Chaos in the unfolding of Hopf-Bogdanov-Takens singularities

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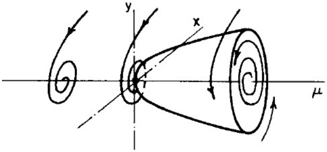
Hopf-Bogdanov-Takens (HBT) singularities

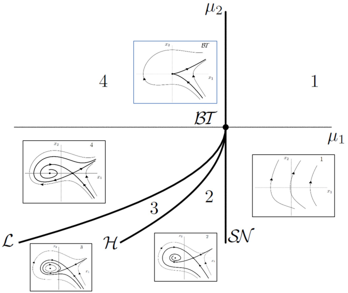
Consider $\dot{x} = f_\mu(x)$ with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$ and f_μ smooth.

Hopf bifurcation (subcritical)



$$\begin{cases} \dot{x} = \mu x - y - (x^2 + y^2)x \\ \dot{y} = x + \mu y - (x^2 + y^2)y \end{cases}$$

Bogdanov-Takens bifurcation



$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu_1 + \mu_2 y + x^2 - xy \end{cases}$$

[1] J. Guckenheimer, P. Holmes, *Applied Mathematical Sciences 42*, (1983).
[2] H. W. Broer, F. Dumortier, S. J. van Strien, and F. Takens, Structures in dynamics, Studies in Mathematical Physics, vol. 2 (1991).
[3] Y. A. Kuznetsov, Applied Mathematical Sciences 112, (1995).

Hopf-Bogdanov-Takens (HBT) singularities

Let X be a $\mathcal{C}^\infty$-vector field on a neighborhood of $p \in \mathbb{R}^4$ with $X(p) = 0$ and

$$DX(p) \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

p (or X) is called a *Hopf-Bogdanov-Takens* singularity.

[1] F. Takens, *Publ. Math. Inst. Hautes udes Sci. 43*, (1974).

[2] F. Dumortier, *J. Differential Equations 23*, (1977).

[3] F. Dumortier, S. Ibáñez, *J. Differential Equations 127*, (1996); *Nonlinearity 11*, (1998); and *Publ. Mat. 43(2)*, (1999).

Motivation to study the HBT singularities

- HBT singularities appear naturally in the context of coupled oscillators.
- Primary local bifurcations up to codimension 2 were obtained under additional symmetry properties.
- A HBT bifurcation curve was detected in a model of two Brusselators linearly coupled by diffusion.

$$\begin{cases} x' = A - (B+1)x + x^2 y \\ y' = Bx - x^2 y \end{cases}$$

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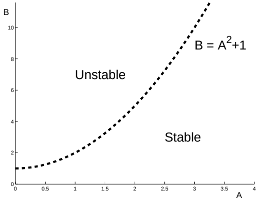

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$$\left\{\begin{array}{rcl} x_1' &=& A-(B+1)x_1+x_1^2y_1+\lambda_1(x_2-x_1) \\ y_1' &=& Bx_1-x_1^2y_1+\lambda_2(y_2-y_1) \\ x_2' &=& A-(B+1)x_2+x_2^2y_2+\lambda_1(x_1-x_2) \\ y_2' &=& Bx_2-x_2^2y_2+\lambda_2(y_1-y_2) \end{array}\right.$$

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\end{cases}
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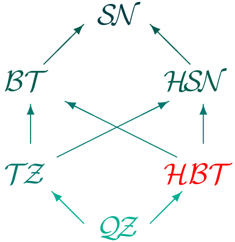

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- H and BT interactions for a fluid-conveying tube, but under symmetry conditions.
- HBT singularities are essential to understand the unfolding of the 4-dimensional nilpotent singularities of codimension 4.

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Part I:

A formal classification of HBT singularities

Part II:

Chaos in the unfolding of HBT singularities

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[2] F. Drubi, S. Ibáñez, D. Rivela, *Discrete Contin. Dyn. Syst. Series B 25*, (2020).

Unfoldings of HBT singularities

Steps for a systematic study of bifurcations of equilibria in the least possible phase-space dimensions:

- Reduction to normal form
- Nondegeneracy and transversality conditions to derive the simplest parameter-dependent form for a "generic" system
- Bifurcation diagrams of the *approximate normal form*
- Influence of higher-order terms

[1] J. Guckenheimer, P. Holmes, *Applied Mathematical Sciences 42*, (1983).

