EXPANDING BAKER MAPS

A first tool to study homoclinic bifurcations of 3-D diffeomorphisms

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Introduction

Strange Attractor

Let $f : \mathcal{M} \to \mathcal{M}$ be a map and $\mathcal{A} \subset \mathcal{M}$. The set $\mathcal{A}$ is said to be an **strange attractor** if

(Attractor) $\mathcal{A}$ is a compact, invariant and transitive set and its stable set has a non-empty interior.

(Strange) $\mathcal{A}$ contains a dense expansive orbit $\mathcal{O}(Q)$ displaying exponential growth, i.e., there exists some constant $c > 0$ such that

$$\|Df^n(Q)\| \geq \exp(cn)$$

for every $n \geq 0$.

The starting point

"Three-dimensional" limit return maps

$$F_{a,b,n}(x,y,z) \underset{n\to\infty}{\longrightarrow} F_{a,b}(x,y,z) = (z, a + by + z^2, y)$$
$$\hookrightarrow T_{a,b}(x,y) = (a + y^2, x + by)$$

Results

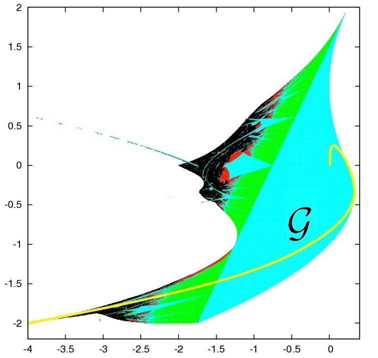
For a positive Lebesgue measure set of parameters, the **numerical analysis** seems to indicate that $T_{a,b}$ exhibits an strange attractor.

References

- **J.C. Tatjer,** *Three-dimensional dissipative diffeomorphisms with homoclinic tangencies*. Ergodic Theory and Dynamical Systems, 21 (2001).
- **A. Pumariño and J.C. Tatjer,** *Attractors for return maps near homoclinic tangencies of three-dimensional dissipative diffeomorphisms*. Discrete and Continuous Dynamical Systems, series B, vol 8, 4 (2007).

1. The family $T_{a,b}$

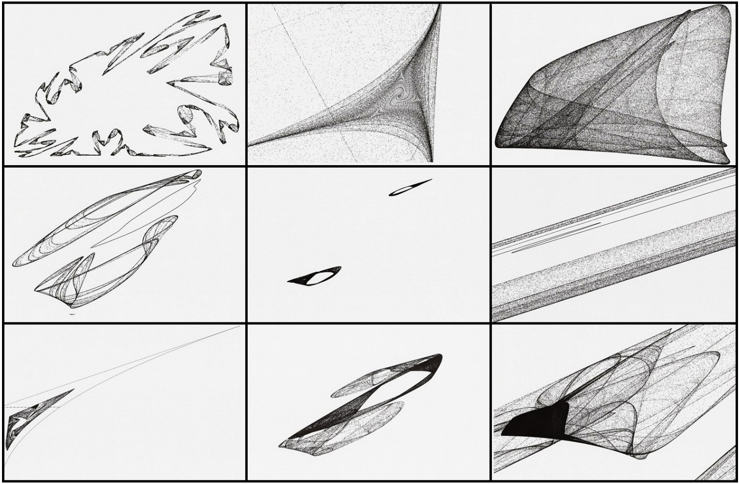
Region in the plane where $T_{a,b}$ has inv. domains



- Blue: sinks
- Green: one of the Lyapounov exponents is zero
- Red: the sum and the product of the two Lyapunov exponents is negative
- Black: the sum of the Lyapunov exponents is positive

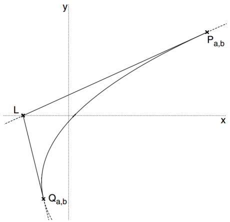
$$\mathcal{G} = \left\{(a(s), b(s)) = \left(-\frac{s^3}{4}(s^3 - 2s^2 + 2s - 2), -s^2 + s\right) : \; s \in [0, 2]\right\}$$

