

MULTI-FREQUENCY OSCILLATIONS

IN DYNAMICAL SYSTEMS

Session 4. Multi-frequency Dynamics in Conservative Systems

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1. KAM theory in Hamiltonian systems

- *Symplectic structure on R^{2d}* : A non-degenerate, closed, differential 2-form ω^2 .
- *Symplectic coordinate*: $(p, q) \in R^d \times R^d$ s.t.

$$\omega^2 = dp \wedge dq.$$

- Symplectic coordinates always exist (Darboux' Theorem).
- *Poisson bracket*: $\{\cdot, \cdot\}$ s.t. $\forall f, g \in C^\infty(R^{2d}, R)$,

$$\{f, g\} = \omega^2(J\nabla f, J\nabla g) = (\nabla f)^\top J \nabla g,$$

where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard *symplectic matrix*.

- *Hamiltonian system*: Given *Hamiltonian*

$$H : R^{2d} \rightarrow R.$$

The associated Hamiltonian system (or flow) reads

$$\dot{z} = J\nabla H(z), \quad z = (p, q),$$

or

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p}. \end{cases}$$

- A Hamiltonian flow ϕ^t on R^{2d} *preserves* ω^2 :

$$(\phi^t)^* \omega^2 = \omega^2,$$

and vice verse.

- A Hamiltonian flow is area (ω^2) and volume (ω^{2d}) preserving.

- *Newtonian systems*: For $i = 1, \dots, d$, let

q_i = position

$p_i = m_i \dot{q}_i$ = momentum.

- The total energy (Hamiltonian):

$$H(p, q) = \underbrace{\sum_i \frac{|p_i|^2}{2}}_{\text{kinetic}} + \underbrace{\sum_i U(q_i)}_{\text{onsite}} + \underbrace{\sum_{i,j} W_{i,j}(q)}_{\text{interacting}}$$

- Newton's equation:

$$m_i \ddot{q}_i = -\nabla U(q_i) - \sum_j \nabla W_{i,j}(q), \quad i = 1, \dots, d.$$

- H is conserved and the Newton's equation is equivalent to the Hamiltonian system associated with H .

- *Physical examples*: N -body problem, mechanical vibrations, rigid bodies rotations, crystal lattice model, biological chains, $\dots$

- *Integrability*: A Hamiltonian H on R^{2d} is *completely integrable* if
 - a) There are d functionally independent first integrals $f_1, \dots, f_d$ (including H);
 - b) $f_1, \dots, f_d$ are in involutions, i.e., $\{f_i, f_j\} = 0, \forall i, j$;
 - c) For each $c = (c_1, \dots, c_d)$ lying in some region $G \subset R^d$, the level set

$$M_c = \{f_i = c_i : i = 1, \dots, d\}$$

is compact and connected.

- *Liouville's Theorem* (Liouville, 1897): If H is completely integrable, then the following holds.

- a) $M_c \simeq T^d, \forall c \in G$;
- b) $\exists$ *action-angle* coordinate $(I, \theta) \in G \times T^d$: $\omega^2 = dI \wedge d\theta$;
- c) Under (I, θ) , $H(p, q) = N(I)$.

- *Integrable system:* Let $H(p, q) = N(I)$, $I \in G \subset \mathbb{R}^d$, be completely integrable. Then

$$\begin{cases} \dot{I} = 0 \\ \dot{\theta} = \omega(I), \end{cases}$$

where $\omega(I) = \frac{\partial N}{\partial I}(I)$ are the *frequencies*.

- *Invariant tori:* For $\forall I_0 \in G$, $T_{I_0} = \{I = I_0\}$ is an invariant d -torus with linear flows $\{\theta_0 + \omega(I_0)t\}$.
- *Foliations:* $G \times T^d = \cup_{I_0 \in G} T_{I_0}$.
- a) $\omega(I_0)$ non-resonant $\Rightarrow$ the flow is *quasi-periodic*.
- b) $\omega(I_0)$ resonant $\Rightarrow T_{I_0}$ is foliated into quasi-periodic lower dimensional tori.