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Unfoldings of HBT singularities

Any $\mathcal{C}^\infty$ family X_ν with $\nu \in \mathbb{R}^3$, such that X_0 is a HBT singularity, can be written in a normal form up to order m:

$$
\begin{cases}
x' = y + \widetilde{F}_1(x, y, u, v, \nu) \\
y' = \displaystyle\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k=0}^{m-2l} a_{k,l}(\nu) x^k (u^2+v^2)^l + y \sum_{l=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{k=0}^{m-2l-1} b_{k,l}(\nu) x^k (u^2+v^2)^l \\
\qquad + \widetilde{F}_2(x, y, u, v, \nu) \\
u' = \displaystyle\sum_{l=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{m-2l+1} (c_{k,l}(\nu) u - d_{k,l}(\nu) v) x^k (u^2+v^2)^{l-1} + \widetilde{F}_3(x, y, u, v, \nu) \\
v' = \displaystyle\sum_{l=1}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=0}^{m-2l+1} (d_{k,l}(\nu) u + c_{k,l}(\nu) v) x^k (u^2+v^2)^{l-1} + \widetilde{F}_4(x, y, u, v, \nu)
\end{cases}
$$

with $F_i(x, y, u, v, \nu) = O(\|(x, y, u, v, \nu)\|^{m+1}),\ i = 1, \ldots, 4.$

Remark: The plane $u = v = 0$ is *formally* invariant.

Unfoldings of HBT singularities

The normal form of HBT unfolding with $m = 2$:

$$\begin{cases} x' = y + O(3) \\ y' = a_{0,0}(\nu) + a_{1,0}(\nu)x + b_{0,0}(\nu)y \\ \qquad\qquad +A_{2,0}(\nu)x^2 + B_{1,0}(\nu)xy + A_{0,1}(\nu)(u^2 + v^2) + O(3) \\ u' = c_{0,1}(\nu)u - d_{0,1}(\nu)v + (C_{1,1}(\nu)u - D_{1,1}(\nu)v)x + O(3) \\ v' = d_{0,1}(\nu)u + c_{0,1}(\nu)v + (D_{1,1}(\nu)u + C_{1,1}(\nu)v)x + O(3) \end{cases}$$

with $a_{0,0}(0) = a_{1,0}(0) = b_{0,0}(0) = c_{0,1}(0) = 0$, $d_{0,1}(0) = 1$.

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Generic conditions: $A_{2,0}(0)$, $B_{1,0}(0)$, $A_{0,1}(0)$, $C_{1,1}(0) \neq 0$.

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Generic conditions: $A_{2,0}(0)$, $B_{1,0}(0)$, $A_{0,1}(0)$, $C_{1,1}(0) \neq 0$.
Transversality condition:

$$|DP(0)| \neq 0, \quad \text{with } P(\nu) = (a_{0,0}(\nu), b_{0,0}(\nu), c_{0,1}(\nu)).$$

Unfoldings of HBT singularities

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$$\begin{cases} x' = y + O(3) \\ y' = \lambda_1 + \lambda_2 y + x^2 + xy + \kappa(u^2 + v^2) + O(3) \\ u' = \mu u - d_{0,1}(\lambda_1, \lambda_2, \mu)v \\ \qquad\qquad +(C_{1,1}(\lambda_1, \lambda_2, \mu)u - D_{1,1}(\lambda_1, \lambda_2, \mu)v)x + O(3) \\ v' = d_{0,1}(\lambda_1, \lambda_2, \mu)u + \mu v \\ \qquad\qquad +(D_{1,1}(\lambda_1, \lambda_2, \mu)u + C_{1,1}(\lambda_1, \lambda_2, \mu)v)x + O(3) \end{cases}$$

with $d_{0,1}(0,0,0), C_{1,1}(0,0,0) \neq 0$ and $\kappa = \pm 1$.

New parameters: $(\lambda_1, \lambda_2, \mu) = (a_{0,0}(\nu), b_{0,0}(\nu), c_{0,1}(\nu))$.

