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# Actions of groups of local biholomorphisms on the space of curves

**Theorem (Shub-Sullivan, 1973)**

Let $f : U \to \mathbb{R}^m$ be a $C^1$ map where $U$ is an open subset of $\mathbb{R}^m$. Suppose that 0 is an isolated fixed point of $f^n$ for every $n \in \mathbb{N}$. Then the sequence $(I(f^n, 0))_{n \geq 1}$ of fixed point indexes is bounded by above by a constant that does not depend on $n$.

$$I(f,0) = \deg\left(\frac{f(x)-x}{|f(x)-x|}\right) : S(\varepsilon) \to S(1)$$

$$\Delta = \{(x,x) : x \in \mathbb{R}^m\}$$

$$I(f,0) = (\Delta, F(\Delta))$$

$$F : U \times U \to \mathbb{R}^m \times \mathbb{R}^m$$
$$(x,y) \mapsto (x, f(y))$$

# Uniform Intersection Property

**Definition**

Let $G$ be a subgroup of $\mathrm{Diff}(\mathbb{C}^n, 0)$. We say that $G$ has the *uniform intersection property* (UI) if for any choice of analytic manifolds $V, W$ of complementary dimension, the set

$$\{(\phi(V), W) : \phi \in G\}$$

is finite.

$\dim \frac{\mathcal{O}_n}{(I, J)}$

$\mathrm{Diff}(\mathbb{C}^n, 0)$ = grupo de difeomorfismos holomorfos en un entorno de 0 en $\mathbb{C}^n$

$V = Z(I)$
$W = Z(J)$

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$G$ satisfies (UI) implies that $G$ is finitely determined.

$\phi \in \mathrm{Diff}(\mathbb{C}^n, 0)$, $\phi : U \to V$, $\phi(0) = 0$

G es finitamente determinado se $\exists K \in \mathbb{N}$ tal que todo $\phi \in G$ está determinado por $j^K \phi$

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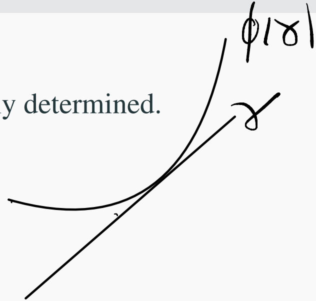


n=2 , v,w curvas

↳ recta

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**Theorem (Binyamini 2015)**

Lie subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ satisfy (UI).

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**Theorem (R. 2018)**

Finite dimensional subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ satisfy (UI).

- Podemos definir el cierre de Zariski $\bar{G}$ de $G$ como un límite inverso.
- $\dim(\bar{G}) < \infty \Rightarrow \bar{G} \sim$ grupo algebraico lineal
- $G$ cíclico, abeliano, f.g., Lie $\Rightarrow \dim(\bar{G}) < \infty$
- $\bar{G}$ es un objeto intrínseco

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$\{(\phi(V), W) \geq k\}$ define un ideal en los coeficientes de $\phi$

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**Theorem (R. 2018)**

Finite dimensional subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ satisfy (UI).

**Corollary (R. 2018)**

Finite generated nilpotent subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ satisfy (UI).

# Uniform Intersection Property

Every finite dimensional group is (UI)

Every (UI) group is finitely determined

**Theorem (R. 2020)**

Let $G$ be a subgroup of $\mathrm{Diff}(\mathbb{C}^2, 0)$. Then $G$ satisfies the uniform intersection property, if and only if, $G$ is finitely determined.

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$\Leftarrow$

Contraejemplo

G no (UI), finitamente determinado, $\dim(G) = \infty$

$\gamma = x$-eje. $X = \frac{\partial}{\partial y}$

$\phi = \exp\left(a(x) \cdot \frac{\partial}{\partial y}\right) = (x, y + a(x))$, $\nu_0(a) < K$

# Rough idea of proof

**Theorem (R. 2020)**

Let $G$ be a subgroup of $\mathrm{Diff}(\mathbb{C}^2, 0)$. Then $G$ satisfies the uniform intersection property, if and only if, $G$ is finitely determined.

$$\phi(x,y) = (x, y + a(x))$$

$$\big(\phi(y=0),\ y=0\big) = V_0(a)$$

$$\phi(y=0) \downarrow (x, a(x)) \qquad y=0 \downarrow (x, 0)$$

# Uniform intersection and curves

$\nu(C_1)$ ‖

The ultrametric

$$d(C_1, C_2) = \frac{m(C_1)m(C_2)}{(C_1, C_2)}$$

defines a topology in the space of curves.