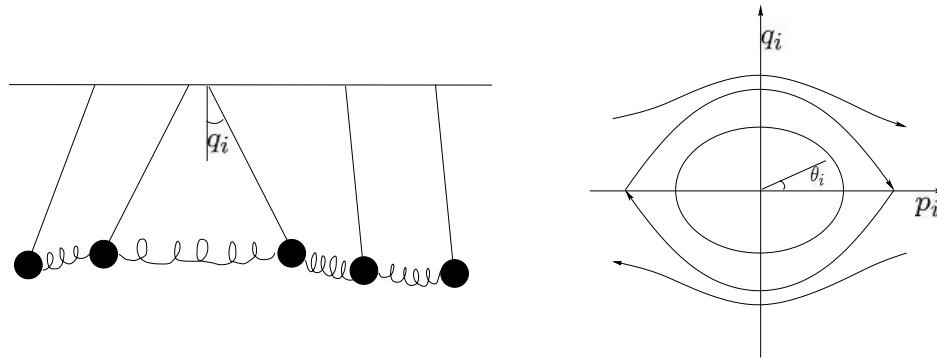
- *Nearly integrable system:*

$$H = N(I) + \varepsilon P(I, \theta, \varepsilon),$$

$$\begin{cases} \dot{I} = -\varepsilon \frac{\partial P}{\partial \theta} \\ \dot{\theta} = \omega(I) + \varepsilon \frac{\partial P}{\partial I}. \end{cases}$$

- *Weakly coupled pendula:*

$$H(p, q) = \sum_i \frac{|p_i|^2}{2} + \sum_i (1 - \cos q_i) + \sum_i k_i (q_{i+1} - q_i)^2.$$



- $I_i = \text{Area}$, $\theta_i = \text{Angle}$, $k_i = O(\varepsilon)$, $\forall i$.

1) Non-resonant tori

- *Non-resonance zone:*

$$\mathcal{O} = \{I \in G : \langle k, \omega(I) \rangle \neq 0, \forall k \in Z^d \setminus \{0\}\}.$$

- *Non-degenerate conditions:*

Kolmogorov:

K) $\partial^2 N(I) = \partial \omega(I)$ is non-singular on G .

Rüssmann: $\exists$ an integer $K \geq 1$ s.t.

R) $\text{rank}\{\partial^\alpha \omega(I) : \forall |\alpha| < K\} = d$ on G .

– *Diophantine set:* Fix $\tau > d - 1$,

$$\mathcal{O}_\gamma = \{I \in G : |\langle k, \omega(I) \rangle| > \frac{\gamma}{|k|^\tau}, \forall k \in Z^d \setminus \{0\}\}.$$

$\Rightarrow \mathcal{O}_\gamma$ - a Cantor set, $|G \setminus \mathcal{O}_\gamma| \rightarrow 0$ as $\gamma \rightarrow 0$.

- *KAM theorem:*

a) If K) holds, then for $\gamma \sim \sqrt{\varepsilon}$ all Diophantine tori T_I , $I \in \mathcal{O}_\gamma$ persists with unchanged frequencies.

(Kolmogorov 54, Arnold 63, Moser 62)

b) If R) holds, then $\exists$ Cantor sets $\mathcal{O}_\varepsilon \subset \mathcal{O}$

$$|\mathcal{O} \setminus \mathcal{O}_\varepsilon| \rightarrow 0$$

s. t. $\forall I \in \mathcal{O}_\varepsilon$, T_I persists.

(Sevryuk 97, Xu-You 97, Chow-Li-Y. 02)

- KAM tori are Floquet;

- If KAM theorem holds, then the nearly integrable Hamiltonian system is *metric stable*, i.e., for any $I(0) \in \mathcal{O}_\varepsilon$,

$\sup_t |I(t) - I(0)| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

- Restricting to $\mathcal{O}_\varepsilon$, the Hamiltonian H becomes integrable.

