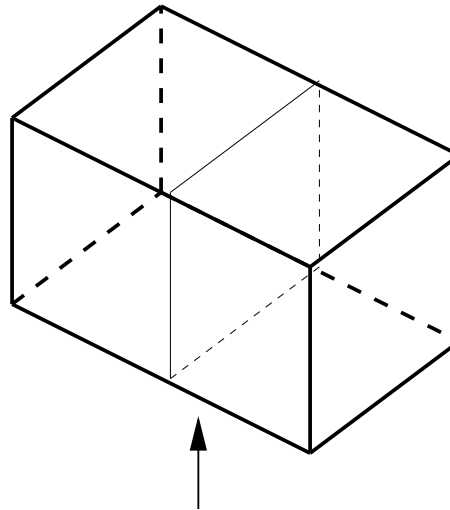


EXAMPLE OF A LARGE SYSTEM

box with fluid-saturated
porous material

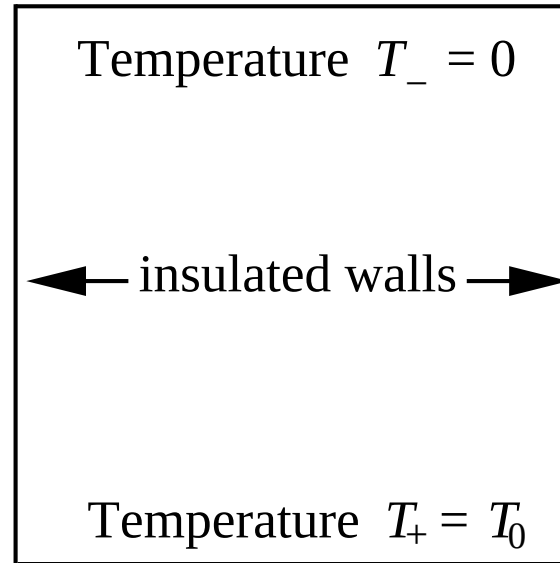


heat from below

Unknowns: Temperature $T = T(t, x, y, z)$,

Fluid velocity $\mathbf{v} = \mathbf{v}(t, x, y, z) = \begin{bmatrix} v_1(t, x, y, z) \\ v_2(t, x, y, z) \\ v_3(t, x, y, z) \end{bmatrix} .$

EXAMPLE OF A LARGE SYSTEM (Continued)



After adimensionalization, Unknowns: u s.t.

Temperature $T(t, x, y) = (1 - y) + u(t, x, y),$

Fluid velocity $\mathbf{v} = \mathbf{v}(t, x, y) = \begin{bmatrix} v_1(t, x, y) \\ v_2(t, x, y) \end{bmatrix}.$

EXAMPLE OF A LARGE SYSTEM (Continued)

$$\Omega = [0, 1] \times [0, 1],$$

$$u_t + \sqrt{\mu} \mathbf{v} \cdot (\nabla u - \mathbf{e}_2) = \Delta u,$$

$$-\nabla p - \mathbf{v} + \sqrt{\mu} u \mathbf{e}_2 = 0,$$

$$\nabla \cdot \mathbf{v} = 0,$$

$$\nabla u \cdot n = 0, \quad x = 0, 1, \quad y \in [0, 1],$$

$$u = 0, \quad x \in [0, 1], \quad y = 0, 1,$$

$$\mathbf{v} \cdot n = 0, \quad (y, z) \in \partial\Omega,$$

$p = p(t, x, y)$ is the pressure

μ is the Rayleigh number, $\mu = \text{Ra} = \frac{g\beta}{\nu\alpha}T_0l^3$

- $g =$ acceleration due to gravity,
- $\beta =$ thermal expansion coefficient,
- $\nu =$ Fluid's kinematic viscosity,
- $\alpha =$ Thermal diffusivity¹,
- $T_0 =$ Temperature jump between top and bottom walls,
- $l =$ Characteristic length of the cavity.

