Qualitative analysis of some models of delay differential equations

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PhD Thesis, DDays 2021

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Quinn & Deriso, 1999; Liz, 2018 (x2).

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Analogously, if $F(x) = x^{\gamma} H(x)$, where $\gamma \geq 0$ and H is nonincreasing, the difference equation $x_{n+1} = F(x_n)$ is a **discrete gamma-model**.

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau))$$
$$\gamma \in (0,1),\ h : [0,\infty) \to (0,\infty) \text{ nonincreasing}$$

An application in **Economics**:

Neoclassical model of economic growth

Solow, 1956; Barro & Sala-i-Martin, 2004; Day, 1982; Matsumoto & Szidarovszky, 2011-2013.

The **capital-labour rate** $x(t)$ is modelled under the equation above.

- γ is the **output elasticity of capital**.
- h is the term related to **pollution effects** due to huge concentrations of capital.
- τ stands for the **production lag**.

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau))$$

$\gamma \in (0,1)$, $h : [0,\infty) \to (0,\infty)$ nonincreasing

An application in **Ecology**:

Nicholson's blowflies model

Gurney, Blythe & Nisbet, 1980

The **size** of the **population** of *Lucilia cuprinia* $x(t)$ is modelled under the equation above with the limit case $\gamma = 1$.

- $h(x) = \beta e^{-\delta x}$ $(\beta, \delta > 0)$ is the **per-capita growth factor**.
- τ represents the **time** required for new individuals to **achieve the adulthood**.

If the constraint $\gamma = 1$ is not imposed, it modules the competition between individuals.

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau))$$

$$\gamma \in (0,1), \; h : [0,\infty) \to (0,\infty) \text{ nonincreasing}$$

An application in **Biology**:

Hematopoiesis

Mackey & Glass, 1977; Mitkowski, 2011.

The **density** of a certain class of **blood cells** $x(t)$ is modelled under the equation above for the cases $\gamma = 0$ and $\gamma = 1$.

- $h(x) = \frac{\beta}{1+\delta x^m}$ $(\beta, \delta, m > 0)$.
- τ represents the **time** required for cellular production.

Generally speaking, γ represents the disturbance on the healthy cellular production.

$$x'(t) = -ax(t) + f(x(t-\tau))$$

This equation is written in terms of three items:

- $a > 0$ is the **destruction rate**.
- $f : I \subset \mathbb{R} \to \mathbb{R}$ is the **production function**.
- $\tau > 0$ is the **discrete delay**.

$$x'(t) = -ax(t) + f(x(t-\tau))$$

This equation is written in terms of the triples (a, f, τ).

For large values of τ, the previous equation might be seen as a **singular perturbation** of the (cont.-time) **difference equation**

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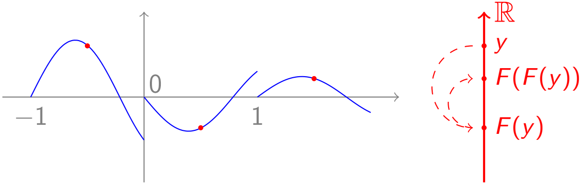

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whose solutions are governed by

$$x_{n+1} = F(x_n)$$

(Mallet-Paret & Nussbaum, 1986; Ivanov & Sharkovsky, 1992).

$$x'(t) = -ax(t) + f(x(t-\tau)) \qquad (E)$$

Equation (E) generates a semiflow $\varphi : \mathbb{R}_+ \times C_I \to C_I$
$x_{n+1} = F(x_n)$ generates a semiflow $\psi : \mathbb{Z}_+ \times I \to I$.

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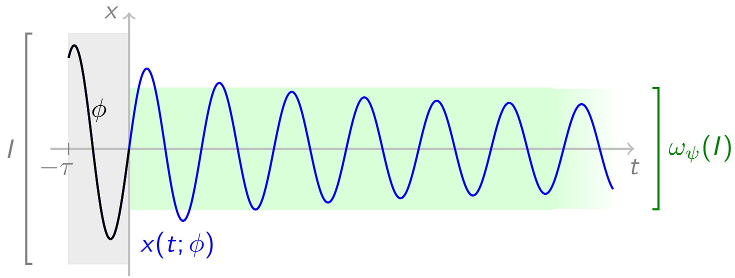
$$\left[\liminf_{t\to\infty} x(t;\phi), \limsup_{t\to\infty} x(t;\phi)\right] \subset \omega_\psi(I), \quad \forall \phi \in C_I.$$



