Oscillatory Dynamics and Neuronal Communication

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Oscillations in the brain

- Oscillatory activity in the brain is widely observed at different organization levels.
- Interactions between neurons lead to macroscopic oscillatory activity.
  - ↳ frequency bands → **Gamma band** (30 ∼ 100 Hz)
- Although oscillations have been associated to specific tasks or behaviours their functional role is not completely understood.
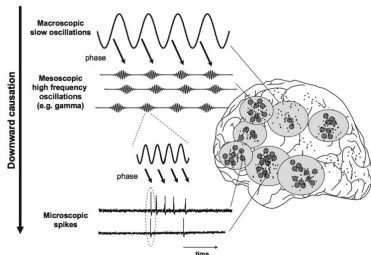
# Communication Through Coherence (P. Fries, 2005, 2015)

E-I networks

- Interaction between excitation and inhibition (E-I) has been proposed as one of the mechanisms for the generation of oscillations.
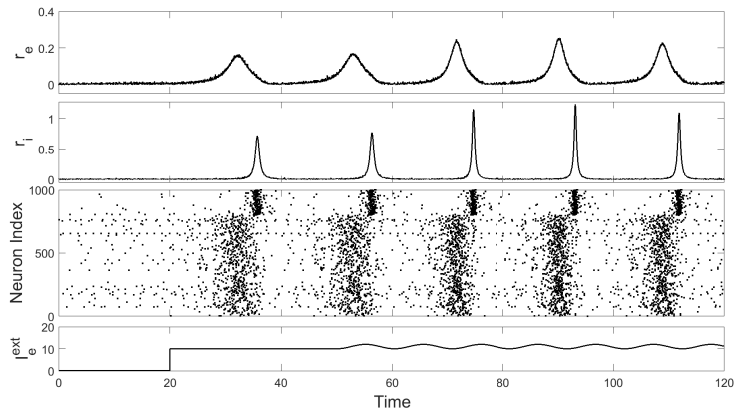- Different excitability properties along a cycle.
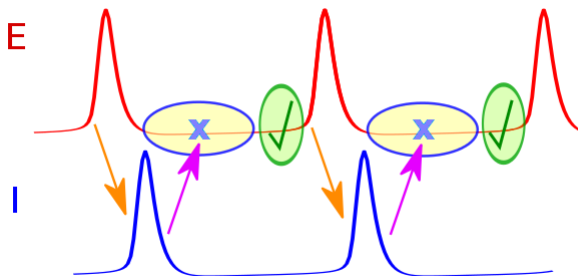
Communication through coherence (P Fries, 2005, 2015)

According to CTC theory two oscillating neuronal groups communicate much effectively when they are properly phase locked so the input reaches the receiving population at its maximum phase of **excitability**
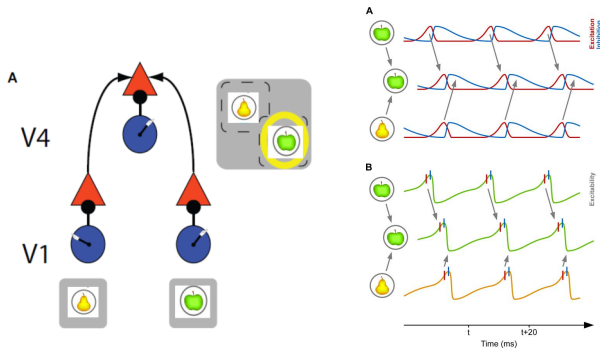
Microscopic (full) model: Simulation

Microscopic (full) model

Population of N neurons with all-to-all coupling (N_e excitatory $+$ N_i inhibitory)

Neuron's voltage V is governed by a QIF model

$$\begin{cases} \tau \dot{V}_j = V_j^2 + \eta_j + I_j \, , \\ \text{if } V_j \geq V_{\text{th}} \text{ then } V_j = V_{\text{reset}} \, , \end{cases} \quad j = 1, \ldots, N$$