– A non-Diophantine, non-resonant torus will not survive from a generic perturbation (Forni 1994, Bessi 2000) and can give rise to a cantorus or Aubry-Mather set (Aubry-Daeron 83, Mather 82).

• *Outline of proof for KAM Theorem:*

Normal form: Let $y = I - \lambda$, $x = \theta$. Consider in a KAM step

$$H_\lambda = e_\lambda + \langle \omega_\lambda, y \rangle + h_\lambda(y) + P_\lambda(y, x), \quad (y, x) \in \mathcal{D}, \quad \lambda \in \mathcal{O},$$

where $h_\lambda = O(|y|^2)$, $\deg h = m \geq 2$, $P_\lambda(y, x) = O(|y|^{m+1})$.

KAM iteration: Find symplectic transformation Φ_λ such that

$$H_\lambda^+ = H_\lambda \circ \Phi_\lambda = e_\lambda^+ + \langle \omega_\lambda^+, y \rangle + h_\lambda^+(y) + P_\lambda^+(y, x), \quad (y, x) \in \mathcal{D}^+, \quad \lambda \in \mathcal{O}^+,$$

where $h_\lambda^+ = O(|y|^2)$, $\deg h = m \geq 2$, and $P_\lambda^+(y, x) = O(|y|^{m+1})$ is smaller.

Write

$$\begin{aligned}
P_\lambda &= \tilde{P} + P^*, \\
\tilde{P} &= \sum_{|j| \leq m, |k| \leq K_+} P_{kj} y^j e^{i\langle k, x \rangle} \\
P^* &= \sum_{|j| > m \text{ or } |k| > K_+ \text{ or both}} P_{kj} y^j e^{i\langle k, x \rangle},
\end{aligned}$$

where $K_+ > 0$ is to be determined.

Assume $\Phi_\lambda = \phi_F^1$ – the time-1 map associated to a unknown Hamiltonian

$$F = F(y, x) = \sum_{|j| \leq m, 0 < |k| \leq K_+} F_{kj} y^j e^{i\langle k, x \rangle}.$$

We wish that Φ_λ averages $\tilde{P}$ (in x) $\implies$

$$\{N_\lambda, F\} = \tilde{P} - [\tilde{P}], \quad N_\lambda = e_\lambda + \langle \omega_\lambda, y \rangle + h_\lambda(y),$$

i.e.,

$$i\langle k, \omega_\lambda + \partial_y h_\lambda(y) \rangle F_{kj} = P_{kj}, \quad |j| \leq m, \quad 0 < |k| \leq K_+,$$

which we solve on the domain $\mathcal{D}^+ \subset \mathcal{D}$,

$$\mathcal{O}^+ = \{\lambda \in \mathcal{O} : |\langle k, \omega_\lambda \rangle| > \frac{\gamma}{|k|^\tau}, \quad \forall 0 < |k| \leq K_+\}.$$

Convergence: Let $\Psi_\lambda^\nu = \Phi_\lambda^\nu \circ \dots \circ \Phi_\lambda^0$,

$$H^{\nu+1} = H^\nu \circ \Psi^\nu = e_\lambda^{\nu+1} + \langle y, \omega_\lambda^{\nu+1} \rangle + h_\lambda^{\nu+1}(y) + P_\lambda^{\nu+1}(y, x, \lambda),$$

$(y, x) \in \mathcal{D}^{\nu+1}$, $\lambda \in \mathcal{O}^{\nu+1}$. $\implies \Psi_\lambda^\nu \rightarrow \Psi_\lambda^\infty$, $H_\lambda^\nu \rightarrow H_\lambda^\infty$ on $\mathcal{D}_* = \cap_\nu \mathcal{D}^\nu$ for all $\lambda \in \mathcal{O}_* = \cap_\nu \mathcal{O}^\nu$, where

$$H_\lambda^\infty = e_\lambda^\infty + \langle \omega_\lambda^\infty, y \rangle + h_\lambda^\infty(y),$$

where $h^\infty = O(|y|^2)$ is a polynomial of order m .