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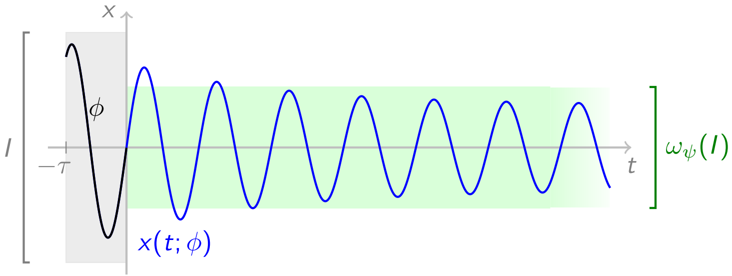

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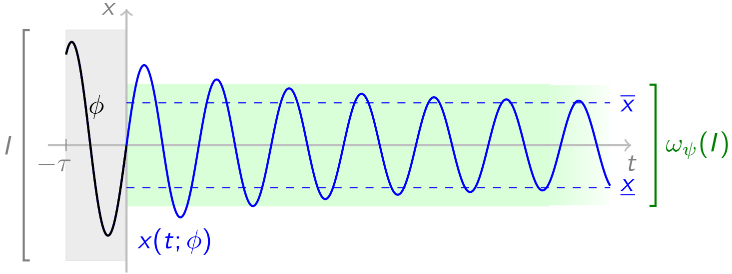

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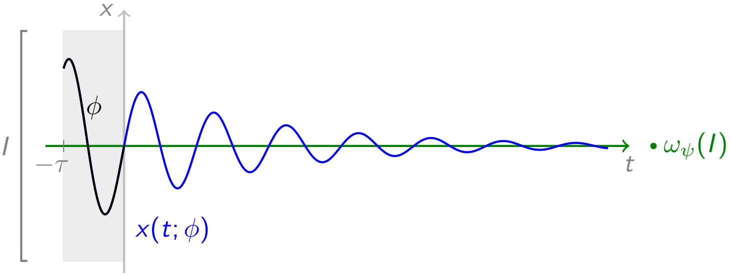

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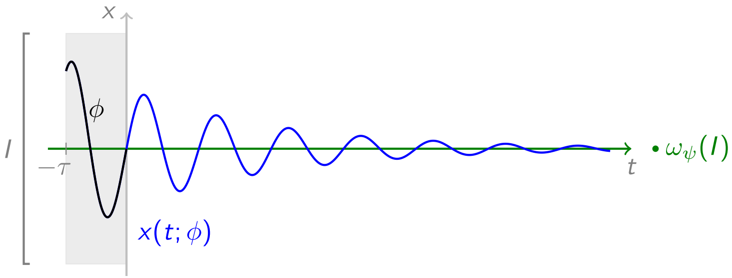

Corollary (Mallet-Paret & Nussbaum, 1986)

Let I be a compact interval and $F : I \to I$ be a continuous function. If p is globally attracting for $x_{n+1} = F(x_n)$, then p is globally attracting for (E).

$$x'(t) = -ax(t) + f(x(t-\tau))$$

Study the global dynamics of $x_{n+1} = F(x_n)$, where $F = \frac{f}{a}$.
In particular, does a **globally attracting equilibrium** exist?

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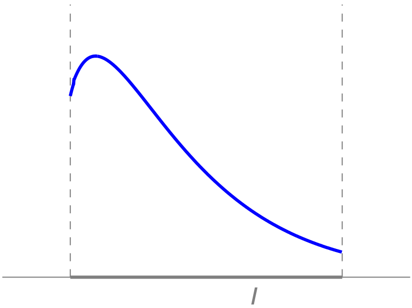
We should handle appropriate hypotheses on the triples (a, f, τ):

(T1) $a, \tau \in \mathbb{R}^+$, $f \in \mathcal{C}^1(I, \mathbb{R})$, $I \subset \mathbb{R}$ is a non-empty open interval



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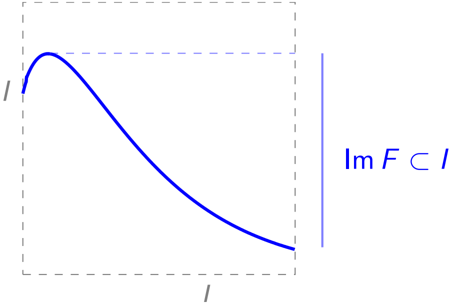
(T2) $\frac{f}{a} : I \to I$ is well-defined



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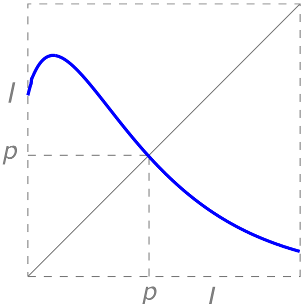
(T3) $F(x) = \frac{f(x)}{a} = x$ has a unique solution $p \in I$