- Time constant τ
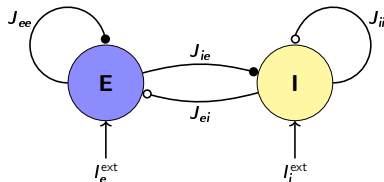- Constant input η_j: introducing heterogeneity in the network by choosing η_j from

$$\text{Lorentzian distr.:} \quad L(\eta) = \frac{1}{\pi} \frac{\Delta}{(\eta - \bar{\eta})^2 + \Delta^2} \quad \text{with} \quad \begin{cases} \text{Half-width } \Delta \\ \text{Center } \bar{\eta} \end{cases}$$

- Input current I_j: common external current $+$ synaptic current
  - Excitatory pop.: $I_e = I_e^{\text{ext}} + \tau_e S_{ee} - \tau_e S_{ei}$
  - Inhibitory pop.: $I_i = I_i^{\text{ext}} + \tau_i S_{ie} - \tau_i S_{ii}$

  with S_{ab} modelling the synapses $\boldsymbol{b}$-to-$\boldsymbol{a}$ as: $\tau_s \dot{S}_{ab} = -S_{ab} + J_{ab} r_b$ (linear ODE)

Mean-field model for a E-I network (I)

Mean-field exact model [*]

E cells
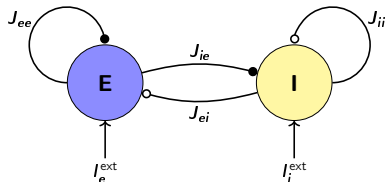$$\begin{cases} \tau_e \dot{r}_e = \dfrac{\Delta_e}{\pi \tau_e} + 2 r_e V_e\,, \\[2mm] \tau_e \dot{V}_e = V_e^2 + \overline{\eta}_e + I_e - (\tau_e \pi r_e)^2\,, \\[2mm] \tau_{s_e} \dot{S}_{ee} = -S_{ee} + J_{ee} r_e\,, \\[2mm] \tau_{s_e} \dot{S}_{ei} = -S_{ei} + J_{ei} r_i\,, \end{cases}$$

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[*] *Montbrió, Pazó & Roxin, Phys X, 2015*
[*] *Dumont & Gutkin, PLoS Comput Biol, 2019*

Variables

- **r**: firing rate (spikes per unit of time)
- **V**: mean membrane potential
- **S**$_{ab}$: synaptic connection from b to a

Parameters

- $\boldsymbol{\tau}$: time constants (speed dynamics)
- $\boldsymbol{J_{ab}}$: $\boldsymbol{b}$-**to**-$\boldsymbol{a}$ connection's strength
- $\boldsymbol{\Delta}$: half-width of Lorentzian distr.
- $\boldsymbol{\overline{\eta}}$: center of Lorentzian distr.

Mean-field model for a E-I network (II)

Mean-field exact model [*]

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$$\begin{cases} \tau_e \dot{r}_e = \dfrac{\Delta_e}{\pi \tau_e} + 2 r_e V_e\,, \\[2mm] \tau_e \dot{V}_e = V_e^2 + \overline{\eta}_e + I_e - (\tau_e \pi r_e)^2\,, \\[2mm] \tau_{s_e} \dot{S}_{ee} = -S_{ee} + J_{ee} r_e\,, \\[2mm] \tau_{s_e} \dot{S}_{ei} = -S_{ei} + J_{ei} r_i\,, \end{cases}$$

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Total input currents

- E-input: $I_e = I_e^{\text{ext}} + \tau_e S_{ee} - \tau_e S_{ei}$
- I-input: $I_i = I_i^{\text{ext}} + \tau_i S_{ie} - \tau_i S_{ii}$

Oscillations ($I_e^{\text{ext}}, I_i^{\text{ext}}$)