Measure estimate: $|\mathcal{O} \setminus \mathcal{O}_*| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

2) Resonant tori

- *Resonance zone:*

$$\mathcal{O}^c = G \setminus \mathcal{O} = \{I \in G : \langle k, \omega(I) \rangle = 0, \text{ for some } k \in \mathbb{Z}^d \setminus \{0\}\}.$$

- Resonant tori tend to be destroyed via generic perturbations $\Rightarrow$ regular orbits (lower dimensional tori) and stochastic layers (chaos or Arnold diffusion)
- *Arnold diffusion:* In the resonance zone, there are orbits whose action variables are drafted arbitrarily far away from the initial positions and wondering around ‘most’ of regular orbits.

- *Poincaré Problem:*

- a) Whether certain fractions of a resonant torus can still survive from a perturbation.
- b) How many and under what mechanism?

- *g-resonant surface*: Fix g – a subgroup of Z^d of rank m . Then g uniquely determines a resonance type and

$$\mathcal{O}_g = \{I \in G : \langle k, \omega(I) \rangle = 0, \forall k \in g\}$$

is a $n = d - m$ dim. submanifold of G , called the *g-resonant surface*.

- *Characterization of resonance zone*: $\mathcal{O}^c = \cup_g \mathcal{O}_g$.
- *Disintegration of resonant tori*: Fix a subgroup g of rank m and let $n = d - m$.
 - g determines a linear transformation: $T^d \rightarrow T^d = T^m \times T^n$ ($n = d - m$):

$$\begin{cases} \phi = K_1 \theta \in T^m, \\ \psi = K_2 \theta \in T^n. \end{cases}$$

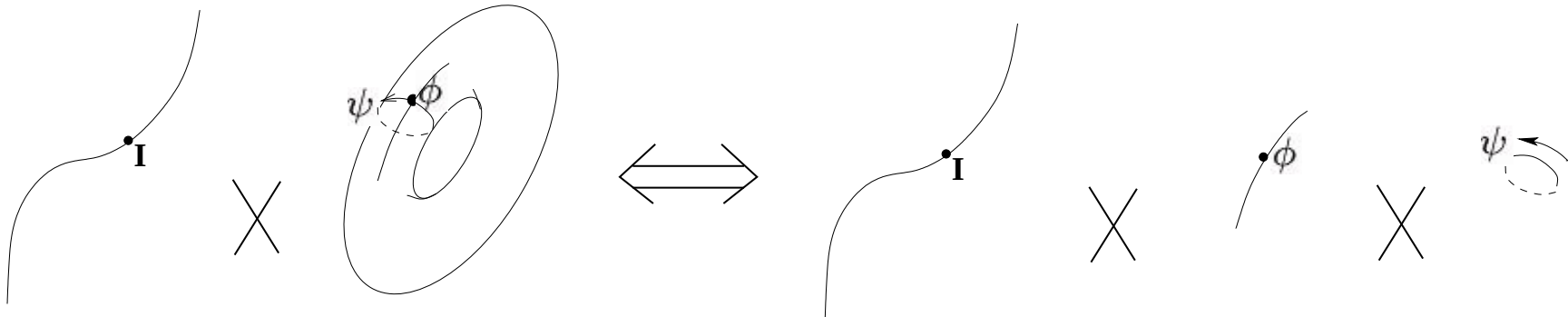
– Under the new coordinate,

$$\mathcal{O}_g = \{I \in G : K_2\omega(I) = 0\}, \quad \omega(I) = (0, K_1\omega(I))^\top$$

$$\begin{cases} \dot{I} = 0 \\ \dot{\phi} = 0 \\ \dot{\psi} = K_1\omega(I). \end{cases}$$

$\implies$ For each $I \in \mathcal{O}_g$, $T_I = \cup_{\phi} T_{I,\phi}$, where

$$T_{I,\phi} = \{I\} \times \{\phi\} \times \overline{\{\psi_0 + K_1\omega(I)t\}} \simeq T^n - \text{invariant}.$$



- *Poincaré non-degeneracy*: $\forall I \in \mathcal{O}_g$, define $h_I : T^m \rightarrow \mathbb{R}$:

$$h_I(\phi) = \int_{T^n} P(I, \phi, \psi) d\psi.$$

$T_{I,\phi}$ is *Poincaré non-degenerate* if ϕ is a non-degenerate critical point of h_I .