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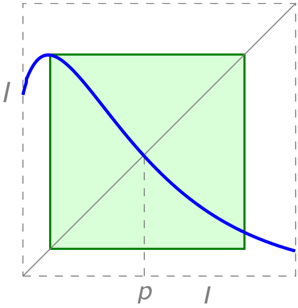
(T4) $K \subset I$ is a non-degenerate compact interval that is globally attracting for $F = \frac{f}{a}$



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Coppel, 1955; Franco, Perán & Segura, 2018

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Let $F : I \to I$ be a continuous function and $p \in I$ be the unique fixed point of F. Then p is GAS for F if and only if

$$(F^2(x) - x)(x - p) < 0, \quad \forall x \in I \setminus \{p\}.$$

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Let $F : I \to I$ be a continuous function and $p \in I$ be the unique fixed point of F. Then p is GAS for F if and only if

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Sufficient condition for its existence:

Use the Schwarzian derivative! (Allwright, 1978; Singer, 1978)

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2, \quad F'(x) \neq 0.$$

Maps F with negative Schwarzian derivative are of great interest:

$$\text{LAS} \implies \text{GAS}$$

Theorem (El-Morshedy & Jiménez López, 2008)

Let $F \in \mathcal{C}^3(I, I)$ *be such that*

$$(F(x) - x)(x - p) < 0, \quad \forall x \in I \setminus \{p\},$$

and having at most one critical point, which would be called c *and would also satisfy* $F''(c) \neq 0$.
Assume that one of the following two conditions holds:

- *The function* F *satisfies* $0 \leq F'(p) < 1$.
- *The function* F *satisfies* $-1 \leq F'(p) < 0$ *and* $SF(x) < 0$, *for every* $x \in I_*$, *where*
  - $I_* = I$, *if there is no extremum,*
  - $I_* = I \cap (c, \infty)$, *if* F *attains a maximum value at* c, *and*
  - $I_* = I \cap (-\infty, c)$, *if* F *attains a minimum value at* c.

Then the equilibrium p *is globally attracting for* F.

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has at most one critical point c (non-inflection)

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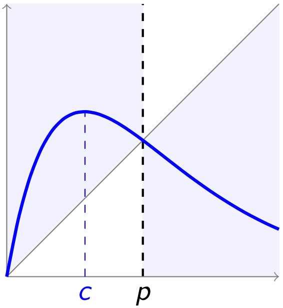

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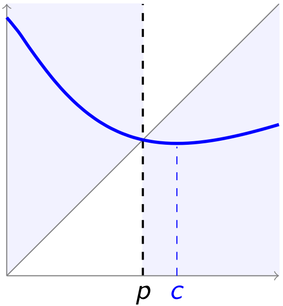

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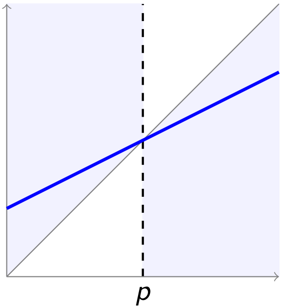

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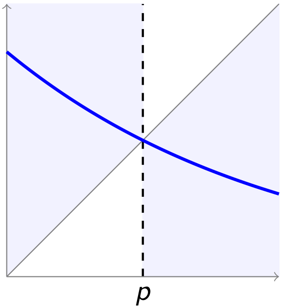

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p is globally attracting when

$0 \leq F'(p) < 1$

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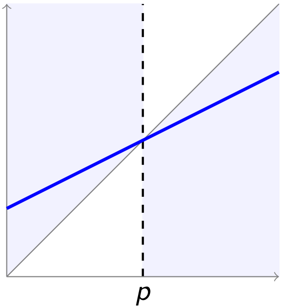

$$x'(t) = -ax(t) + f(x(t-\tau)) \qquad (E_P)$$

Study the global dynamics of $x_{n+1} = F(x_n)$, where $F = \frac{f}{a}$.
In particular, does a **globally attracting equilibrium** exist?

Sufficient condition for its existence:

El-Morshedy & Jiménez López, 2008

Thesis:

p is globally attracting when

$0 \leq F'(p) < 1$

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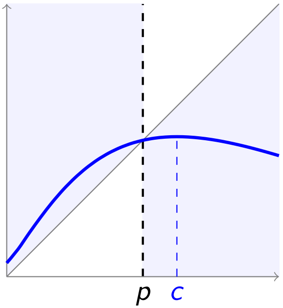

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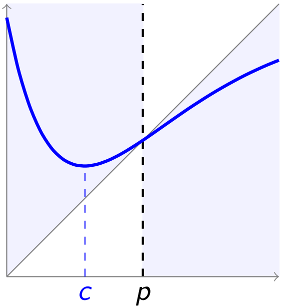

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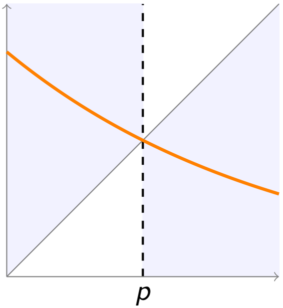
(I_* subinterval of decrease)