Possible strange attractors outside $\mathcal{G}$

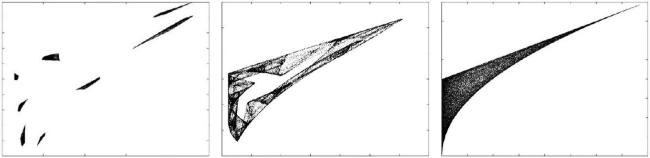


Possible strange attractors along $\mathcal{G}$

Invariant domain



Numerically obtained strange attractors



$s = 1.8909$ $s = 1.8939$ $s = 1.99$

The special value $s = 2$

Let $\mathcal{T}$ be the triangle with vertices $(0,0)$, $(1,1)$ and $(2,0)$ and let $\Lambda_1 : \mathcal{T} \to \mathcal{T}$ be the map defined by

$$\Lambda_1(x,y) = \begin{cases} (x+y, x-y) & \text{, if } x \leq 1 \\ (2-x+y, 2-x-y) & \text{, if } x \geq 1 \end{cases}$$

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Proposition

The map $\Lambda_1|_{\mathcal{T}}$ is conjugated to the shift with two symbols and to $T_{a(2),b(2)}$.
Therefore, $T_{a(2),b(2)}$ has an unique ergodic ACIM and a dense orbit with two positive Lyapounov exponents.

Reference

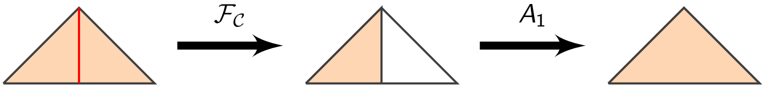
A. Pumariño and J.C. Tajter, *Dynamics near homoclinic bifurcations of three-dimensional dissipative diffeomorphisms,* Nonlinearity, 19 (2006).

Two-dimensional tent map

We can write Λ_1 as follows: $\Lambda_1 = A_1 \circ \mathcal{F}_{\mathcal{C}}$, being

$$\mathcal{F}_{\mathcal{C}}(x,y) = \begin{cases} (x,y) & \text{, if } x \leq 1 \\ (2-x,y) & \text{, if } x \geq 1 \end{cases}$$

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$



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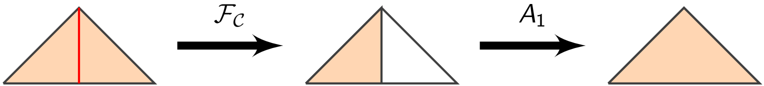
From the geometry of this map, the term *Expanding Baker Map* arises.



2. Expanding Baker Maps

Folds and good folds

Let $\mathcal{K} \subset \mathbb{R}^2$ be a polygonal domain, $P \in \mathcal{K}$ an $\mathcal{L}$ a straight line dividing $\mathcal{K}$ into two subsets $\mathcal{K}_0$ and $\mathcal{K}_1$ (assume that $P \in \mathcal{K}_0$). We define the **fold** of $\mathcal{K}$ by $\mathcal{L}$ as

$$\mathcal{F}_{\mathcal{L}}(x,y) = \begin{cases} (x,y) & \text{, if } (x,y) \in \mathcal{K}_0 \\ (\overline{x},\overline{y}) & \text{, if } (x,y) \in \mathcal{K}_1 \end{cases}$$

being $(\overline{x},\overline{y})$ the symmetric point of (x,y) with respect to $\mathcal{L}$.

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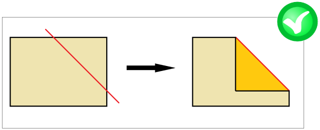
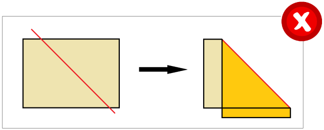
The map $\mathcal{F}_{\mathcal{L}}$ is said to be a **good fold** if $\mathcal{F}_{\mathcal{L}}(\mathcal{K}) = \mathcal{K}_0$.



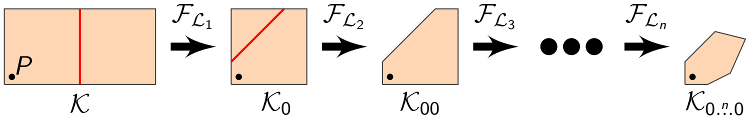


Sequence of folds

Let $\mathcal{K} \subset \mathbb{R}^2$ be a polygonal domain and let P be a point in $\mathcal{K}$.

- Let $\mathcal{F}_{\mathcal{L}_1}$ be a good fold defined in the domain $\mathcal{K}$. We obtain the set $\mathcal{K}_0$.
- Let $\mathcal{F}_{\mathcal{L}_2}$ be a good fold defined in the domain $\mathcal{K}_0$. We obtain the set $\mathcal{K}_{00}$.
- We can repeat the process folding by $\mathcal{F}_{\mathcal{L}_3}$ ... $\mathcal{F}_{\mathcal{L}_n}$ ($n \in \mathbb{N}$)

After n good folds we obtain a set $\mathcal{K}_{0\overset{n}{\dots}0} \subset \mathcal{K}$ with $P \in \mathcal{K}_{0\overset{n}{\dots}0}$.