- *Poincaré's theorem*: Assume that N satisfies the Kolmogorov non-degenerate condition K). Consider a maximal resonant d -torus T_{I_*} , i.e., T_{I_*} is foliated into invariant periodic orbits with the same period. Then all Poincaré non-degenerate periodic orbits persist.

(Poincaré 1892)

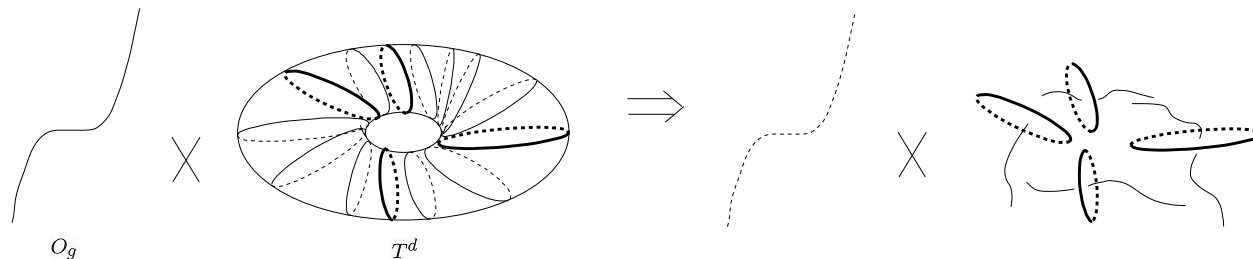
- *Treshchév theorem*: Assume that N satisfies the Kolmogorov non-degenerate condition K) and the g - non-resonant condition G) $K_2 \partial_I^2 N(I) K_2^\top$ is non-singular on $\mathcal{O}_g$.

Then all hyperbolic, Diophantine, Poincaré non-degenerate n -tori on $\mathcal{O}_g$ persist.

(Treshchév 1991)

- $\text{rank } g = 1$ (elliptic case): Eliasson 94, Chiechia 94, Cheng 96, Rudnev-Wiggins 97
- General g -non-resonant tori: Cong et al 97, Jorba-de la Llave-Zou 99.

- *Quasi-periodic Poincaré's theorem:*
 - a) Assume that N satisfies the Kolmogorov non-degenerate condition K). Then $\exists$ Cantor sets $\mathcal{O}^\varepsilon \subset \mathcal{O}_g$, $|\mathcal{O}_g \setminus \mathcal{O}^\varepsilon| \rightarrow 0$, such that $\forall I \in \mathcal{O}^\varepsilon$ all Poincaré non-degenerate n -tori on $\mathcal{O}_g$ persist.
(Li-Y. 2002)
 - b) Assume that N satisfies the g -non-degenerate condition G) and $K_1\omega$ satisfies the Rüssmann non-degenerate condition R) on $\mathcal{O}_g$. Then a) holds.
(Li-Y. 2004)



3) Poisson-Hamilton systems

- *Poisson structure on R^d* : A Poisson bracket $\{\cdot, \cdot\}$ - bi-linear, skew-symmetric which satisfies the Jacobi identity and Leibniz rule.
- *Structure matrix*: $J(z) = (\{z_i, z_j\})$.
 - If $d = \text{odd}$, then $J(z)$ is everywhere singular.
 - If $d = \text{even}$ and J is non-singular on G , then the Poisson structure becomes symplectic.
- *Poisson - Hamilton systems*: Given