[*] *Montbrió et al., Phys X, 2015*
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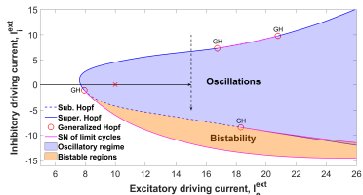
Gamma rhythms: PING mechanism
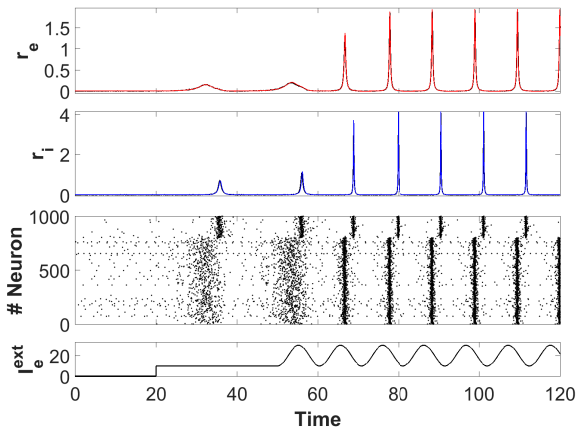
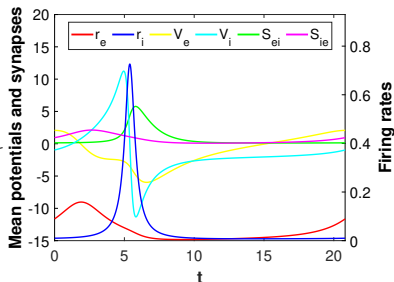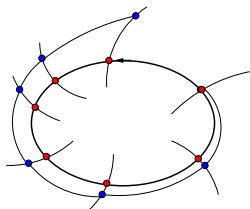
Parameters: $J_{ee} = J_{ii} = 0$, $J_{ei} = J_{ie} = 15$

Phases of an oscillator

Let Γ be the hyperbolic limit cycle of period T^* parametrized by the phase variable $\theta = t + \theta_0 \ (\mathrm{mod}\ T^*)$ as

$$\gamma : \mathbb{T} \to \mathbb{R}^n$$
$$\theta \to \gamma(\theta)$$

* Phase function $\Theta : \Gamma \subset \mathbb{R}^n \to \mathbb{S}^1$, $\Theta(x) = \theta$
  ↳ Basin of attraction Γ: $\|x(t) - y(t)\| \overset{t \to \infty}{\longrightarrow} 0 \implies \Theta(y_0) = \Theta(x_0)$

Phase response curve and Adjoint method

The *Phase response curve* (PRC) measures the phase variation due to a given perturbation $\Delta \vec{x}$ applied at different phases of the cycle
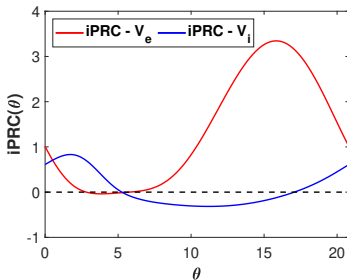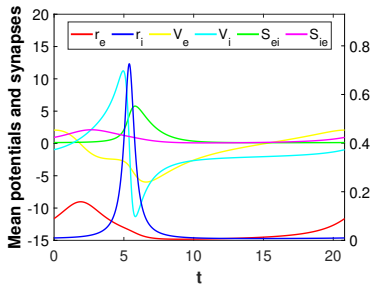
$$PRC(\theta) = \theta_{\text{new}} - \theta, \quad \theta \in [0, T^*)$$

* Sign: $PRC > 0 \rightarrow$ phase advance whereas $PRC < 0 \rightarrow$ phase delay

iPRC (infinitesimal perturbation) $PRC(\theta) = \theta_{\text{new}} - \theta \approx \nabla \Theta(\gamma(\theta)) \cdot \Delta \vec{x} = Z(\theta) \cdot \Delta \vec{x}$
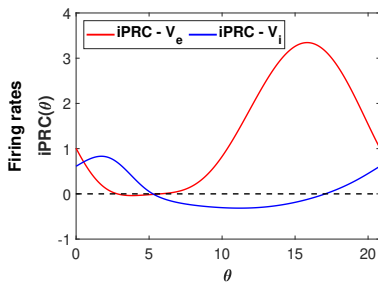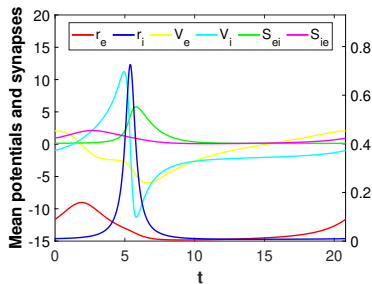
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Adjoint Method: The iPRC (linear response of an oscillator), $Z(t) \in \mathbb{R}^n$, is a periodic solution of

