

MULTI-FREQUENCY OSCILLATIONS

IN DYNAMICAL SYSTEMS

Session 5. Transient Bio-Chemical Oscillations

Yingfei Yi

Georgia Tech & Jilin University

DANCE Winter School RTNS2011

1. Chemical Oscillations

- **BZ-Reaction:** A chemical oscillatory reaction, fluctuating in color, discovered by B. P. Belousov (1951) and continued by A. M. Zhabotinsky (1964).
- **Chemical-Reaction:** Following BZ-reaction, general chemical reactions have been studied using
 - rate equations (ODE models assuming high concentration of species),
 - reaction-diffusion equations (adding diffusion mechanism),
 - chemical master equations (lattice models assuming lower concentration of species).

- **Rate Equations:**

- *Law of Mass Action:* Rate of any given chemical reaction is proportional to the product of the concentrations of the reactants.

- Example:

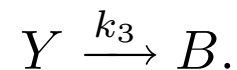
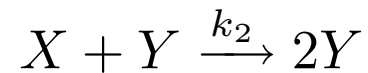
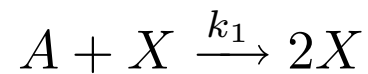


$$\begin{cases} \dot{x} &= & -kxy, \\ \dot{y} &= & -kxy, \\ \dot{z} &= & kxy. \end{cases}$$

2. Two Type of Systems

- **Open Systems:**

- Exchange of energy with surrounding environment.
- Conservative.
- Wide range of dynamics.
- e.g. (Irreversible Lotka-Volterra System)



$$\begin{cases} \dot{x} &= k_1 c_A x - k_2 xy \\ \dot{y} &= k_2 xy - k_3 y \end{cases}$$

- **Closed Systems:**

- No exchange of energy with surrounding environment is allowed.

- Dissipative.

- Solutions converge to an equilibrium (2nd law of Thermodynamics).

- Almost all existing chemical-reaction models assume constant sources and sinks and thus are open systems.

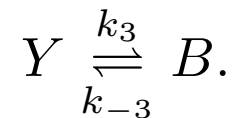
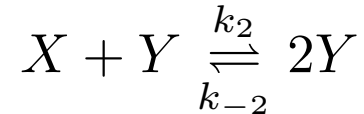
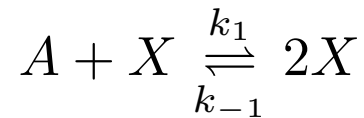
But in reality, a chemical reaction takes place in a closed environment, so can this oscillatory reaction really go on forever? The answer is no! In fact, the oscillatory reaction consumes energy. If the oscillation were possible going forever, it would be a perpetual motion machine! That is against the first law of thermodynamics.

3. Biochemical Oscillations

- Kruse and Julicher (2005): “oscillations play an important role in many dynamic cellular processes. They can emerge as the collective dynamic behavior of an ensemble of interacting components in the cell. Examples include oscillations in cytoskeletal structures ... All these oscillations share many general features. Models and theoretical approaches are essential for an understanding of the principles underlying these dynamic cellular processes”.
- To understand bio-chemical oscillations, it seems appropriate to
 - use closed system models, due to the lacking of natural source terms;
 - not to neglect a backward reaction, since energy between a reactant and a product is directly related to the ratio of the forward and backward rate constant.

4. Reversible Lotka-Volterra System

(Li-Qian-Y. 08, 10)



$$k_{-i} \ll k_i \quad \text{and} \quad k_1 \ll k_2.$$

$$\left\{ \begin{array}{l} \dot{x} = k_1 c_A x - k_{-1} x^2 - k_2 xy + k_{-2} y^2, \\ \dot{y} = k_2 xy - k_{-2} y^2 - k_3 y + k_{-3} c_B, \\ \dot{c}_A = -k_1 c_A x + k_{-1} x^2, \\ \dot{c}_B = k_3 y - k_{-3} c_B. \end{array} \right.$$

- **Rescaling:**

$$u = \frac{k_2}{k_3}x, \quad v = \frac{k_2}{k_3}y, \quad w = \frac{k_1}{k_3}c_A, \quad z = \frac{k_2}{k_3}c_B, \quad \tau = k_3t,$$
$$\varepsilon = \frac{k_{-1}}{k_1} = \frac{k_{-2}}{k_2} = \frac{k_{-3}}{k_3}, \quad \sigma = \frac{k_1}{k_2}, \quad \beta = \varepsilon z, \quad \delta = \varepsilon\sigma$$

- **Dimensionless Form:**

$$\left\{ \begin{array}{l} \dot{u} = u(w - v) - \varepsilon(\sigma u^2 - v^2) \\ \dot{v} = v(u - 1) - \varepsilon v^2 + \varepsilon z \\ \dot{w} = -\sigma(wu - \varepsilon\sigma u^2), \\ \dot{z} = v - \varepsilon z. \end{array} \right.$$

where $0 < \varepsilon \ll \sigma \ll 1$.