$$H : R^d \rightarrow R \quad - \quad a \text{ Hamiltonian function.}$$

The associated Poisson-Hamilton system reads

$$\dot{z} = J(z)\nabla H(z),$$

$\Longleftrightarrow$

$$\dot{z}_i = \{z_i, H\}(z), \quad i = 1, 2, \dots, d.$$

- A Poisson-Hamilton flow preserves the Poisson structure.
- Poisson-Hamilton systems arise naturally in celestial mechanics, fluid dynamics, plasma physics, mean field theory, chemical and biological population, optics, etc. (bi-Hamiltonian structures).
- *Integrability*: A Hamiltonian H on R^d is *completely integrable* if
 - a) $\exists$ a set of l functionally independent first integrals f_i ,
 $i = 1, 2, \dots, l$.
 - b) $\exists$ a set of $n = d - l$ Hamiltonian symmetry functions s_j ,
 $j = 1, \dots, n$, whose Hamiltonian vector fields $\{J\nabla s_j\}$ are
 linearly independent on R^d .
 - c) $\forall i = 1, 2, \dots, l$ and $j, k = 1, 2, \dots, n$,

$$\{f_i, f_j\} = 0, \{s_j, s_k\} = \text{consts}, \{s_j, f_i\} = 0;$$
 - d) $\forall c = (c_1, c_2, \dots, c_l) \in G \subset R^l$, the level set
 $M_c = \{f_i = c_i, i = 1, 2, \dots, l\}$ is compact connected.

• *Extended Liouville Theorem*: If H is completely integrable, then the following holds.

1) $\forall I = (I_1, I_2, \dots, I_l) \in G \subset R^l$, called *action variable*,

$$M_I = \{f_i = I_i, i = 1, 2, \dots, l\} \simeq T^n;$$

2) $\exists$ a global coordinate $\theta \in M_I \simeq T^n$, called *angle variable*, with respect to which each M_I admits parallel flows;

3) $H(z) = N(I)$.

(Bogoyavlenskij 98)

– Also require that

$$J(z) = J(I) = (\{I_i, I_j\}) = \begin{pmatrix} O & B(I) \\ -B(I)^\top & C(I) \end{pmatrix}$$

where $B = B_{l,n}$ and $C = C_{n,n}$ is skew-symmetric. If $l > n$, then J is singular everywhere.

- *Integrable Poisson-Hamilton system:*

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \end{pmatrix} = J(I) \nabla N(I),$$

or

$$\begin{cases} \dot{I} = 0 \\ \dot{\theta} = \omega(I), \end{cases}$$

where

$$\omega(I) = -B^\top(y) \frac{\partial N}{\partial I}(I) \quad - \text{ frequency.}$$

- *Invariant tori:* $\forall I \in G, T_I = \{I\} \times T^n$ is an invariant n -torus with linear flows $\{\theta_0 + \omega(I)t\}$.
- *Foliations:* $G \times T^n = \cup_{I \in G} T_I$.
 - $\omega(I)$ non-resonant $\implies$ the flow is quasi-periodic.
 - $\omega(I)$ resonant $\implies T_I$ is foliated into q.-p. lower dim. tori.

- *Nearly integrable Poisson-Hamilton system:*

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \end{pmatrix} = J(I) \nabla H(I, \theta) = J(I) \nabla (N(I) + \varepsilon P(I, \theta, \varepsilon)).$$

- KAM Theorem (Li-Y., 2003):

a) If ω satisfies the Rüssmann non-degenerate condition R) on G , then a similar KAM theorem holds.

b) If J is a constant matrix and N satisfies the Kolmogorov non-degenerate condition K) on G , then the majority of Diophantine tori persists with unchanged frequencies.

– For $l + n = \text{even}$, I non-singular: (Parayuk 84, Herman 91, Moser 95, Cong-Li 98).

– For volume preserving maps: (Herman 91, Xia 92, Yoccoz 92, Cheng-Sun, 94, Cong et al 96).