$$\dot{Z} = -D_x F^T (\gamma(t)) Z\,, \quad \text{subject to} \quad Z(t) \cdot F(\gamma(t)) = 1$$

Phase Reduction

Consider $\dot{x} = F(x) + Ap(t)\vec{v}$, with T-periodic perturbation $p(t)$ in the direction $\vec{v}$:

$$\frac{d\theta}{dt} = \nabla\Theta(x(t)) \cdot \frac{dx}{dt} = 1 + Ap(t)\underbrace{[\nabla\Theta(x(t)) \cdot \vec{v}]}_{\textbf{①}} \approx 1 + Ap(t)[Z(\theta) \cdot \vec{v}]$$

> **①** *Weak coupling hypothesis*: if $\|A\,p(t)\| << 1 \implies \nabla\Theta(x(t)) \approx Z(\theta)$

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$$\theta_{n+1} = P(\theta_n) = \Phi_T(\theta_n) \mod T^*.$$

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$p : q$ phase-locking: Two oscillators are $p : q$ phase-locked if oscillator 1 (forced) rotates p times as oscillator 2 (the forcing) rotates q.

> $p : q$ phase-locking $\longleftrightarrow$ fixed/periodic points of P: $\theta + pT^* = P^q(\theta)$

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Rotation number: Defined as (with $\bar{P}$ the lift of P)

$$\rho(P) := \lim_{n \to \infty} \frac{\bar{P}^n(x) - x}{n}$$

↳ if $\rho = \frac{p}{q} \to p : q$ phase-locking.

External drive - Von Mises distribution

Assume that external inputs arrive more or less clustered around some phases of the cycle.

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Periodic perturbation: Von Mises density function

$$p(t) = \frac{e^{\kappa \cos\left(\frac{2\pi(t-\mu)}{T}\right)}}{2\pi\, I_0(\kappa)}, \quad x \in \left[-\frac{T}{2}, \frac{T}{2}\right]$$

$\kappa > 0$: concentration of the distribution about the location μ

- ∗ $\mu \in \mathbb{R}$: the location where the distribution clusters around
- ∗ $\kappa > 0$: a measure of concentration of the distribution about the location μ
- ∗ T: period of the distribution
- ∗ I_0: modified Bessel function (of first kind) of order 0
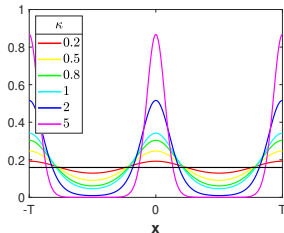
Arnold tongues and entrainment properties

Rotation number

Arnold tongues
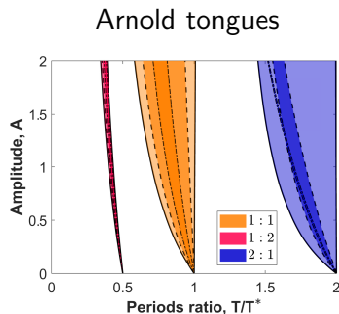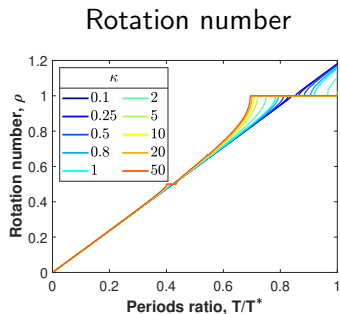
Increasing the coherence κ of the periodic input enlarges the input frequency range that entrains the network

[*Reyner-Parra, H., BioRXiv, 2021*]

Implications for CTC

Communication between brain regions is much effective if oscillators are properly phase-locked.