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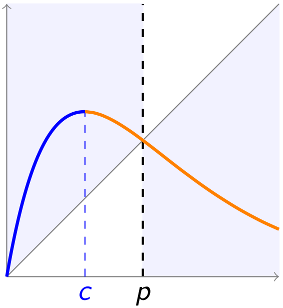

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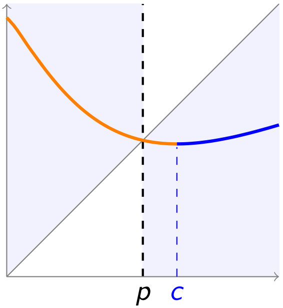

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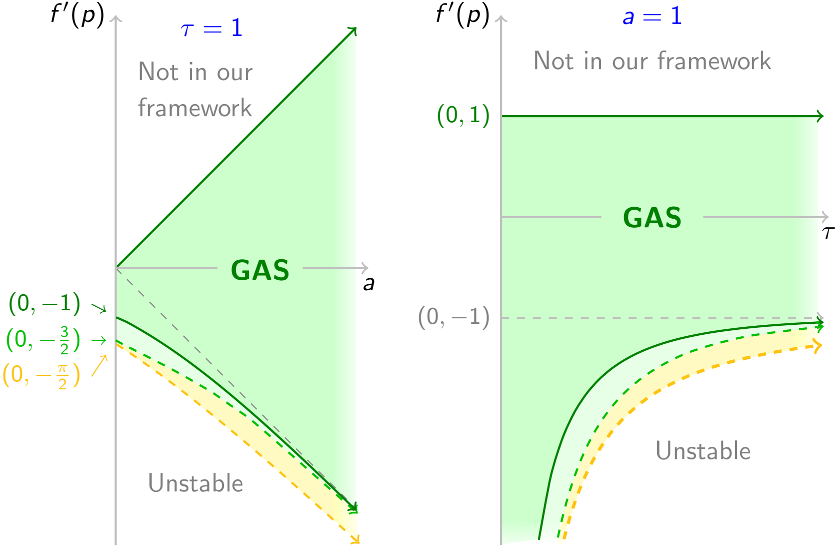
Definition

Maps on the former context

- with $SF < 0$ on the interval of decrease are called S_*-**maps**.
  [El-Morshedy & Jiménez López]
- with $SF < 0$ everywhere are called S-**maps**.
  [Allwright, Singer]

Space of triples $(a, f'(p), \tau)$, with negativity-type hypothesis on Sf.



We analyse the **role** of $\gamma \in (0,1)$ on the dynamics of $x_{n+1} = F(x_n)$, where $F(x) = \frac{1}{a} x^{\gamma} h(x)$, for **different choices of the function** h.

Theoretical framework

Theorem

Let $a > 0$ *and* $h : [0,\infty) \to (0,\infty)$ *be a nonincreasing function of class* $\mathcal{C}^1$*. Then, for each* $\gamma \in (0,1)$*,* We analyse the **role** of $\gamma \in (0,1)$ on the dynamics of $x_{n+1} = F(x_n)$, where $F(x) = \frac{1}{a} x^{\gamma} h(x)$, for **different choices of the function** h.

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[Graph] $(F(x) - x)(x - p) < 0, \quad \forall x \in (0,\infty) \setminus \{p\}$;

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[Graph] $(F(x) - x)(x - p) < 0, \quad \forall x \in (0,\infty) \setminus \{p\}$*;*

[Value equilibrium] *one of the following cases holds:*

- $p(\gamma) < 1$ *is decreasing on* $(0,1)$*, or, equivalently,* $H(1) < 1$*;*
- $p(\gamma) = 1$*, for every* $\gamma \in (0,1)$*, or, equivalently,* $H(1) = 1$*;*
- $p(\gamma) > 1$ *is increasing on* $(0,1)$*, or, equivalently,* $H(1) > 1$*;*

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- $p(\gamma) > 1$ *is increasing on* $(0,1)$, *or, equivalently,* $H(1) > 1$;

[Local dynamics] $F'(p) = \gamma + p^\gamma H'(p) < 1$.