4) Partially integrable systems

- A model problem: Pendula coupled by soft-hard springs:
- Normal form: Let $(y, x, z) \in R^n \times T^n \times R^{2m}$ be *action-angle-normal* coordinates.

$$H = e_\lambda + \langle \omega_\lambda, y \rangle + \langle A_\lambda z, z \rangle + P_\lambda(y, x, z).$$

- *Melnikov problem*: Persistence of quasi-periodic n -tori under Melnikov conditions - non-resonant conditions between ω_λ and eigenvalues of JA_λ .

(Melnikov 65, Moser 67, Graff 74, Zehnder 75, Kuksin 87, Eliasson 88,94, Pöschel 89, Treshchev 89, Broer-Huitema-Takens 90, Bourgain 94,96,98, Chiechia 94, Cheng 96, Rudnev-Wiggins 97, Jorba-Villanueva 97, You 99, Jorba-de la Llave-Zou 99, Li-Y. 99,02, Gallavotti-Gentile 02)

5) ∞ -dim Hamiltonian systems

- *Abstract form:*

$$\dot{u} = \mathcal{J} \delta H(u).$$

- Physical models: Nonlinear Schrödinger equations, wave equations, beam equations, Kdv, Euler equations, Hamiltonian networks, etc.

- *Nonlinear Schrödinger equations:*

$$\begin{cases} iu_t + Au + \frac{\partial F}{\partial \bar{u}}(u, \bar{u}) = 0, \\ \text{Dirichlet or periodic boundary conditions,} \end{cases}$$

where $A = -\Delta + V(x)$.

- Hamiltonian setting:

$$u_t = i \frac{\partial H}{\partial \bar{u}}, \quad H = \langle Au, u \rangle + \int F(u, \bar{u}) dx.$$

– Lattice form: Let μ_n, ϕ_n be eigenvalues, eigenvectors of A and consider

$$u(x, t) = \sum_n q_n(t) \phi_n(x).$$

Then q_n are described by the Hamiltonian

$$H = \sum_n \mu_n |q_n|^2 + P(q, \bar{q}),$$

where

$$P = \int F\left(\sum_n q_n \phi_n, \sum_n \bar{q}_n \bar{\phi}_n\right) dx.$$

– Normal form: Given n , choose parameter $\omega \in R^n$,
action-angle-normal coordinate $(y, x, z, \bar{z}) \in R^n \times T^n \times \ell^1 \times \ell^1$ s.t.

$$H = e + \langle \omega, y \rangle + \sum_j \Omega_j(\omega) z_j \bar{z}_j + P(y, x, z, \bar{z}).$$

– Existence of (time) quasi-periodic solutions:

KAM or CWB (Craig-Wayne-Bourgain) methods (Kuksin 93, Bourgain 94, 98, 05, Kuksin-Pöschel 96, Geng-You 04, Geng-Y. 06)

• Other ∞ -dim Hamiltonians:

– *Nonlinear Waves*: Kuksin-Pöschel 96, Pöschel 96, Bourgain 00, Chierchia-You 00, Yuan 07

– *Kdv*: Kuksin 98, Kappeler-Pöschel 03

– *Beam equations*: Geng-You 03, Liang-Geng 05

– *Hamiltonian networks*: Yuan 02, Geng-Y. 05, Geng-Viveros-Y. 07

2. Aubry-Mather Theory

I. Mather's Approach

- Poincaré map: $\phi: A = S^1 \times [\alpha', \alpha''] \rightarrow A$:

$$\begin{cases} x_1 = f(x, y), \\ y_1 = g(x, y), \end{cases} \quad x \in S^1, y \in [\alpha', \alpha''],$$

is area preserving, boundary preserving, and monotone, i.e., $\frac{\partial f}{\partial y} > 0$.