* If external input **precedes** inhibition $\implies$ increase excitatory activity (more responsive to that stimulus)

* If external input **follows** inhibition $\implies$ input may be ignored (less responsive to that stimulus)

Factors

* Timing inhibition-perturbation: $\Delta\tau = \dfrac{t_{\text{inh}} - t_p}{T}$ $\begin{cases} \Delta\tau < 0.5 \implies \text{inhibition } \textbf{follows} \\ \Delta\tau > 0.5 \implies \text{inhibition } \textbf{precedes} \end{cases}$

* Impact perturbation onto r_e:
$$\begin{cases} \Delta\alpha = \dfrac{R_e^A}{R_e^0}, \ R_e^A : \textbf{maximum} \text{ exc. activity for a given } A \\ \\ \Delta\bar{\alpha} = \dfrac{\bar{R}_e^A}{\bar{R}_e^0}, \ \bar{R}_e^A := \dfrac{1}{T}\int_0^T r_e(t)dt \\ \textbf{time-average} \text{ exc. activity for a given } A \end{cases}$$

Implications for CTC

Measures of effective communication inside 1:1 phase-locking region. Colors correspond to different amplitude values.
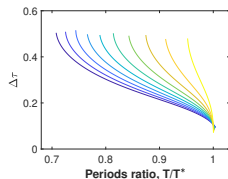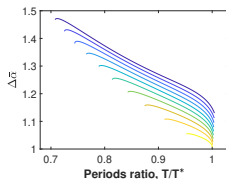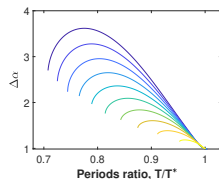
Higher input frequencies produce more effective communication

Implications for CTC
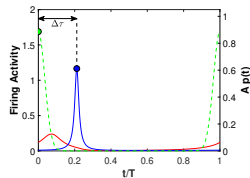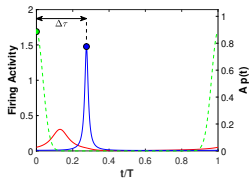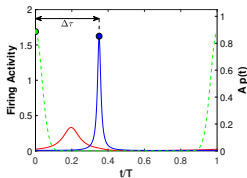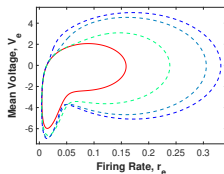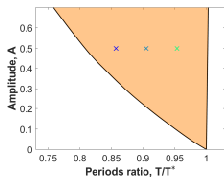
Timing inhibition-perturbation $\Delta\tau$ and impact onto activity of E-cells (r_e) depends on input frequency.
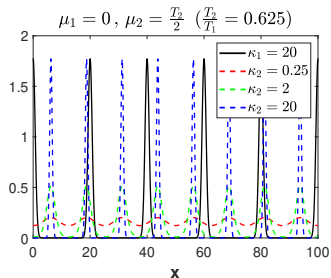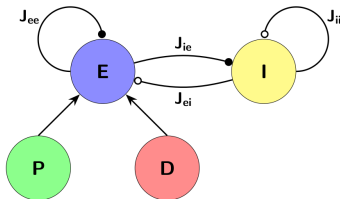
Two competing inputs - Selective communication
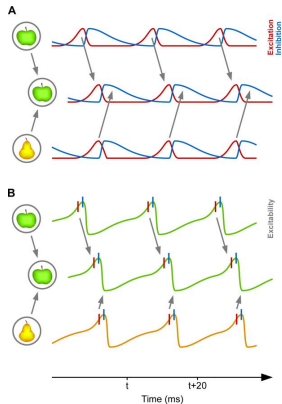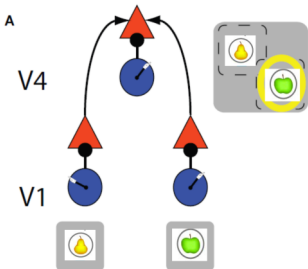
Two competing inputs - Selective communication

Goal

To study how the phase-locked states are affected by the presence of a distracting stimulus.