- Conservation of mass:

$$u + v + \frac{w}{\sigma} + z = \xi.$$

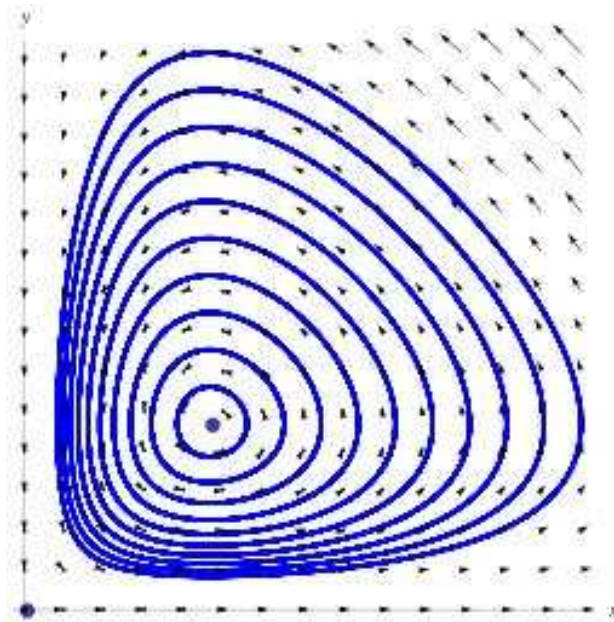
- 3D Version of Reversible LV System:

$$\left\{ \begin{array}{l} \frac{du}{d\tau} = u(w - v) - \varepsilon(\sigma u^2 - v^2) \\ \frac{dv}{d\tau} = v(u - 1) - \varepsilon v^2 + \varepsilon \left(\xi - u - v - \frac{w}{\sigma} \right) \\ \frac{dw}{d\tau} = -\sigma(wu - \varepsilon \sigma u^2). \end{array} \right.$$

- **Unperturbed system:**

$$\begin{cases} \frac{du}{d\tau} = u(w - v) \\ \frac{dv}{d\tau} = v(u - 1) \\ \frac{dw}{d\tau} = 0. \end{cases}$$

– A family of irreversible LV system.



- **Reaction zone:**

$$\mathcal{T} = \left\{ (u, v, w) \in \mathbb{R}^{3+}, \quad u + v + \frac{w}{\sigma} \leq \xi \right\}$$

is positively invariant.

- **Long term dynamics:**

- The system is dissipative;

- \exists a unique positive equilibrium P_* which is a global attractor in \mathcal{T} .

- **Important issues:**

- Quasi-steady-state;

- Transient dynamics;

- Nature and mechanism of oscillations.

- **Oscillating Zone:** Away from the equilibrium, \exists an oscillating zone $\mathcal{T}^o \subset \mathcal{T}$, $\{w \geq \sigma\} \subset \mathcal{T}^o \subset \{w \geq \sigma^2\}$, with the following properties:
 - (Oscillating Axis) \exists 1D, asymptotically stable, locally invariant manifold $W_{\sigma, \varepsilon}^o \subset \mathcal{T}^o$ s. t. each orbit in \mathcal{T}^o oscillates around $W_{\sigma, \varepsilon}^o$ in a finite number of times (multi-frequency) with decreasing diameters;
 - (Nature of Oscillation) Each round of oscillation exhibits both singular and regular behaviors: It consists of a “horizontal” portion with nearly unchanged energy and w -value which shadows a periodic orbit of the unperturbed system on the same energy level, and a “vertical” portion with exponentially decay energy and w -values;
 - (Winding Number) The closer an orbit is to the oscillating axis initially, the less number of oscillation it makes.

-- If $\xi \sim \frac{1}{\sigma}$, then the oscillation is regular (no jumping).