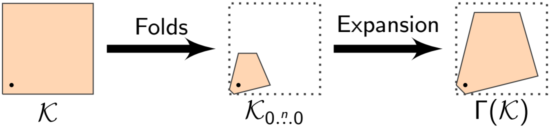
Expanding Baker Maps

Let us consider a polygonal domain $\mathcal{K}$ and a point $P \in \mathcal{K}$.

- $\{\mathcal{F}_{\mathcal{L}_1} \dots \mathcal{F}_{\mathcal{L}_n}\}$ is a sequence of good folds of $\mathcal{K}$.
- $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an expanding linear map centered in P such that $A(\mathcal{K}_{0 \overset{n}{\dots} 0}) \subset \mathcal{K}$.

We define the **Expanding Baker Map** associated to $\mathcal{K}, \mathcal{L}_1, \dots, \mathcal{L}_n$, P and A as the map $\Gamma : \mathcal{K} \to \mathcal{K}$ given by

$$\Gamma = A \circ \mathcal{F}_{\mathcal{L}_n} \circ \dots \circ \mathcal{F}_{\mathcal{L}_1}.$$



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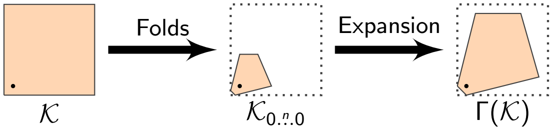


For short, $\Gamma = EBM(\mathcal{K}, \mathcal{L}_1, \dots, \mathcal{L}_n, P, A)$.

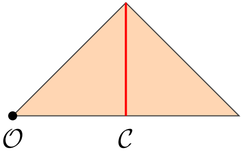
3. The family $\{\Lambda_t\}_t$

Notation

From now on, we will consider:

- $\mathcal{T} \equiv$ triangle with vertices $(0,0)$, $(1,1)$ and $(2,0)$.
- $\mathcal{O} \equiv$ origin of the plane.
- $\mathcal{C} \equiv$ straight line $\{(x,y) \in \mathcal{T} : \ x = 1\}$.
- $A_t \equiv$ linear map defined by

$$A_t = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}.$$

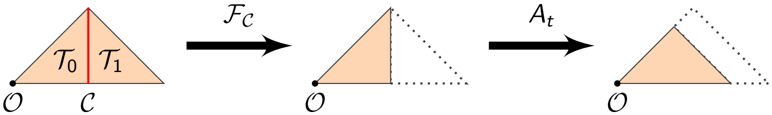


The family $\{\Lambda_t\}_t$

For each $t \in [0,1]$, we define the map $\Lambda_t = A_t \circ \mathcal{F}_{\mathcal{C}}$, i.e.,

$$\Lambda_t(x,y) = \begin{cases} (t(x+y), t(x-y)) & , \text{ if } x \leq 1 \\ (t(2-x+y), t(2+x-y)) & , \text{ if } x > 1 \end{cases}$$

Dynamics



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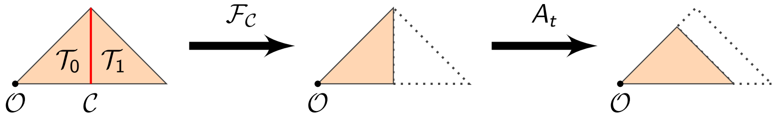
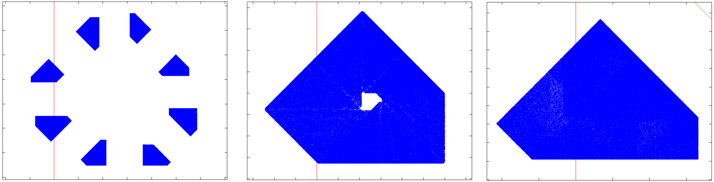


Proposition

For every $\frac{1}{\sqrt{2}} < t \leq 1$, $\Lambda_t = EBM(\mathcal{T}, \mathcal{C}, \mathcal{O}, A_t)$.

Dynamics of Λ_t

$\Lambda_t = EBM(\mathcal{T}, \mathcal{C}, \mathcal{O}, A_t)$ displays three kinds of non-trivial attractors: non-connected, connected but non simply-connected and convex attractors.