* Does the distractor prevent the oscillator from following the primary stimulus?
* Which stimulus is effective and which one is ignored?

- Vary frequency of the first input but ensuring 1:1 phase-locking.
- Vary distractor coherence and distractor frequency.

Effects of a distractor

Compute the **synchronization index or vector strength** as a function of the distractor coherence κ_2 and period T_2.

$$r = \frac{1}{N}\sqrt{\left(\sum_i x_i\right)^2 + \left(\sum_i y_i\right)^2}, \quad \text{with } \begin{cases} x_i = \cos\theta_i \\ y_i = \sin\theta_i \end{cases}$$

* θ_i: phase for the i-th iterate of the stroboscopic map (P_{T_1}).
* N: total number of iterations

If $r = 1 \Rightarrow$ perfect phase-locking. If $r = 0 \Rightarrow$ totally desynchronized
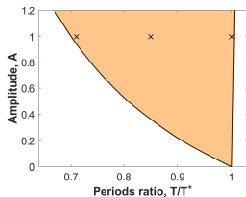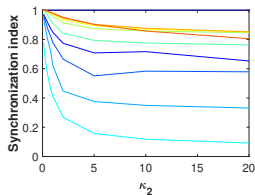
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Effects of a distractor

Network entrainment by higher frequency inputs is more robust to distractors.

$\mu_1 = 0$, $\mu_2 = \frac{T_1}{2}$ $(\frac{T_2}{T_1} = 0.625)$

$\mu_1 = 0$, $\mu_2 = \frac{T_1}{2}$ $(\frac{T_2}{T_1} = 1.2)$
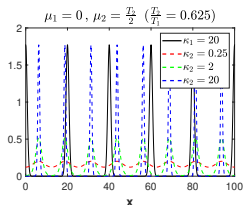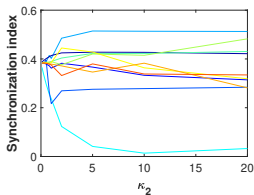
$p = T_1/T^* = 0.71$
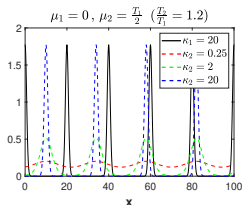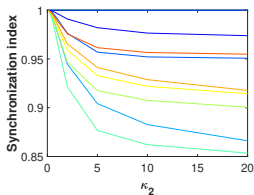
$p = T_1/T^* = 0.85$

$p = T_1/T^* = 1$

The role of inhibition

- $p = T_1/T^*$ (high input frequencies / close to left boundary of the Arnold tongue)

  ↳ Inhibition **is present** near distractor volleys, therefore suppressing its activity

- $p = T_1/T^*$ (closer to right boundary of the Arnold tongue)

  ↳ Inhibiton **is less present** around distractor volleys, giving the competitor the opportunity to weaken synchrony
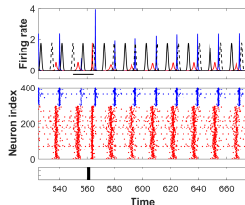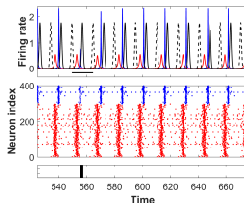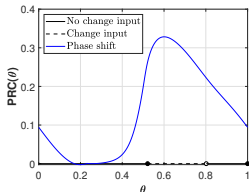
Switch between inputs