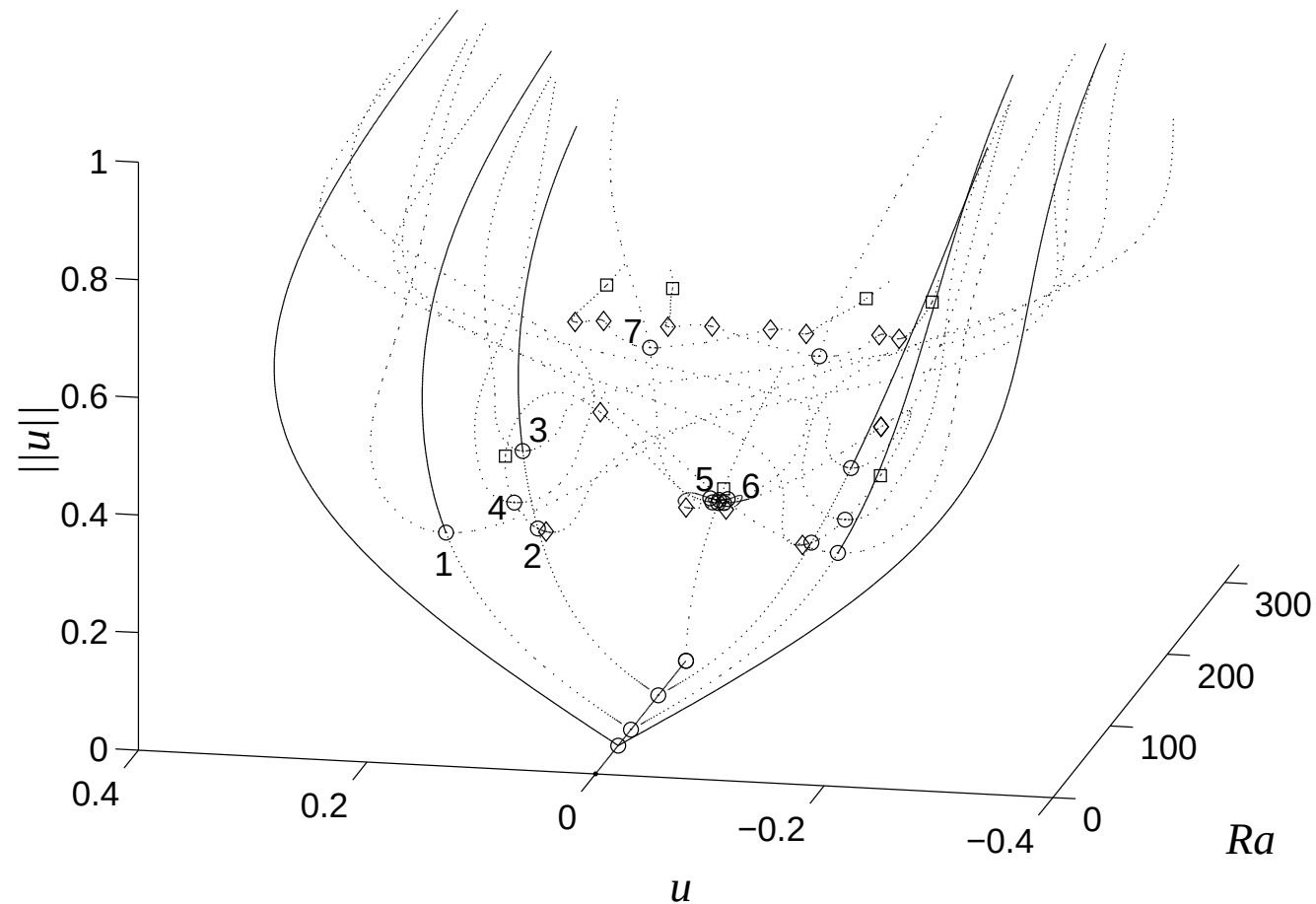
¹Equal to thermal conductivity over the product of density and specific heat capacity.