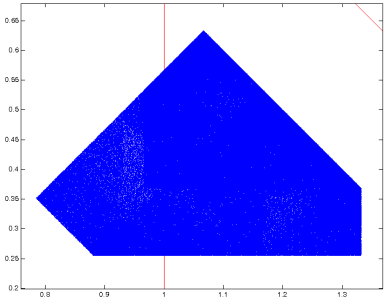


Reference

A. Pumariño, J. A. Rodríguez, J.C. Tatjer and E. Vigil, *Expanding Baker Maps as models for the dynamics emerging from 3D-homoclinic bifurcations*. Contin. Dyn. Syst. Ser. B, 19, 2 (2014).

Convex attractors

In the case $\frac{1}{\sqrt[3]{2}} \leq t \leq 1$ the attracting set is formed by a unique piece without holes.



Convex attractors

We have proved that if $t \in (t_0, 1]$, being $t_0 \approx 0.882$, then:

(1) Λ_t has a **strange attractor** $\mathcal{R}_t$ with two positive Lyapounov exponents.

(2) Λ_t is **strongly topologically mixing** on $\mathcal{R}_t$.

(3) $\mathcal{R}_t$ supports a **unique ergodic ACIM** μ_t.

(4) The family $\{\Lambda_t\}_t$ is **statistically stable**.

References

- **A. Pumariño, J. A. Rodríguez, J.C. Tatjer and E. Vigil,** *Chaotic dynamics for 2-D tent maps*, Nonlinearity, 28, 407–434 (2015).
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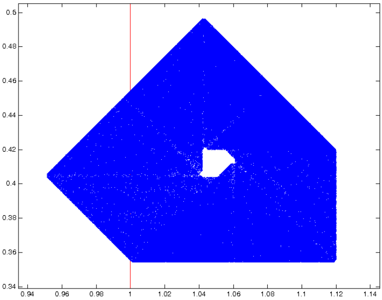
In fact, these results hold for $t \in \left[\frac{1}{\sqrt[3]{2}}, 1\right]$.

Connected but non simply-connected attractors

In the case $\frac{1}{\sqrt[5]{4}} \leq t < \frac{1}{\sqrt[3]{2}}$, the attracting set is formed by a single piece with a hole.

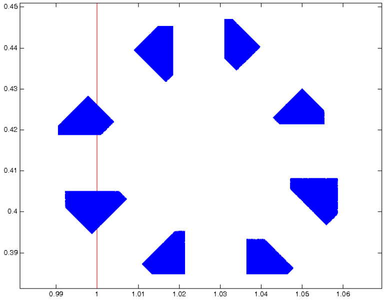


Connected but non simply-connected attractors

- The hole appears when the symmetric point of P_t with respect to $\mathcal{C}$ leaves the attractor, so there are no preimages of P_t in the attractor.
- The hole is determined by the first images of the critical line.
- The attractor becomes an octagon when $t < t_1 \approx 0.771$

Non-connected attractors

In the case $\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt[5]{4}}$, the attracting set is formed by several pieces.



Non-connected attractors

How many pieces are obtained? Is there only one attractor?

(a) If $0.723 \approx t_2 \leq t < \frac{1}{\sqrt[5]{4}}$ we obtain a unique 8−pieces attractor.

(b) If $0.717 \approx t_3 \leq t < t_2$ we obtain a unique 32−pieces attractor.

(c) If $0.711 \approx t_4 \leq t < t_3$ we obtain **TWO** attractors formed by 32−pieces.

(d) . . .

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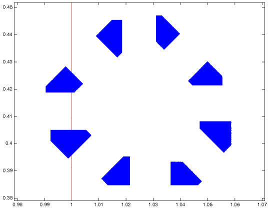

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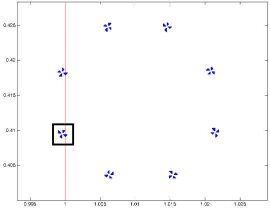

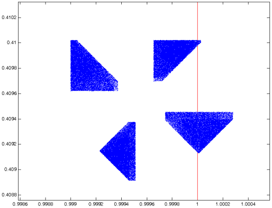

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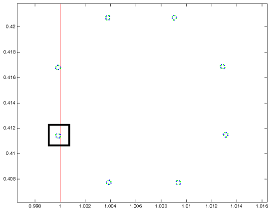

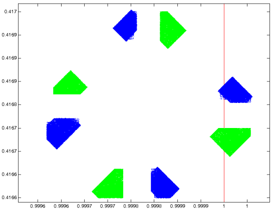

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Non-connected attractors

Suppose that an attractor has k pieces and let $\mathcal{P}$ be one of them.

