+1} \text{ coefficients.}$$

Using the FWT To Estimate Regularities

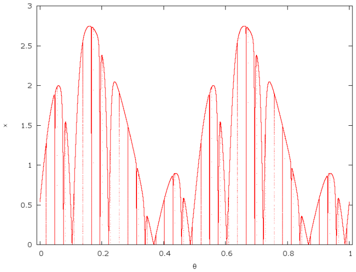
The above algorithm gives the following regularity graph for the one-parameter family of skew products, with $\varphi \neq 0$, given by the System (2):

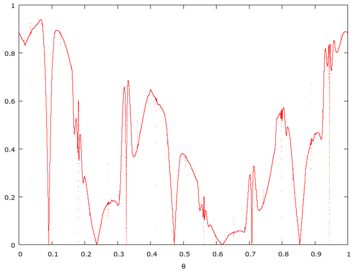


Regularity along $\varepsilon(\sigma)$.

The results are obtained by using a sample of 2^{30} points, a transient of 10^5 iterates and the Daubechies Wavelet with 10 vanishing moments. We can detect in a *correct way* the regularity leap in "$\mathcal{O}(N)$" time.

Using the Invariance Equation, we get *similar pictures* and *time*.





Y esto es todo.