Oscillator perturbed by **two-equally strong** stimuli

> **Goal**
>
> Study the effects of brief/short stimulus (**pulse**) to the resulting coupled system.
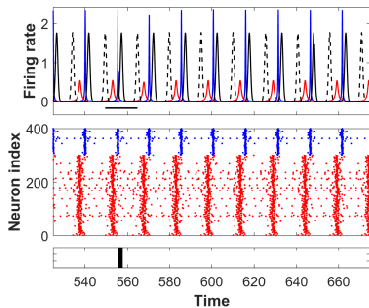>
> Brief stimulus could
>
> * lengthen or shorten the cycle period (i.e. phase shifting)
> * force the oscillator to phase-lock to the other input
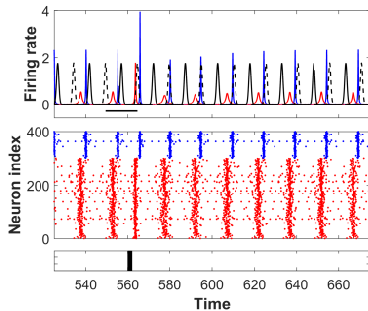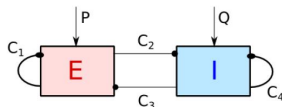>   ↳ **Change in the effective input**
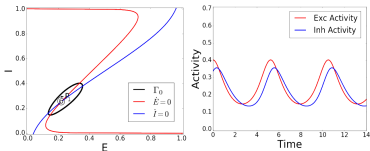
Switch between inputs

Beyond the weak coupling hypothesis

Wilson-Cowan equations for an E-I network

$$\tau_e \dot{E} = -E + S_e(c_1 E - c_3 I + P)$$
$$\tau_i \dot{I} = -I + S_i(c_2 E - c_4 I + Q)$$

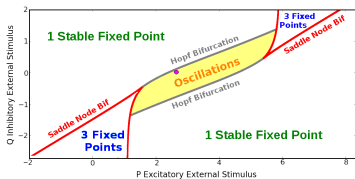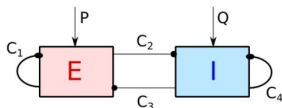where $S_k(x) = 1/(1 + e^{-a_k(x-\theta_k)})$.

Beyond the weak coupling hypothesis

Wilson-Cowan equations for an E-I network

$$\tau_e \dot{E} = -E + S_e(c_1 E - c_3 I + P)$$
$$\tau_i \dot{I} = -I + S_i(c_2 E - c_4 I + Q)$$
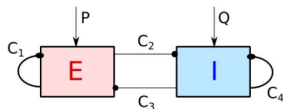
where $S_k(x) = 1/(1 + e^{-a_k(x-\theta_k)})$.

Beyond the weak coupling hypothesis

Wilson-Cowan equations for an E-I network

$$\tau_e \dot{E} = -E + S_e(c_1 E - c_3 I + P + Ap(t))$$
$$\tau_i \dot{I} = -I + S_i(c_2 E - c_4 I + Q)$$
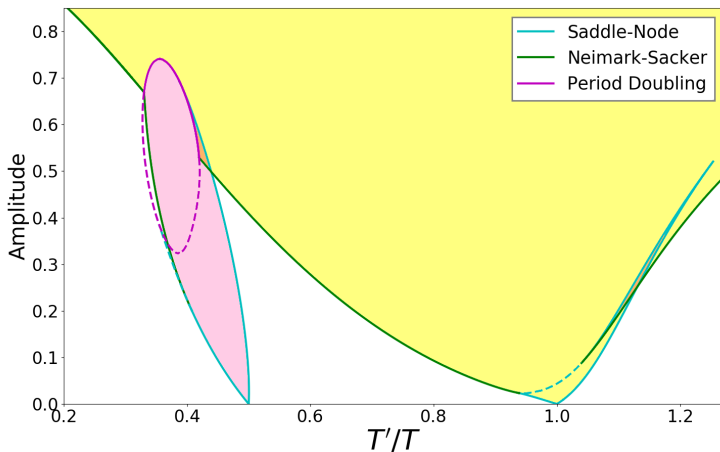
where $S_k(x) = 1/(1 + e^{-a_k(x-\theta_k)})$.