- The other pieces can be obtained as $\Lambda_t^n(\mathcal{P}),\ n = 1\ \ldots\ k-1$
- $\Lambda_t^k(\mathcal{P}) = \mathcal{P}$.

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- $\Lambda_t^k(\mathcal{P}) = \mathcal{P}$.

This is the main reason behind the idea of using a **renormalization** scheme.

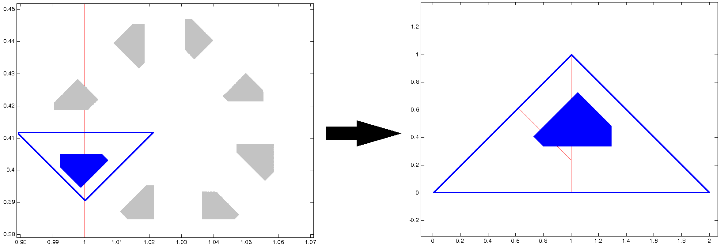
Definition

An *EBM* Γ is said to be **renormalizable** if there exists a domain $\mathcal{D}$ and a natural number k such that $\Gamma^k_{|\mathcal{D}}$ is, up to an affine change in coordinates, an *EBM* defined on $\mathcal{K}$.
If $\Gamma^k_{|\mathcal{D}}$ is *renormalizable*, we call Γ **twice renormalizable**.
In general, we can speak about **n times renormalizable EBMs** or even **infinitely renormalizable EBMs**.

Renormalization scheme: The First Renormalization

There exists an interval of parameters $\mathcal{I}_1$ and a domain $\mathcal{T}_1$ such that Λ_t^8 is, up to an affine change in coordinates, the *EBM*

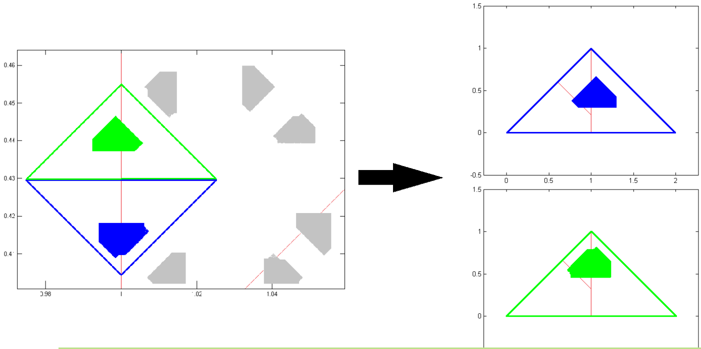
$$\Gamma_{1,t} = EBM(\mathcal{T}, \mathcal{C}, \mathcal{L}_{1,t}, \mathcal{O}, B_{1,t}).$$



Renormalization scheme: The Second Renormalization

Moreover, there exists an interval of parameters $\mathcal{I}_2 \subset \mathcal{I}_1$ and two domains $\mathcal{T}_{2,1}$, $\mathcal{T}_{2,2}$ such that $\Gamma^4_{1,t}$ (and therefore Λ_t^{32} restricted to each domain is, up to an affine change in coordinates, an *EBM* defined on $\mathcal{T}$. In other words,

- Λ_t is a **twice renormalizable EBM**.
- **Two** strange attractors are coexisting.



Conjectures

Conjecture 1

For every natural number n there exists an interval of parameters I_n such that Λ_t is a *n times renormalizable EBM* displaying, at least, n different strange attractors for every $t \in I_n$.

Conjecture 2

There is no value of t for which Λ_t is infinitely many renormalizable.

Reference

A. Pumariño, J. A. Rodríguez and E. Vigil, *Renormalizable Expanding Baker Maps: Coexistence of Strange Attractors*. To appear in Discrete and Continuous Dynamical System - A (2016).

Final remarks

Outstanding work

1. To complete the renormalization scheme.
2. To study the “hole case”.
3. To study new kinds of attractors.
4. (For the afterlife) By using the possible results obtained, prove the existence of two-dimensional strange attractors for the return maps associated to a neighbourhood of a generalized homoclinic tangency.

Bread is ready . . .

Thank you so much!