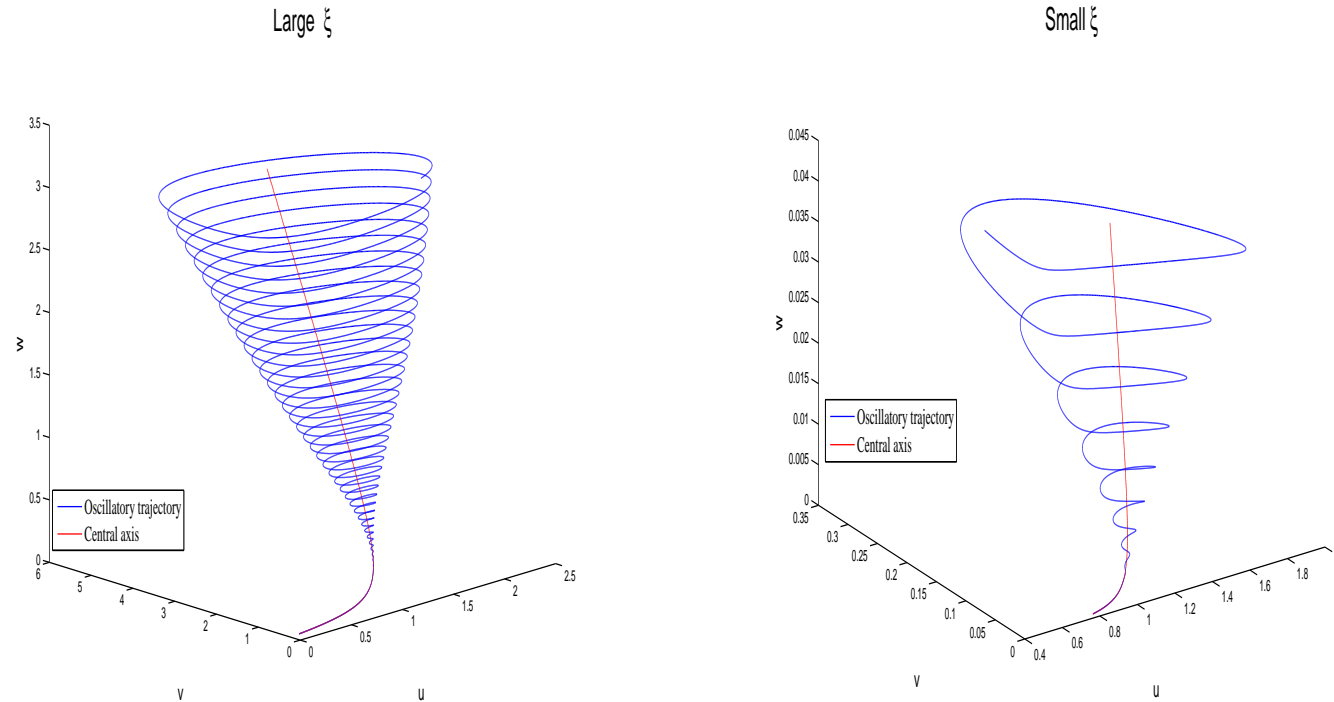


Figure 1: Oscillation zone. In the left panel, $\xi = 500$, $\sigma = 0.01$, $\varepsilon = \sigma^2$ with initial $(2, 2, 3)$. In the right panel, $\xi = 10$, $\sigma = 0.005$, $\varepsilon = \sigma^2$ with initial $(0.5, 0.1, 0.045)$.

• **Non-oscillating Zone:** Near the equilibrium, \exists a non-oscillating zone $\mathcal{T}^n \subset \mathcal{T}$, $\{w \leq \sigma\varepsilon\} \subset \mathcal{T}^n \subset \{w \leq \sigma\}$, with the following properties:

— (Non-oscillating) $P_* \in \mathcal{T}^n$ is an asymptotically stable node;

— (Strongly Stable Manifold) \exists a 2D, exponentially stable, positively invariant manifold $M_{\sigma,\varepsilon}^n \subset \mathcal{T}^n$ containing P_* ;

— (Stable Manifold) Within $M_{\sigma,\varepsilon}^n$, \exists an 1D, exponentially stable, positively invariant manifold $M_{\sigma,\varepsilon}^n$ containing P_* along with 2D stable foliations.