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau))$$

$\gamma \in (0,1)$, $h : [0,\infty) \to (0,\infty)$ nonincreasing

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Lasota equation

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Lasota equation

Pollution effects, $h(x) = \beta e^{-\delta x}$, $\beta, \delta > 0$.

$$x'(t) = -ax(t) + \beta x^{\gamma}(t-\tau) e^{-\delta x(t-\tau)} \quad (E)$$

$$x_{n+1} = \frac{\beta}{a} x_n^{\gamma} e^{-\delta x_n} \quad (D)$$

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Theorem (B.-F. & Liz, 2018)

Let p be the unique equilibrium of (E). If

$$\frac{\beta}{a} \leq e^{\gamma+1} \left(\frac{\gamma+1}{\delta} \right)^{1-\gamma} \qquad (*)$$

then,

$$\lim_{t\to\infty} x(t;\phi) = p, \quad \forall \phi \in C_{(0,\infty)}$$

regardless the value of τ. Condition $(*)$ is the **sharpest delay-independent stability condition** *for (E).*

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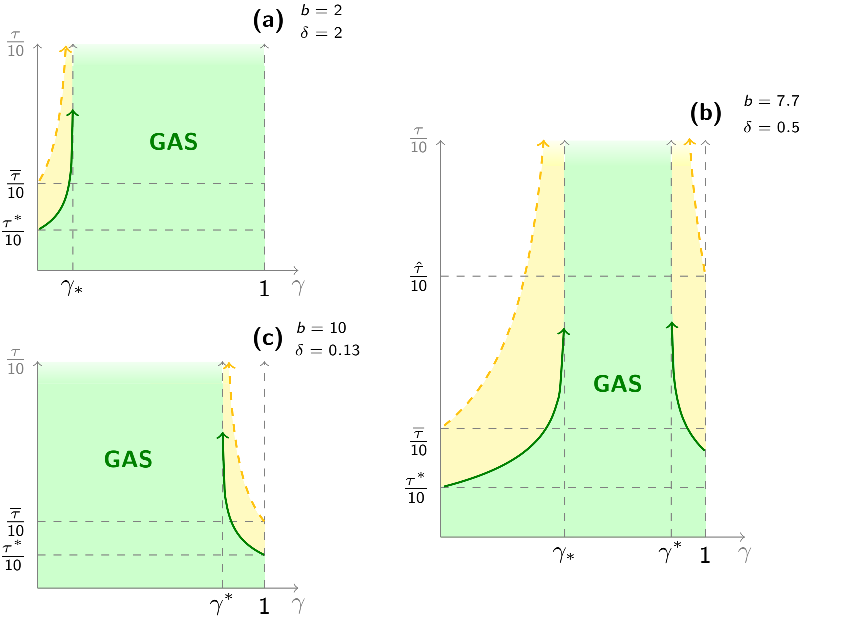
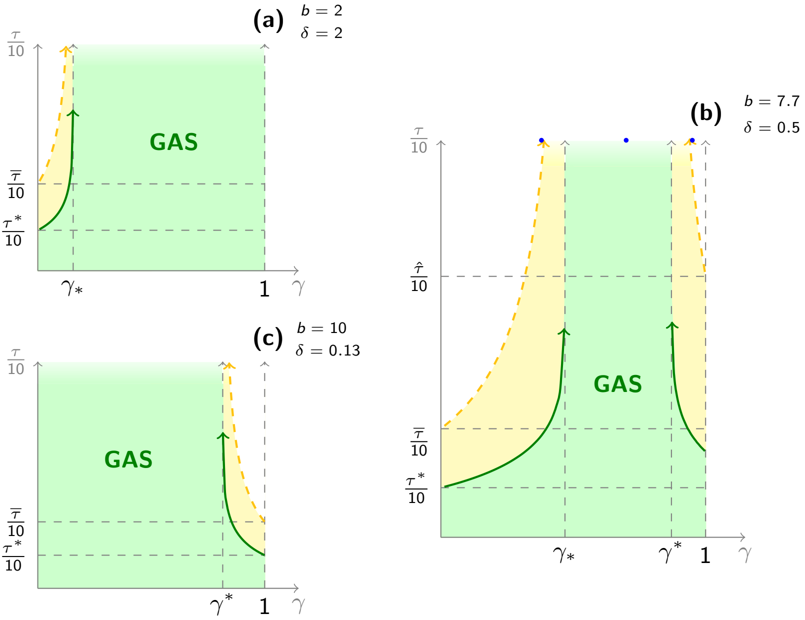

(a)
b = 2
δ = 2
τ/10
τ̄/10
τ*/10
GAS
γ*
1
γ
(b)
b = 7.7
δ = 0.5
τ/10
τ̂/10
τ̄/10
τ*/10
GAS
γ*
γ*
1
γ
(c)
b = 10
δ = 0.13
τ/10
τ̄/10
τ*/10
GAS
γ*
1
γ


