Stable manifolds of biholomorphisms in $\mathbb{C}^n$ asymptotic to formal curves

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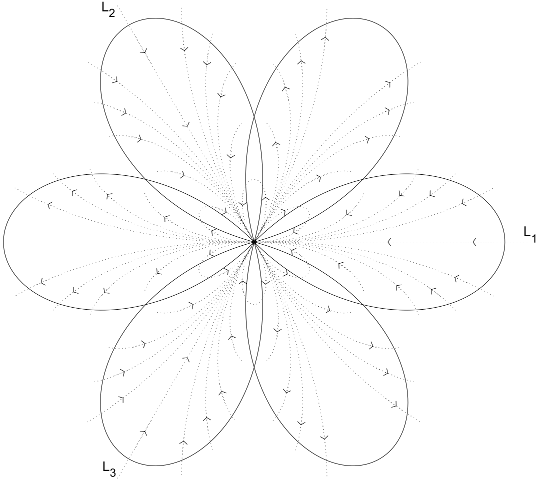
Dynamics in dimension 1

Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ be a germ of biholomorphism of $\mathbb{C}^n$ with $F(0) = 0$. We are interested in the existence of stable manifolds for F.

In the case $n = 1$, $F(z) = az + ...$ with $a \neq 0$. The existence of stable manifolds is determined by the multiplier a:

- If $|a| < 1$, every orbit of F converges to 0.
- If $|a| > 1$, no orbit of F converges to 0.
- If a is a root of unity, either F is periodic or there exist stable manifolds (Leau-Fatou flower theorem)
- If $|a| = 1$ and a is not a root of unity, there are no orbits converging to 0 (Pérez-Marco).

Leau-Fatou flower theorem



Dynamics of $z \mapsto z - z^4 + z^5 (\dots)$

Dynamics in dimension $n \geq 2$

For $n \geq 2$, we study the orbits which converge and are asymptotic to a given formal curve Γ. This means that the orbit converges tangentially to Γ and that property is preserved under blow-ups.

(Equivalently (for $n = 2$ and Γ non-singular), if $\gamma(s) = (s, a_1 s + a_2 s^2 + \dots)$ is a parametrization of Γ then an orbit $\{(x_j, y_j)\}$ is asymptotic to Γ if for any $N \geq 0$ there exists $C_N > 0$ such that for all j

$$\left| y_j - (a_1 x_j - a_2 x_j^2 - \cdots - a_N x_j^N) \right| \leq C_N |x_j|^{N+1}.)$$

In this case, Γ is necessarily invariant for F, i.e. $F(\gamma(s)) = \gamma(\theta(s))$ for some $\theta(s)$.

Necessary condition: Because of the one-dimensional results, we need that either

$$\left| (F|_\Gamma)'(0) \right| < 1 \quad \text{or} \quad (F|_\Gamma)'(0) \text{ is a root of unity and } F|_\Gamma \text{ is not periodic}$$

Existence of stable manifolds

Theorem

Consider $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ *and let* Γ *be a formal invariant curve such that*

$$\left|(F|_\Gamma)'(0)\right| < 1 \quad \textit{or} \quad (F|_\Gamma)'(0) \textit{ is a root of unity and } F|_\Gamma \textit{ is not periodic}$$

then there exist stable manifolds $M_1, M_2, \ldots, M_r$ *with* $\dim M_i \in \{1, \ldots, n\}$ *in which every orbit is asymptotic to* Γ. *Moreover, every orbit of* F *which converges and is asymptotic to* Γ *eventually lies in* $M_1 \cup \cdots \cup M_r$.

- For $n = 2$: L-H, J. Ribón, J. Raissy, F. Sanz Sánchez, *Stable manifolds of two-dimensional biholomorphisms asymptotic to formal curves*, Int. Math. Res. Not. 2021.
- In the general case: L-H, J. Ribón, F. Sanz Sánchez, L. Vivas, *Stable manifolds of biholomorphisms in* $\mathbb{C}^n$ *asymptotic to formal curves*, arXiv:2002.07102.

Idea of the proof

If $\left|(F|_\Gamma)'(0)\right| < 1$, after some blow-ups following Γ we have that the other eigenvalues of $DF(0)$ have modulus greater than 1, and by the stable manifold theorem Γ is analytic.

We assume then that $(F|_\Gamma)'(0)$ is a root of unity and that $F|_\Gamma$ is not periodic. We look for a convenient normal form for the pair (F, Γ) which allows to describe the behavior of the orbits. In the two-dimensional case, we get this normal form with blow-ups following Γ. In the higher dimensional case we use a normal form for linear ODEs due to Turrittin, that can be adapted to vector fields, and we apply it to a formal vector field associated to F.

Existence of stable manifolds

Up to an iteration of F, there exists a formal vector field X whose time-1 flow is F (Binyamini). We show that this vector field is geometrically significant: for example, the set of fixed points of F coincides with the singular locus of X and the formal invariant curves of F and X coincide.