Steady state solutions: $\mathbf{v} = \mathbf{v}(u)$,

$$f(u, \mu) \equiv -\Delta u + \sqrt{\mu} \mathbf{v}(u) \cdot (\nabla u - \mathbf{e}_2) = 0, \quad \text{in } \Omega$$

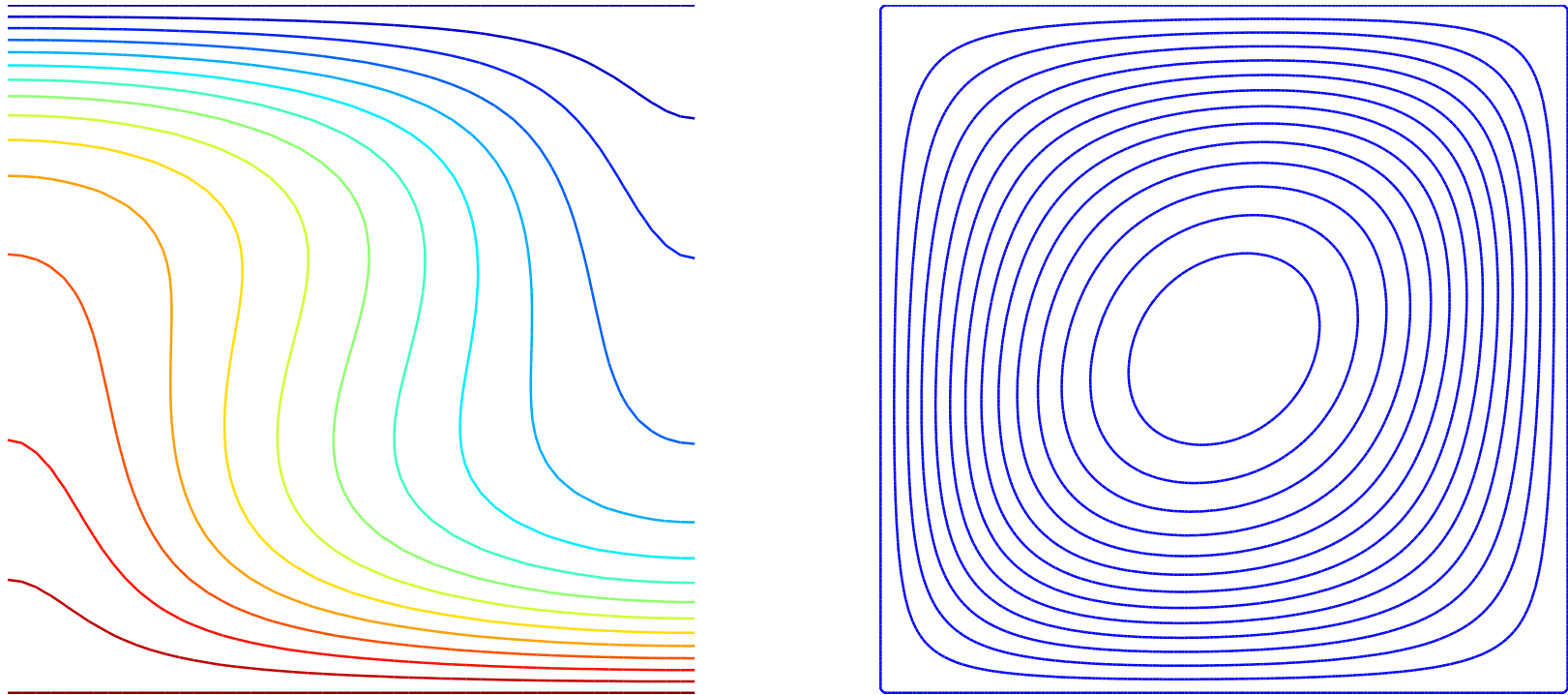
$$\begin{aligned} \nabla u \cdot n &= 0, & x = 0, 1, \quad y \in [0, 1], \\ u &= 0, & x \in [0, 1], \quad y = 0, 1, \end{aligned}$$