- Goal: For $\alpha \in [\rho_{\alpha'}, \rho_{\alpha''}] \cap Q^c$, find “invariant curve”

$$\mathcal{C}_\alpha = \{(u(\xi), v(\xi)) : \xi \in S^1\},$$

such that u is monotone, $u(\xi) = \xi + U(\xi)$, and,

$$\phi(u(\xi), v(\xi)) = (u(\xi + \alpha), v(\xi + \alpha)).$$

- Generating function: $h = h(x, x_1)$ s.t.

$$y = \frac{\partial h}{\partial x}(x, x_1),$$

$$y_1 = -\frac{\partial h}{\partial x_1}(x, x_1).$$

- E-L equation:

$$\frac{\partial h}{\partial x}(u(\xi), u(\xi + \alpha)) + \frac{\partial h}{\partial x_1}(u(\xi - \alpha), u(\xi)) = 0.$$

- Minimizing:

$$\min \int_0^1 h(u(\xi), u(\xi + \alpha)) d\xi$$

over a space of monotone, left continuous functions.

- Verify the E-L equation.
- Remark: An Aubry-Mather set is an almost 1-cover of a circle and a Denjoy cantor set, which is not necessarily contained in an invariant circle.

- Forced Pendulum:

$$\ddot{u} + V_u(u, t) = 0,$$

$$V(u, t) = V(u + 1, t) = V(u, t + 1).$$

Using Poincaré map and verify monotonicity $\implies$:

- a) For $\rho \notin Q$ in some range, Aubry-Mather solutions have the form:

$$u(t) = \alpha + \rho t + U(\beta + \rho t, \alpha + t),$$

$U : T^2 \rightarrow R$ discontinuous, but $U(\beta + \rho t, \alpha + t)$ has convergent Fourier series for some α, β .

- b) In $T^1 \times R^1 \times T^1$, $cl\{(u(t), \dot{u}(t), t)\} = \text{Aubry-Mather set (Denjoy Cantorus)}$.

- *Order to chaos via Cantori:*

Consider a periodically forced Duffing equation

$$\ddot{u} + \alpha_1 u - \alpha_2 u^3 - \alpha_3 V(t)u = 0, \quad V(t) = V(t + 1).$$

$\exists \rho_0 > 0$ s. t.

– $\rho < \rho_0$: Chaos.

– $\rho > \rho_0$:

ρ Diophantine of constant type: Quasi-periodic 2-tori.

ρ rational: Periodic orbits.

ρ other case (e.g., Liouville): Aubry-Mather sets (Cantori).

(van Noort-Porter-Chow-Y., 07)

II. Moser's Approach

$$\ddot{x} + V_x(x, t) = 0,$$

where $V(x, t) = V(x + 1, t) = V(x, t + 1)$.

- Lagrangian:

$$L(x, \dot{x}, t) = \frac{\dot{x}^2}{2} - V(x, t).$$

- Action-minimizing: Global minimal function $x(t)$ exists, i.e.,

$$\int [L(x + \phi, \dot{x} + \dot{\phi}, t) - L(x, \dot{x}, t)] dt \geq 0$$

for all $\phi \in Lip_{comp}$.

— E-L equation is satisfied by a global minimum automatically.

- Rotation number: For any global minimal function x ,

$$\rho(x) = \lim_{t \rightarrow \infty} \frac{x(t)}{t}$$

exists and for $\forall \rho, \exists$ global minimal function x such that $\rho(x) = \rho$.
Denote $\mathcal{M}_\rho = \{\text{global minimal functions with rotation number } \rho\}$.

- If $\rho \notin Q$, then each $x \in \mathcal{M}_\rho$ has the form

$$x(t) = u(t, \theta), \quad \theta \in S_t = \{\rho(t + j) - k : (j, k) \in \mathbb{Z}^2\},$$

where $u(t, \theta) = u(t + 1, \theta)$, $u(t, \theta + 1) = u(t, \theta) + 1$, and u is increasing in θ .

- Aubry-Mather solution:

$$x(t) = u^\pm(\alpha + t, \beta + \rho t),$$

where

$$u^\pm(t, \theta) = \lim_{\theta_n \rightarrow \theta, \theta_n - \theta \in R_\pm} u(t, \theta_n).$$

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