We apply Turrittin reduction to the vector field X: after a finite number of holomorphic changes of coordinates, blow-ups and ramifications, the transformed vector field is in a convenient reduced form. In terms of the pair (F, Γ), there exist coordinates $(x, \underline{y}) \in \mathbb{C} \times \mathbb{C}^{n-1}$ such that Γ is non-singular and transverse to $x = 0$ and

$$x \circ F(x, \underline{y}) = x - x^{k+p+1} + O(x^{k+p+2})$$
$$\underline{y} \circ F(x, \underline{y}) = \exp\big(x^k \left(D(x) + x^p C\right)\big)\underline{y} + O(x^{k+p+1}),$$

where $k, p \geq 0$, $k + p \geq 1$, $D(x)$ is a diagonal matrix of polynomials of degree at most $p - 1$ with $D(0) \neq 0$ and C is a constant matrix commuting with $D(x)$.

Existence of stable manifolds

$$x \circ F(x, \underline{y}) = x - x^{k+p+1} + O(x^{k+p+2})$$
$$\underline{y} \circ F(x, \underline{y}) = \exp\big(x^k \left(D(x) + x^p C\right)\big)\underline{y} + O(x^{k+p+1})$$

If $\|\underline{y}\|$ is small, in the variable x we have $k + p$ attracting directions and attracting petals. Let ℓ be one those attracting directions.

We use the time-1 flow of the vector field

$$x^k \left[-x^{p+1}\frac{\partial}{\partial x} + \left(D(x) + x^p C\right)\underline{y}\frac{\partial}{\partial \underline{y}}\right]$$

as a toy model for the dynamics. The orbits are

$$\underline{y} = c \exp\left(-\int \frac{D(x) + x^p C}{x^{p+1}} dx\right)$$

and their behavior around ℓ depends on the real part of $D(x)$.

Existence of stable manifolds

We say that $\ell = \mathbb{R}^+$ is a *node direction* for the variable y_j if

$$(\mathrm{Re}(A_{j0}), \mathrm{Re}(A_{j1}), \ldots, \mathrm{Re}(A_{j,p-1})) < 0$$

in the lexicographic order, where $A_{j0} + A_{j1}x + \cdots + A_{j,p-1}x^{p-1}$ is the j-th entry of $D(x)$. Otherwise, we say that $\ell = \mathbb{R}^+$ is a *saddle direction*.

If $\ell = \mathbb{R}^+$ is a node direction for exactly s variables, $0 \leq s \leq n-1$, we find a stable manifold of dimension $s+1$.

$$\begin{aligned} x \circ F(x, \underline{y}) &= x - x^{k+p+1} + O(x^{k+p+2}) \\ \underline{w} \circ F(x, \underline{y}) &= \exp\big(x^k\,(D_1(x) + x^p C_1)\big)\underline{w} + O(x^{k+p+1}) \\ \underline{z} \circ F(x, \underline{y}) &= \exp\big(x^k\,(D_2(x) + x^p C_2)\big)\underline{z} + O(x^{k+p+1}) \end{aligned}$$

where $(x, \underline{y}) = (x, \underline{w}, \underline{z}) \in \mathbb{C} \times \mathbb{C}^s \times \mathbb{C}^{n-s-1}$, $\underline{w}$ are the node variables and $\underline{z}$ are the saddle variables. The stable manifold we find is given by a graph $\underline{z} = \varphi(x, \underline{w})$ over a domain of the form

$$S = \{(x, \underline{w}) \in \mathbb{C} \times \mathbb{C}^s : x \in \text{“petal” bisected by } \mathbb{R}^+, \|\underline{w}\| \leq |x|^m\}.$$

Existence of stable manifolds

The map $\varphi : S \to \mathbb{C}^{n-s-1}$ must satisfy the invariance equation

$$\varphi(F_1(x, \underline{w}, \varphi(x, \underline{w}), \overline{F_2}(x, \underline{w}, \varphi(x, \underline{w}))) = \overline{F_3}(x, \underline{w}, \varphi(x, \underline{w})),$$

where $F_1 = x \circ F$, $\overline{F_2} = \underline{w} \circ F$ and $\overline{F_3} = \underline{z} \circ F$.

We find φ as a fixed point of the map:

$$T\varphi(x_0, \underline{w}_0) = \sum_{j\geq 0} E(x_0) \left[\varphi(x_j, \underline{w}_j) E(x_j)^{-1} - \overline{F}_3(x_j, \underline{w}_j, \varphi(x_j, \underline{w}_j)) E(x_{j+1})^{-1}\right]$$

where $(x_{j+1}, \underline{w}_{j+1}) = \left(F_1(x_j, \underline{w}_j, \varphi(x_j, \underline{w}_j)), \overline{F}_2(x_j, \underline{w}_j, \varphi(x_j, \underline{w}_j))\right)$ and

$$E(x) \simeq \exp\left(-\int \frac{D_2(x) + x^p C_2}{x^{p+1}} dx\right)$$