Contributions to mathematical analysis of non-linear models with applications in population dynamics

Cristina Lois-Prados

Supervisors: E. Liz Marzán & R. Rodríguez López

Sesión de Tesis - DDays 2021

Research Line I: Fixed point theory & Differential equations

Generalizations of Krasnosel'skii f.p.t.

- C. Lois-Prados & R. Rodríguez-López (2020). A generalization of Krasnosel'skii compression fixed point theorem by using star convex sets. *Proceedings of the Royal Society of Edinburgh, Section A: Mathematics* 150:1, pp. 277-303
- C. Lois-Prados, R. Precup & R. Rodríguez-López (2020). Krasnosel'skii type compression-expansion fixed point theorem for set contractions and using star convex sets. *Journal of Fixed Point Theory and Applications* 22:63, pp. 1-20

Application to periodic predator-prey models

- C. Lois-Prados & R. Precup (2020). Positive periodic solutions for Lotka-Volterra systems with a general attack rate. *Nonlinear Analysis: Real World Applications* 52, 103024

Research Line II: Qualitative study & 1-D difference equations

Blood cell production

E. Liz & C. Lois-Prados (2020). A note on the Lasota discrete model for blood cell production. *Discrete & Continuous Dynamical Systems - B* 25:2, 701-713

Fisheries Management

E. Liz & C. Lois-Prados (2020). Dynamics and bifurcations of a family of piecewise-smooth maps arising in population models with threshold harvesting. *Chaos* 30:7, 1-16

C. Lois-Prados & F. Hilker (submitted). Bifurcation sequences in a discontinuous piecewise-smooth map combining constant-catch and threshold-based harvesting strategies. *SIAM Journal on Applied Dynamical Systems*

Krasnosel'skii f.p.t. for set contractions & star convex sets

$$\begin{cases} x'(t) = f(t, x(t)), \ t \in [0,1] \\ x(0) = 0 \end{cases}$$

f continuous and bounded

$$T(x)(t) = \int_0^t f(s, x(s))ds$$

T continuous and compact

Theorem (Krasnosel'skii, 1964): Compressive Case.
$(X, \|\cdot\|)$ Banach, C cone, $T : C \to C$ continuous and compact, $T(0) = 0$

$$x - T(x) \notin C, \forall\, x \in \{x \in C, \ \|x\| = r\}$$
$$T(x) - (1+\varepsilon)x \notin C, \forall\, \varepsilon > 0, \ x \in \{x \in C, \ \|x\| = R\}$$

Then, T has a fixed point in $\{x \in C, \ r \leq \|x\| \leq R\}$

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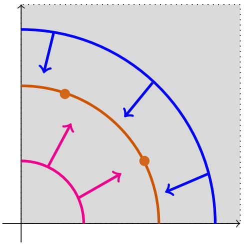
T continuous and compact

Theorem (Krasnosel'skii, 1964): Compression conditions.
$(X, \|\cdot\|)$ Banach, C cone, $T : C \to C$ continuous and compact, $T(0) = 0$

$$x - T(x) \notin C, \forall\ x \in \{x \in C,\ \|x\| = r\}$$
$$T(x) - (1+\varepsilon)x \notin C, \forall\ \varepsilon > 0,\ x \in \{x \in C,\ \|x\| = R\}$$



$T(x) = x$
$\{x \in C,\ r \leq \|x\| \leq R\}$

LIMITATIONS

Krasnosel'skii f.p.t. for set contractions & star convex sets

Theorem (Lois-Prados & Precup & Rodríguez-López, 2020)
$(X, \|\cdot\|)$ Banach, C *cone*, E_1, E_2 satisfying P1, P2 and $T : C \cap (E_2 \backslash \mathring{E}_1) \to C$ a set contraction such that

$$x - T(x) \notin C, \forall\, x \in C \cap E_1$$
$$T(x) - (1+\varepsilon)x \notin C, \forall\, \varepsilon > 0,\ x \in C \cap E_2$$

Then, T has a fixed point in $C \cap (E_2 \backslash \mathring{E}_1)$

$$\begin{cases} x'(t) = f(t, x(t)) + g(t, x'(t)),\ t \in [0,1] \\ x(0) = 0 \end{cases}$$

f continuous
g Lipschiptz

$$T(x')(t) = f\left(t, \int_0^t x'(s)ds\right) + g(t, x'(t))$$

T set contraction

Krasnosel'skii f.p.t. for set contractions & star convex sets

Theorem (Lois-Prados & Precup & Rodríguez-López, 2020)
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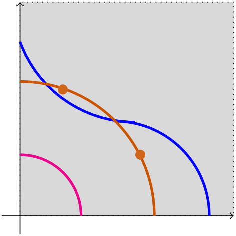