(a)
b = 2
δ = 2
τ/10
τ̄/10
τ*/10
GAS
γ*
1
γ
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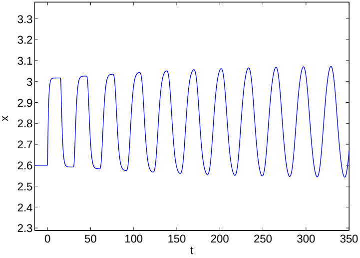
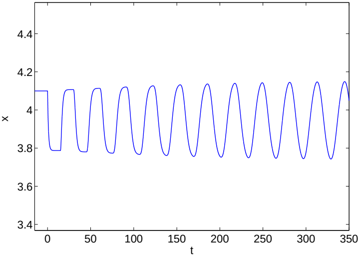
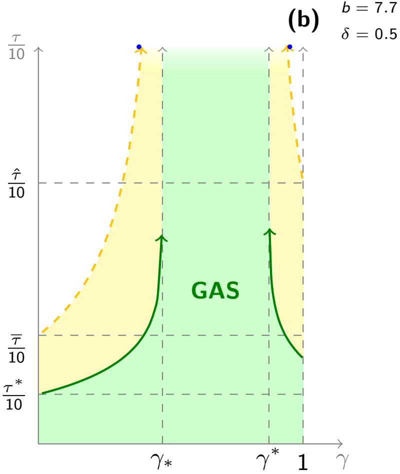
b = 7.7
δ = 0.5
τ/10
τ̂/10
τ̄/10
τ*/10
GAS
γ*
γ^*
1
γ
(c)
b = 10
δ = 0.13
τ/10
τ̄/10
τ*/10
GAS
γ^*
1
γ

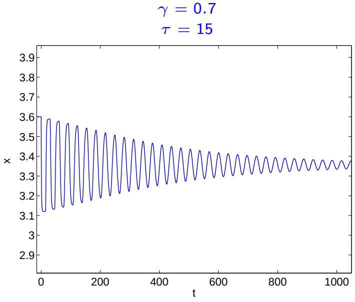
$\gamma = 0.38$
$\tau = 15$

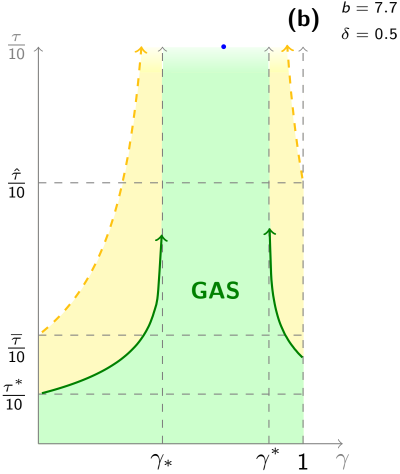


$\gamma = 0.95$
$\tau = 15$






$\gamma = 0.7$
$\tau = 15$
3.9
3.8
3.7
3.6
3.5
3.4
3.3
3.2
3.1
3
2.9
x
0
200
400
600
800
1000
t


(b)
$b = 7.7$
$\delta = 0.5$
$\frac{\tau}{10}$
$\frac{\hat{\tau}}{10}$
$\frac{\overline{\tau}}{10}$
$\frac{\tau^*}{10}$
GAS
γ_*
γ^*
1
γ

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γ-logistic DDE

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$$\gamma \in (0,1),\ h : [0,\infty) \to (0,\infty) \text{ nonincreasing}$$

γ-logistic DDE

Pollution effects, $h(x) = \beta(1-x)$, $\beta > 0$.

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Corollary

The unique positive equilibrium p of (E) decreases for increasing $\gamma \in (0,1)$.

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Theorem (B.-F., 2019)

Let the consistency condition hold and p be the unique equilibrium of (E). If

$$\frac{\beta}{a} \leq \gamma + 1 \left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma} \qquad (*)$$