Primary bifurcations in the unfolding of a HBT singularity

Second-order truncation of the normal form:

$$X^2_{(\lambda_1,\lambda_2,\mu)} := \begin{cases} x' = y \\ y' = \lambda_1 + \lambda_2 y + x^2 + xy + \kappa(u^2 + v^2) \\ u' = \mu u - d_{0,1} v + (C_{1,1} u - D_{1,1} v)x \\ v' = d_{0,1} u + \mu v + (D_{1,1} u + C_{1,1} v)x \end{cases}$$

where $d_{0,1}$, $D_{1,1}$ and $C_{1,1}$ are functions of $(\lambda_1, \lambda_2, \mu)$.

Close to the origin and for $\|(\lambda_1, \lambda_2, \mu)\|$ small enough, the family has

- no equilibrium points if $\lambda_1 > 0$;
- a unique equilibrium point at $P^0 = (0, 0, 0, 0)$ if $\lambda_1 = 0$;
- two equilibria $P^{\pm} = (\pm\sqrt{-\lambda_1}, 0, 0, 0)$ if $\lambda_1 < 0$.

Primary bifurcations in the unfolding of a HBT singularity

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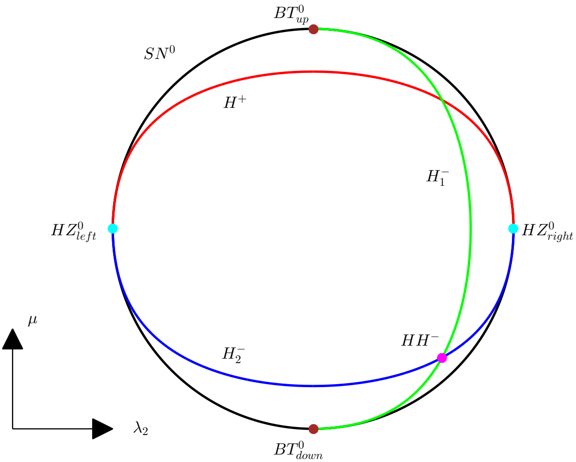

A sphere $\lambda_1^2 + \lambda_2^2 + \mu^2 = \delta$ is fixed and the front view corresponds to $\lambda_1 < 0$. We also assume $C_{1,1} < 0$.

Chaos in the generic unfoldings of HBT singularities

Hopf-Zero bifurcations:

Theorem

When $\lambda_1 = \mu = 0$ and $\lambda_2 \neq 0$, family $X^2_{(\lambda_1,\lambda_2,\mu)}$ exhibits a Hopf-Zero singularity. Four cases can be distinguished:

	$\lambda_2 C_{1,1} < 0$	$\lambda_2 C_{1,1} > 0$
$\kappa = -1$	*Case II*	*Case I*
$\kappa = 1$	*Case IV*	*Case III*

The emergence of chaotic behavior can be expected in case III, when higher order terms are considered.

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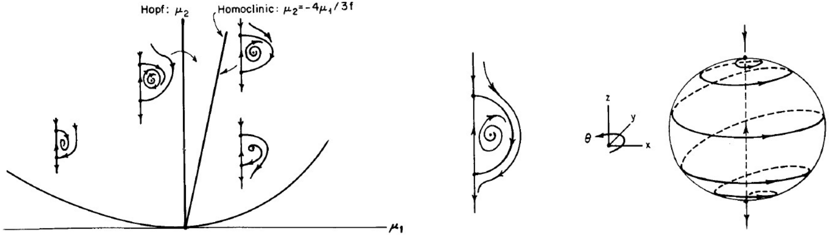
Chaos in the generic unfoldings of HBT singularities

Hopf-Zero bifurcations: Case III



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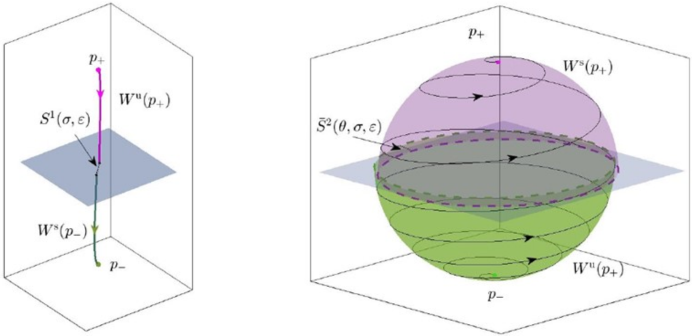
Hopf-Zero bifurcations: Effect of higher-order terms