Add periodic forcing modelling oscillatory input from the emitting population to the E receiving population
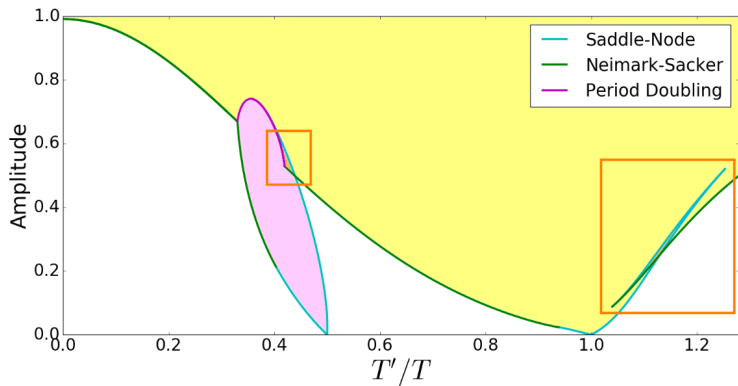
$$p(t) = 1 + \cos\left(\frac{2\pi t}{T}\right)$$

Bifurcation diagram

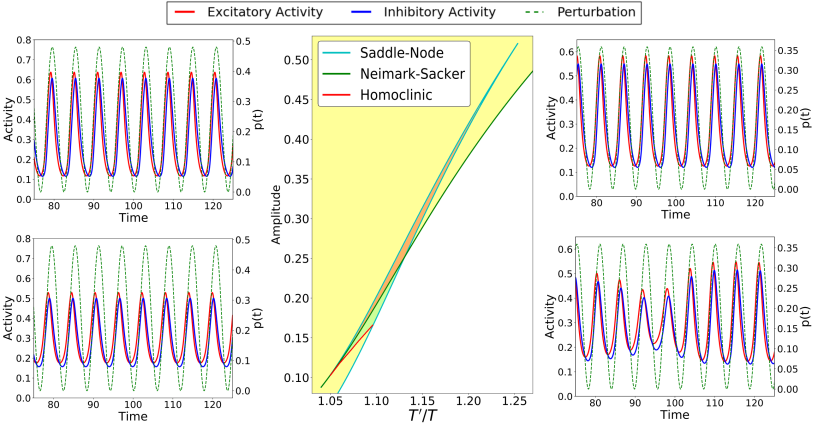Bifurcations of fixed points of the stroboscopic map (time-T map of the system).

1:1 phase locking (yellow), 1:2 phase-locking (pink)

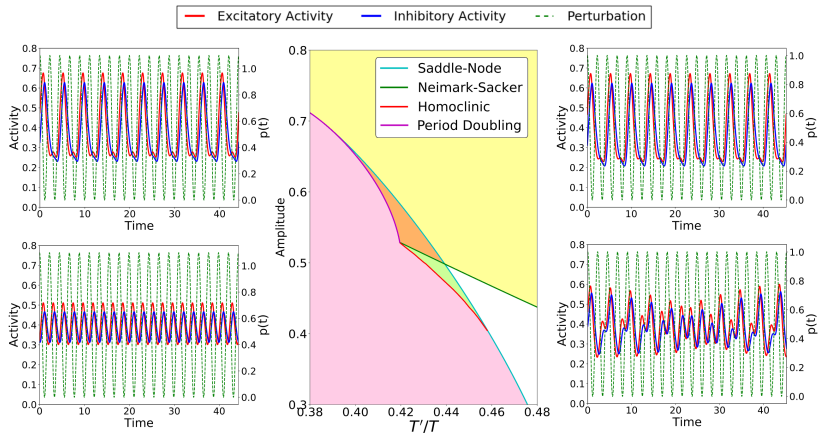[*Pérez-Cervera, Seara, H., CNSNS, 2020*]

Bistability areas

Bistability

Bistability between two synchronous solutions (left column) and between synchronous and asynchronous solutions (right column).

Bistability

Bistability between two synchronous solutions 1:1 and 1:2 PL (left column) and between synchronous and asynchronous solutions (right column).

Conclusions

- Simplified mathematical setting for CTC.
- The phase relationship that naturally emerges has an impact on the robustness of the entrainment.
- Both input frequency and coherence play a relevant role in establishing effective communication
- Phase reduction must be assumed cautiously. What happens for large amplitudes?

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