then,

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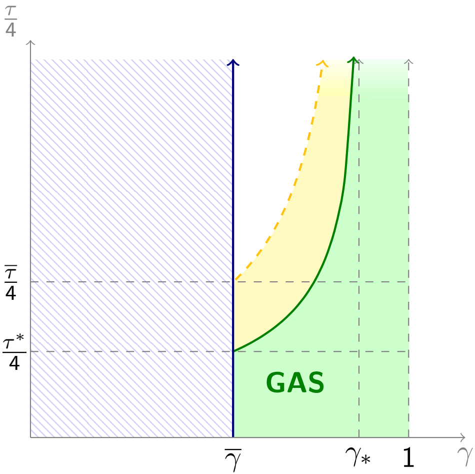
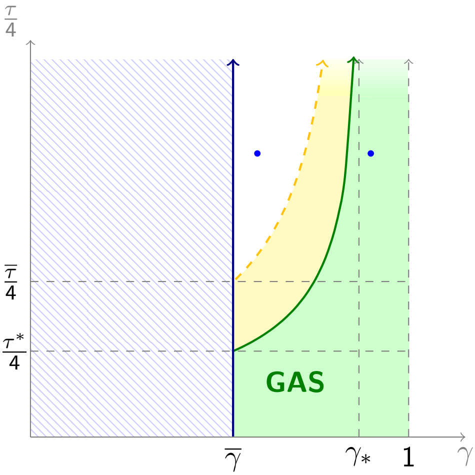
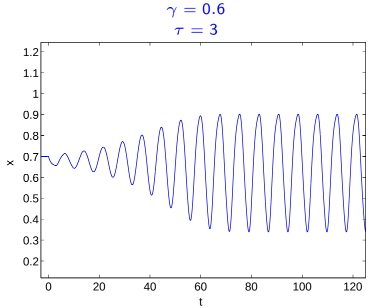
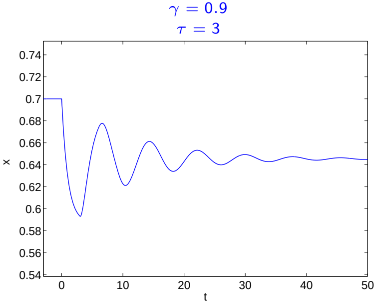
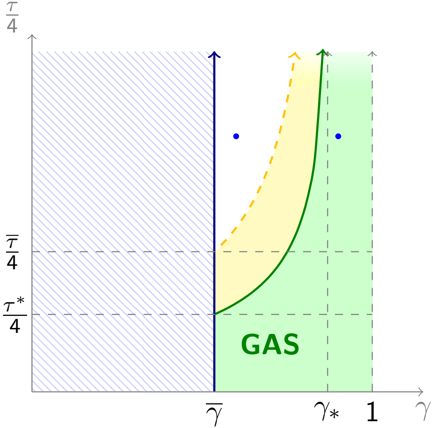

$\frac{\tau}{4}$
$\frac{\overline{\tau}}{4}$
$\frac{\tau^*}{4}$
GAS
$\overline{\gamma}$
γ_*
1
γ


$\frac{\tau}{4}$
$\frac{\overline{\tau}}{4}$
$\frac{\tau^*}{4}$
GAS
$\overline{\gamma}$
γ_*
1
γ


$\gamma = 0.6$
$\tau = 3$
x
t
0 20 40 60 80 100 120
0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2


$\gamma = 0.9$
$\tau = 3$
x
t
0 10 20 30 40 50
0.54 0.56 0.58 0.6 0.62 0.64 0.66 0.68 0.7 0.72 0.74


$\frac{\tau}{4}$
$\frac{\overline{\tau}}{4}$
$\frac{\tau^*}{4}$
GAS
$\overline{\gamma}$
γ_*
1
γ

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γ-Mackey-Glass DDE

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Let $m = 1$ and p be the unique equilibrium of (E). Then,

$$\lim_{t\to\infty} x(t;\phi) = p, \quad \forall \phi \in C_{(0,\infty)}.$$

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau))$$

$$\gamma \in (0,1),\ h : [0,\infty) \to (0,\infty) \text{ nonincreasing}$$

γ-Mackey-Glass DDE

Pollution effects, $h(x) = \frac{\beta}{a}\frac{1}{1+\delta x^m}$, $\beta, \delta, m > 0$.

$$x'(t) = -ax(t) + \frac{\beta x^{\gamma}(t-\tau)}{1+\delta x^m(t-\tau)} \quad (E)$$

$$x_{n+1} = \frac{\beta}{a}\frac{x_n^{\gamma}}{1+\delta x_n^m} \quad (D)$$

Theorem (B.-F. & Liz, 2018)

Let $m = 1$ and p be the unique equilibrium of (E). Then,

$$\lim_{t\to\infty} x(t;\phi) = p, \quad \forall \phi \in C_{(0,\infty)}.$$

What happens if $m \neq 1$?

Key idea: a change of variables from Liz, 2007.

$$\begin{matrix} x_{n+1} = F(x_n) \\ F(x) = x^{\gamma} H(x) \end{matrix} \quad \xrightarrow{y=\Lambda(x):=-\ln(x/p)} \quad \begin{matrix} y_{n+1} = T(y_n) \\ T(y) = \gamma y + G(y) \end{matrix}$$

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Here, $G(y) := -\ln(p^{\gamma-1}H(pe^{-y}))$.