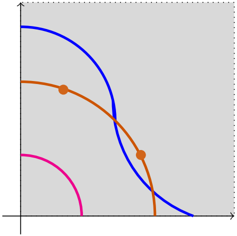

Direct application of f.p. theorems to periodic predator-prey models

Periodic Lotka-Volterra type system with general prey growth g and functional response of predators φ

$$\begin{cases} x' = a(t)xg(x) - \varphi(t,x,y)xy \\ y' = -b(t)y + c(t)\varphi(t,x,y)xy \end{cases}$$

$a, b, c \in \mathcal{C}_\omega(\mathbb{R}, \mathbb{R}_+)$, $a, b \neq 0, c > 0$
$g \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ decreasing, $g(0) \leq 1$
$\varphi \in \mathcal{C}_\omega(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$

Particular cases: $g(x) = 1$ (linear growth), $g(x) = 1 - x/K$ (logistic growth)
$\varphi(t,x,y) = \lambda(t) + \alpha(t)y$ (linear hunting cooperation)

Direct application of f.p.t:

Theorem (O'Regan & Precup, 2001): Expansive Case.
$(X, \|\cdot\|)$ Banach, C cone, $0 < r < R$, $T : C \cap B(0,R) \to C$ continuous and compact

$$T(x) \neq lx, \quad \forall\, x \in C, \|x\| = r, l > 1,$$
$$\exists\, v \in C \backslash \{0\} : x - T(x) \neq lv, \quad \forall\, x \in C, \|x\| = R, l > 0$$

Then, T has a fixed point in $\{x \in C : r \leq \|x\| \leq R\}$

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Then, T has a fixed point in $\{x \in C \;:\; r \leq \|x\| \leq R\}$

Qualitative study of one-dimensional difference equations

$h : J \longrightarrow J,\ J \subseteq \mathbb{R},\ x_0 \in J$

$x_{n+1} = h(x_n),\ n \in \mathbb{N} \cup \{0\}$

Determine the long term behavior of

$\{x_n\}_{n\in\mathbb{N}} = \{h(x_{n-1})\}_{n\in\mathbb{N}} = \{h^n(x_0)\}_{n\in\mathbb{N}}$

$h^0 = id,\ h^n = h \circ h^{n-1},\ n \in \mathbb{N}$

We deal with some particular expression of h arising in models of:

- Blood cell production
- Fisheries management

We are interested in:

- (Global) stability and bifurcations of fixed points
- Regions of complex dynamics and related bifurcations
- Other interesting phenomena of the dynamics

Qualitative study of one-dimensional difference equations

Blood cell production

$$x_{n+1} = (1-\sigma)x_n + (cx_n)^\gamma e^{-x_n}$$

In the time interval $[n, n+1]$

σx_n: cells destroyed

$p_n = (cx_n)^\gamma e^{-x_n}$: cells produced in the bone marrow

(Lasota, 1977) fixed $c = 0.47$ and $\gamma = 8$ to study the influence of σ on the dynamics

He considered 3 clinical cases

Normal conditions ($\sigma = 0.1$): bistability (extinction and positive eq.)

Non-severe disease ($\sigma = 0.4$): 2-periodic attractor

Severe disease ($\sigma = 0.8$): chaotic behavior (3-periodic orbit)

Main aim: Carry out a detailed study of asymptotic dynamics

- Fixed points: stability and local bifurcations
- Increasing σ can be (de)stabilizing
- Essential extinction: collision of chaotic attractor with extinction ($\sigma \approx 0.87$)

Qualitative study of one-dimensional difference equations

Blood cell production

$$x_{n+1} = (1 - \boldsymbol{\sigma})x_n + \boldsymbol{(cx_n)^\gamma e^{-x_n}}$$

In the time interval $[n, n+1]$

$\boldsymbol{\sigma x_n}$: cells destroyed

$\boldsymbol{p_n = (cx_n)^\gamma e^{-x_n}}$: cells produced in the bone marrow

(Lasota, 1977) fixed $\boldsymbol{c = 0.47}$ and $\boldsymbol{\gamma = 8}$ to study the influence of $\boldsymbol{\sigma}$ on the dynamics