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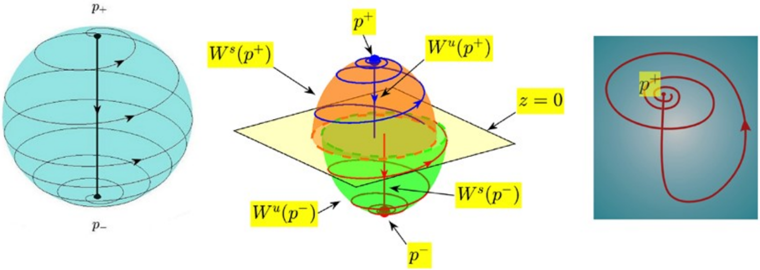

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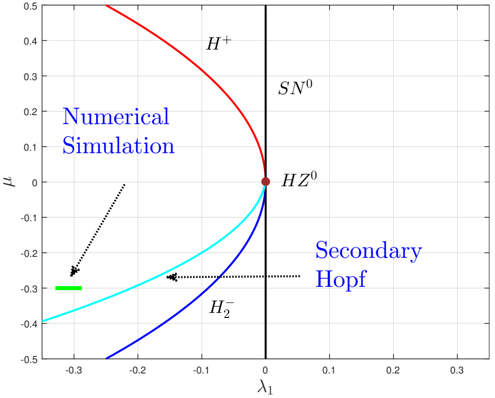
Chaos in the generic unfoldings of HBT singularities

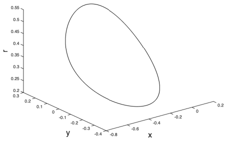
Partial bifurcation diagram close to the HZ point: Case III with $C_{1,1} = -1$, $\kappa = 1$ and $\lambda_2 = -0.1$.



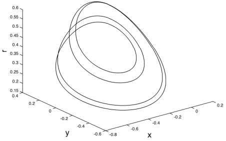
Note that the birth of an invariant torus when parameters cross the secondary Hopf bifurcation.

Chaos in the generic unfoldings of HBT singularities

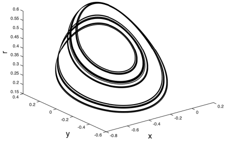
Period-doubling and chaotic dynamics close to a HZ bifurcation ($\lambda_2 = -0.1$, $\mu = -0.3$ and $C_{1,1} = -1$):



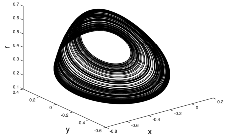
$\lambda_1 = -0.291$



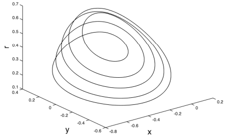
$\lambda_1 = -0.313$



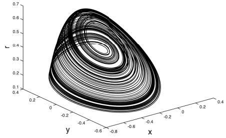
$\lambda_1 = -0.315$



$\lambda_1 = -0.319$ (Lyap. Exp. 0.042)



$\lambda_1 = -0.321$



$\lambda_1 = -0.326$ (Lyap. Exp. 0.052)

Chaos in the generic unfoldings of HBT singularities

Hopf-Hopf bifurcations:

Theorem

When $\lambda_2 = \sqrt{-\lambda_1}$ *and* $\mu - \widehat{C}_{1,1}\sqrt{-\lambda_1} = 0$*, family* $X^2_{(\lambda_1,\lambda_2,\mu)}$ *exhibits a Hopf-Hopf singularity. Eight cases can be distinguished:*

	$C_{1,1} < 0$	$0 < C_{1,1} < \frac{1}{4}$	$\frac{1}{4} < C_{1,1} < \frac{1}{2}$	$\frac{1}{2} < C_{1,1}$
$\kappa = -1$	*Case IVb*	*Case VIIa*	*Case VIIb*	*Case V*
$\kappa = 1$	*Case VIa*	*Case Ib*	*Case Ia*	*Case III*

The emergence of chaotic behavior can be expected in case VIa, when higher order terms are considered.

[1] J. Guckenheimer, P. Holmes, *Applied Mathematical Sciences 42*, (1983).

[2] F. Drubi, S. Ibáñez, D. Rivela, *Discrete Contin. Dyn. Syst. Series B 25*, (2020).