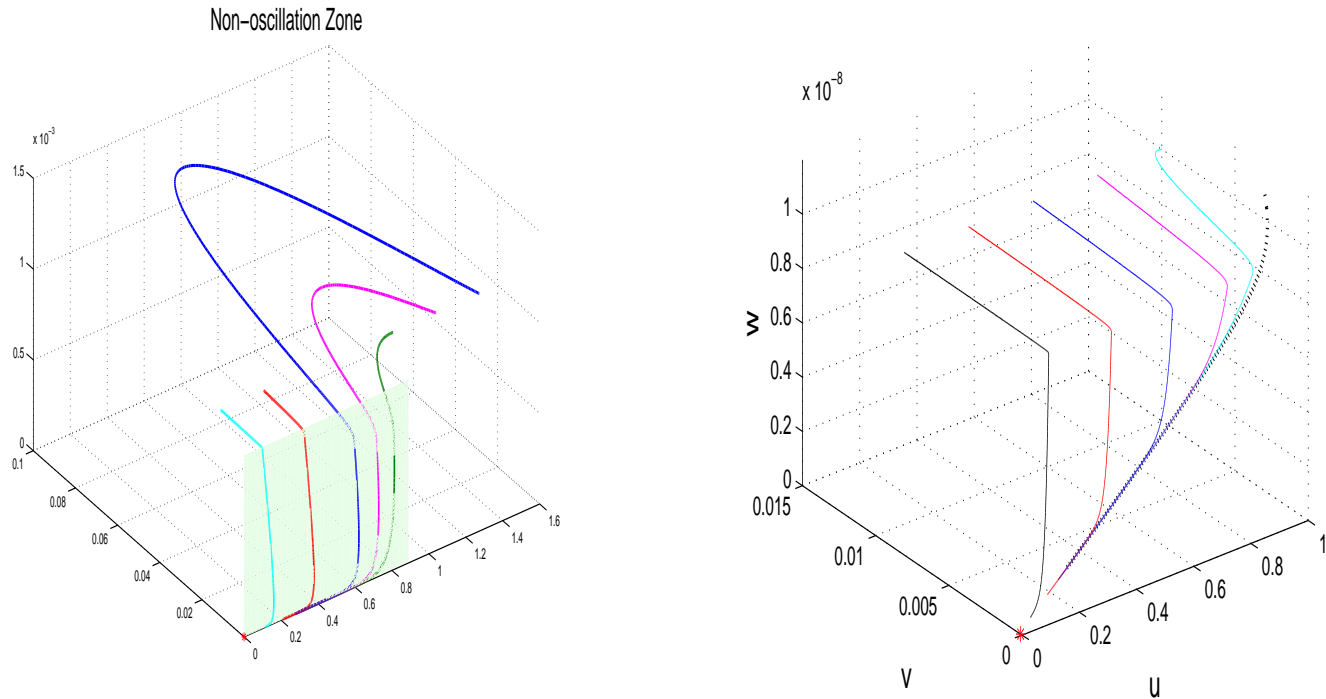


Figure 2: Nonoscillating zone. In both panel, $\xi = 100$; $\sigma = 0.01$, $\varepsilon = \sigma^3$. In the left one, the initials are $(u_0, 0.02, 0.001)$ with $u_0 = 0.1, 0.333, 1.033, 1.267, 1.5$. In the right one, the initials are $(u_0, \sigma, \sigma\varepsilon)$ with $u_0 = 0.1, 0.325, 0.55, 0.775, 1$.

5 . Remark

- The dynamics is a mixture of dissipative, conservative, and monotone ones.
- The system is a dissipative, singular perturbation to conservative ones.
- The oscillation is neither electrical nor mechanical.
- The long term dynamics are trivial, but the transient ones are rich.

6. Open Problems

- *Higher dimensional models:* More complicated transient, oscillating dynamics are expected.
- *Stochastic counterparts:* Dynamical connections between a rate equation and its stochastic counterparts.
 - When noise is added, the probability density ρ satisfies a Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\varepsilon D(x) \nabla \rho - \rho v(x)), \quad x \in R^N, \quad \int_{R^N} \rho dx = 1.$$

- When the total volume is low, the probability distribution $p(n, t) = \Pr(x_1(t)V = n_1, \dots, x_N(t)V = n_N)$, where V is the total volume, satisfies a Chemical Master Equation

$$\dot{p}(n, t) = \nabla \cdot (A(n) \nabla p(n, t) - p(n, t)v(n)).$$

REFERENCES

- Y.-F. Li, H. Qian, and Y. Yi, Nonlinear oscillations and multiscale dynamics in a closed chemical reaction system, *J. Dynam. Differential Equations*, Vol 22 No 3 (2010).
- Y.-F. Li, H. Qian, and Y. Yi, Oscillation and multiscale dynamics in a closed chemical reaction system: Second law of thermodynamics and temporal complexity, *J. Chem. Phys.* Vol. 129 No. 15 (2008).