$C_1, C_2 \rightarrow$ curvas lisas

$d(C_1, C_2) \leq \frac{1}{k} \Longleftrightarrow$ $C_1, C_2$ tienen $k-1$ tangentes comunes

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$G < \mathrm{Diff}(\mathbb{C}^2, 0)$ has the uniform intersection property

$$\Updownarrow$$

any $G$-orbit of a curve is discrete and closed

$(\phi_n(\gamma), \gamma') \to \infty \Leftrightarrow \phi_n(\gamma) \to \gamma'$

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any $G$-orbit of a curve is discrete

$\phi_n(\gamma) \to \gamma'$ ; $\phi_n(\gamma) \sim \phi_{n+1}(\gamma)$

$\gamma \sim \phi_n^{-1} \circ \phi_{n+1}(\gamma)$

# Actions on the space of curves

**Theorem (R. 2020)**

Let $G$ be a subgroup of $\mathrm{Diff}(\mathbb{C}^2, 0)$. Then $G$ acts on the space of curves by discrete orbits, if and only if, $G$ is finitely determined.

# Example

We denote

$$\phi_j(x_1, x_2, x_3) = (x_1, x_2 + d_j x_1^2 + x_3^{j+2}, x_3) \in \text{Diff}\,(\mathbb{C}^3, 0)$$

for any $j \in \mathbb{N}$ where $\{d_1, d_2, \cdots\}$ is linearly independent over $\mathbb{Q}$. We define the group $G = \langle \phi_1, \phi_2, \cdots \rangle$. It is abelian and finitely determined but non-(UI).

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Denote $\alpha = \{x_1 = x_2 = 0\}$ and $\beta = \{x_2 = 0\}$. We have

$$(\phi_j^{-1}(\alpha), \beta) = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(x_1, x_2 + d_j x_1^2 + x_3^{j+2}, x_2)}$$

$$= \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(x_1, x_2, x_3^{j+2})} = j + 2$$

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$$\phi_j(0, 0, x_3) = (0, x_3^{j+2}, x_3)$$

The orbit of the $x_3$-axis is non-discrete.

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$$\phi_j(x_1, x_2, x_3) = (0, x_3^{j+2}, x_3)$$

The orbit of the $x_3$-axis is non-discrete.

$$(\phi_j)_{|x_1=0}(x_2, x_3) = (x_2 + x_3^{j+2}, x_3)$$

# Actions on the space of curve

We study the actions of finitely determined subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ (with $n \geq 3$) on the space of curves.

Given a curve $\gamma$, we want to describe the possible obstructions to discreteness of $\mathcal{O}_G(\gamma)$.

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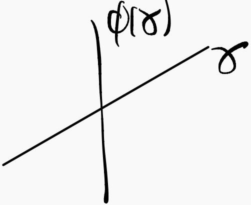
$$\mathcal{O}_G(\gamma) \text{ is discrete} \Leftrightarrow \mathcal{O}_{G_{\gamma,p}}(\gamma) \text{ is discrete}$$




$\phi \notin G_{\gamma,1}$

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**Definition**

We say that $\gamma$ is *$G$-weakly trascendent* if $\mathcal{O}_G(\gamma)$ is not contained in a proper analytic set.

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We say that $\gamma$ is $G$-*weakly trascendent* if $\mathcal{O}_G(\gamma)$ is not contained in a proper analytic set.

We say that $\gamma$ is $G$-*trascendent* if is $G_{\gamma,p}$–*trascendent* for any $p \geq 1$

# The trichotomy

**Theorem (R.)**

Let $G < \mathrm{Diff}(C^n, 0)$ be a finitely determined subgroup and $\gamma$ a curve. Suppose that $\mathcal{O}_G(\gamma)$ is non-discrete. Then at least one of the following properties holds:

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- $\gamma$ is non-trascendent
- There exists an abelian normal subgroup $H$ of some $G_{\gamma,p}$ such that $\mathcal{O}_H(\gamma)$ is non-discrete

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- $\gamma$ is non-trascendent
- There exists an abelian normal subgroup $H$ of some $G_{\gamma,p}$ such that $\mathcal{O}_H(\gamma)$ is non-discrete
- There exists $G_{\gamma,p}$ such that $G_{\gamma,p} \cap \mathrm{Diff}_1(\mathbb{C}^n, 0) = \{\mathrm{Id}\}$ and is non-virtually solvable.

# The abelian case

**Theorem (R.)**

Let $G < \mathrm{Diff}(C^n, 0)$ be a finitely determined abelian subgroup and $\gamma$ a curve. Suppose that $\mathcal{O}_G(\gamma)$ is non-discrete. Then $\gamma$ is non-weakly transcendent.

$\mathrm{Diff}(\mathbb{C}, 0)$

$$\phi(z) = z + z^2 = \exp\left( (z^2 + \cdots) \frac{\partial}{\partial z} \right)$$

$$\psi(z) = z + z^3 = \exp\left( (z^3 + \cdots) \frac{\partial}{\partial z} \right)$$

$$\langle \phi, \psi \rangle = \exp\left( (z^4 + \cdots) \frac{\partial}{\partial z} \right)$$

no-resoluble

Thank you