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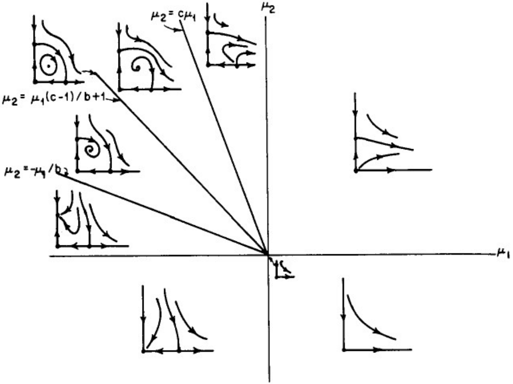
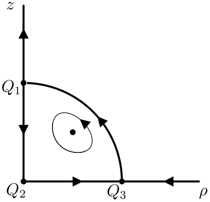
Chaos in the generic unfoldings of HBT singularities

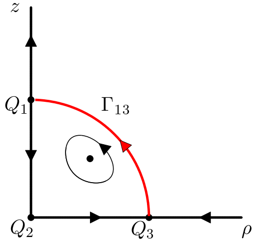
Hopf-Hopf bifurcations: Case VIa





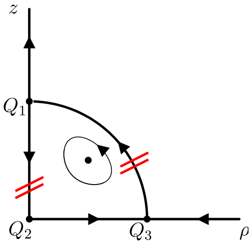
[1] J. Guckenheimer, P. Holmes, *Applied Mathematical Sciences 42*, (1983).

Hopf-Hopf bifurcations: Case VIa



[1] J. Guckenheimer, P. Holmes, *Applied Mathematical Sciences 42*, (1983).

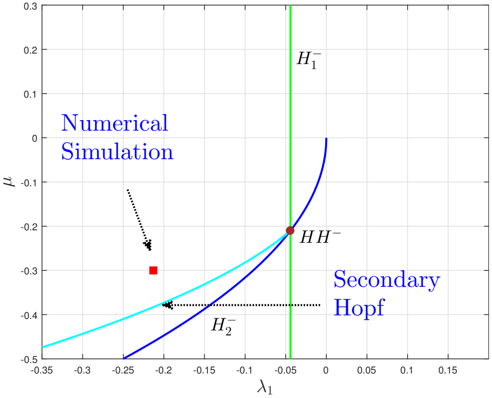
Hopf-Hopf bifurcations: Effect of higher-order terms



[1] J. Guckenheimer, P. Holmes, *Applied Mathematical Sciences 42*, (1983).

Chaos in the generic unfoldings of HBT singularities

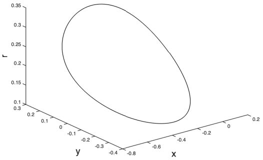
Partial bifurcation diagram close to the HH point: Case VIa with $C_{1,1} = -1$, $\kappa = 1$ and $\lambda_2 = 0.21$.



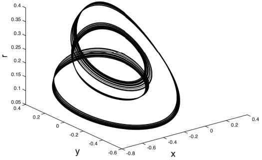
Note that the birth of an invariant torus when parameters cross the secondary Hopf bifurcation.

Chaos in the generic unfoldings of HBT singularities

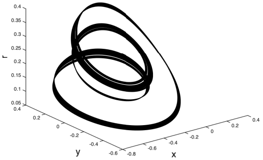
Period-doubling and chaotic dynamics close to a HH bifurcation ($\lambda_2 = 0.21$, $\mu = -0.3$ and $C_{1,1} = -1$):



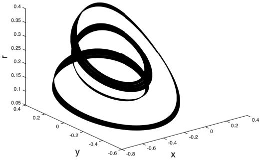
$\lambda_1 = -0.2129725000$



$\lambda_1 = -0.2129725798$



$\lambda_1 = -0.2129792959$ (Lyap. Exp. 0.011)



$\lambda_1 = -0.2129794796$ (Lyap. Exp. 0.012)

Conclusions and Future Research

1. A deep understanding of the bifurcation diagram of HBT singularities: analytical and numerical studies.
2. How do all the bifurcation surfaces arising at the bifurcation curves of codimension two glue together?
3. We have to check that higher order terms in the truncated normal form do not affect to our arguments.
4. Additional efforts are needed to prove that the topological types described here are indeed those exhibited by these singularities.

Acknowledgements

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Thank you for your attention!