$\Lambda : (0, \infty) \to \mathbb{R}$ is a **topological conjugacy** between both equations

$$\Lambda \circ F = T \circ \Lambda.$$

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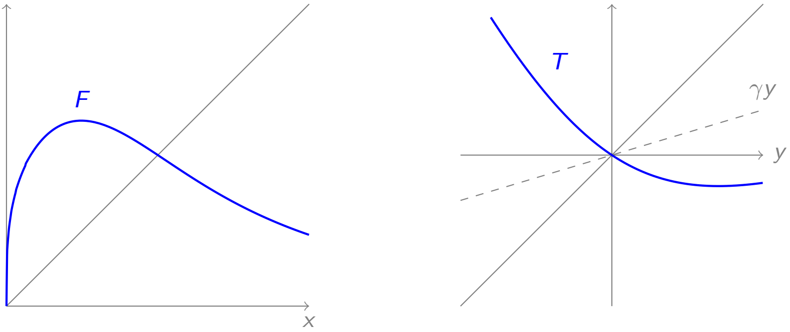

$$x_{n+1} = x_n^{\gamma} H(x_n), \quad H \text{ is decreasing and } \mathcal{C}^3 \quad \text{(D)}$$

Theorem (Liz & B.-F., 2019)

Let $\gamma \in [0,1]$ *and assume that the following condition holds:*

$$x^2 \left(2\, SH(x) + \left(\frac{H'(x)}{H(x)} \right)^2 \right) < 1, \quad \forall x > 0. \qquad (*)$$

Then, the local asymptotic stability of the unique positive equilibrium p *in equation (D) implies its global stability. This stability condition is*

$$-p^{\gamma} H'(p) \leq 1 + \gamma.$$

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau)) \quad (E)$$

$\gamma \in [0,1]$, $h : [0,\infty) \to (0,\infty)$ decreasing and $\mathcal{C}^3$

$$x_{n+1} = x_n^{\gamma} H(x_n), \quad H = h/a$$

γ-**Mackey-Glass DDE** (also for $m \neq 1$): $h(x) = \frac{\beta}{1+\delta x^m}$

$$H(x) = \frac{\beta/a}{1+\delta x^m}, \quad SH(x) = \frac{1-m^2}{2x^2}, \quad \frac{H'(x)}{H(x)} = \frac{-\delta m x^{m-1}}{1+\delta x^m}.$$

Thus, the new Schwarzian-type condition $(*)$ holds:

$$x^2\left(2SH(x) + \left(\frac{H'(x)}{H(x)}\right)^2\right) = 1 - m^2 + m^2\left(\frac{\delta x^m}{1+\delta x^m}\right)^2 < 1, \quad \forall x > 0$$

$$x'(t) = -ax(t) + x^{\gamma}(t-\tau)\, h(x(t-\tau)) \quad (E)$$

$\gamma \in [0,1]$, $h : [0,\infty) \to (0,\infty)$ decreasing and $\mathcal{C}^3$

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γ-**Mackey-Glass DDE** (also for $m \neq 1$): $h(x) = \frac{\beta}{1+\delta x^m}$

Theorem (Liz & B.-F., 2019)

Let p be the unique positive equilibrium of equation (E). If

$$m \leq 1+\gamma \ \text{ or } \ \left[m > 1+\gamma, \quad \frac{\beta}{a} \leq \frac{m}{m-1-\gamma} \left(\frac{\gamma+1}{\delta(m-1-\gamma)} \right)^{\frac{1-\gamma}{m}} \right],$$

then

$$\lim_{t\to\infty} x(t;\phi) = p, \quad \forall \phi \in C_{(0,\infty)}.$$

The former is the **sharpest delay-independent stability condition** *for (E).*

Further comments:

- Another way to write $(*)$ was suggested by V. Jiménez-López:

$$SH(x) < \frac{H(x) - xH'(x)}{2(xH(x))^2}(xH(x))', \quad \forall x > 0. \qquad (*)$$

- Our result can be adapted to $\gamma = 0$ in order to complement the classical Allwright-Singer one. Note that $F(x) = x^0 H(x) = H(x)$.

Theorem ($\gamma = 0$; Liz & B.-F., 2019)

Assume that the $\mathcal{C}^3$ map $F : (0,\infty) \to (0,\infty)$ has at most one critical point c, which would be a local extremum, and $(F(x) - x)(x - p) < 0$ on $(0,\infty) \setminus \{p\}$.

If $F'(p) \geq -1$ and $(*)$ holds for all $x > 0$ such that $F'(x) \neq 0$, then p is globally attracting for

$$x_{n+1} = F(x_n).$$

Systems of DDEs

$$x_i'(t) = -x_i(t) + f_i(x_1(t-\tau), \dots, x_s(t-\tau)), \quad \forall i \in \{1, \dots, s\},$$

$$[x_{n+1}]_i = f_i([x_n]_1, \dots, [x_n]_s), \quad \forall i \in \{1, \dots, s\}.$$

DDEs with integral production terms

$$x'(t) = -a(t)x(t) + \int_{\sigma}^{t} \eta(t,s)x(\nu(s))\,ds, \quad t \geq \sigma,$$

Joint work with Rosana Rodríguez López (USC)

Impulsive periodic DDEs

$$\begin{cases} x'(t) = -a(t)x(t) + g(t, x_t), & t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases}$$

Joint work with Teresa Faria (Univ. Lisboa)

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