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Qualitative study of one-dimensional difference equations

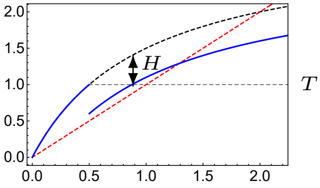
Fisheries management

$$x_n \xrightarrow{Stock-Recruitment} f(x_n) \xrightarrow{\textbf{Harvesting}} F(x_n)$$

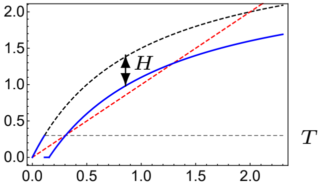
$x_{n+1} = f(x_n)$ Unharvested population, T: threshold, H: constant quota

TCC: Threshold constant catch

$$x_{n+1} = F(x_n) = \begin{cases} f(x_n), & f(x_n) < T \\ \max\{0, f(x_n) - H\}, & f(x_n) \geq T \end{cases}$$



$f(x) = \frac{r\,x_n}{1+x_n}$, $r > 1$, $H < T$



$f(x) = \frac{r\,x_n}{1+x_n}$, $r > 1$, $H \geq T$

Qualitative study of one-dimensional difference equations

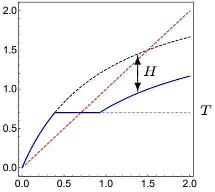
Fisheries management

$$x_n \xrightarrow{Stock - Recruitment} f(x_n) \xrightarrow{\textbf{Harvesting}} F(x_n)$$

$x_{n+1} = f(x_n)$ Unharvested population, T: threshold, H: constant quota

PTCC: Precautionary threshold constant catch

$$x_{n+1} = F(x_n) = \begin{cases} f(x_n), & f(x_n) \leq T \\ T, & T < f(x_n) \leq T + H \\ f(x_n) - H, & f(x_n) > T + H \end{cases}$$



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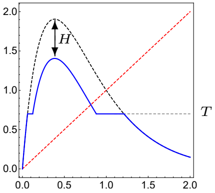


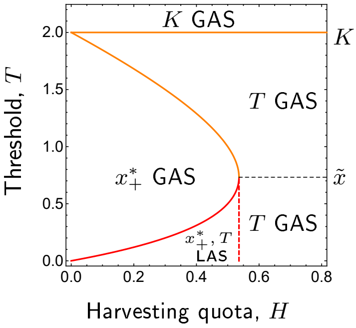
$f(x) = x_n\, e^{r(1-x_n), r>0}$

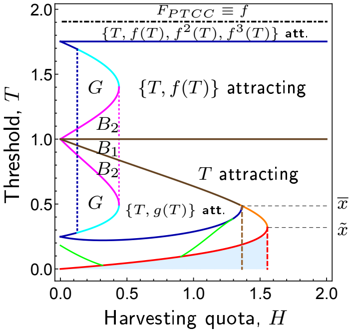
Qualitative study of one-dimensional difference equations

Fisheries management: PTCC

For some stock-recruitment maps f we classify the asymptotic dynamics

- **Compensatory models (PTCC)**
- **Overcompensatory models (PTCC)**



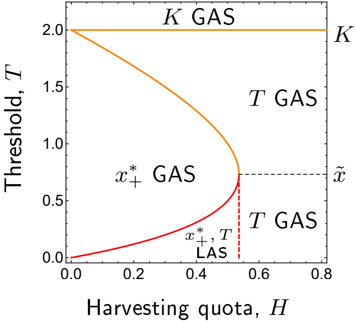


Qualitative study of one-dimensional difference equations

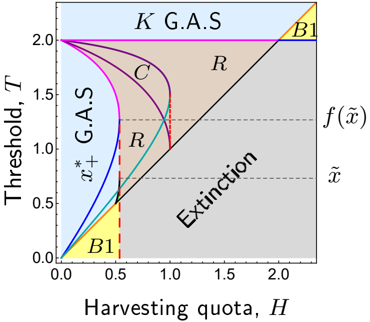
Fisheries management: PTCC vs TCC

For some stock-recruitment maps f we classify the asymptotic dynamics

- **Compensatory models (PTCC)**



- **Increasing models (TCC)**



Thank you very much!

¡Muchísimas gracias!