EXAMPLE OF A LARGE SYSTEM (Continued)



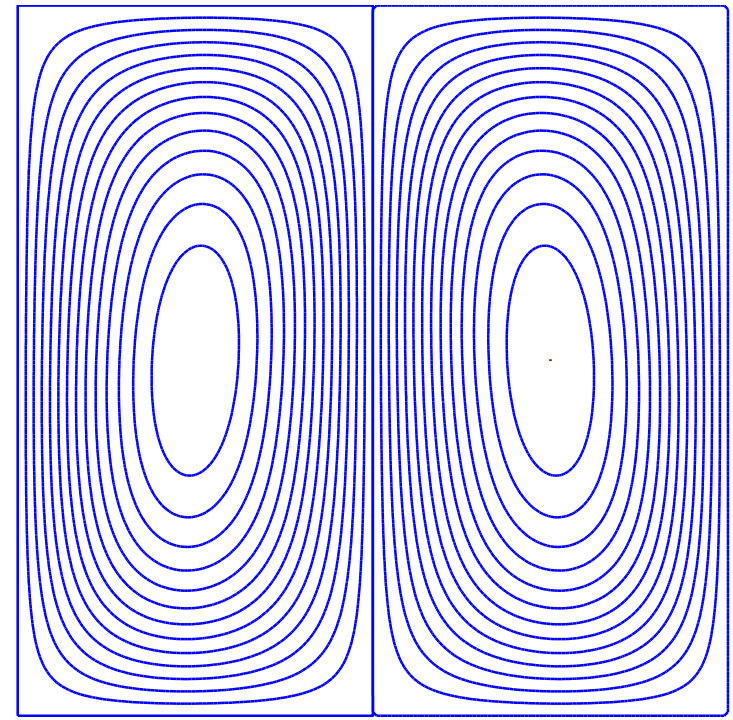
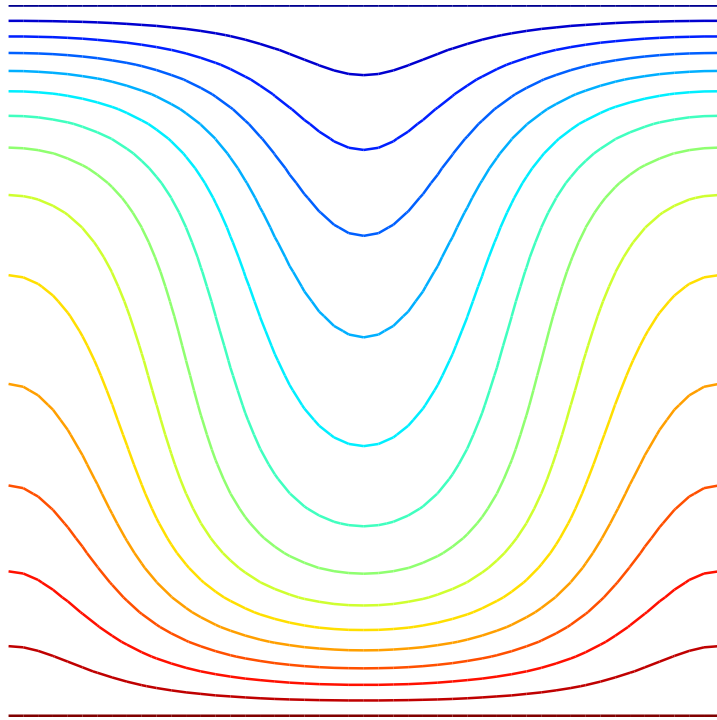
Bifurcation diagram of steady-state solutions.

EXAMPLE OF A LARGE SYSTEM (Continued)

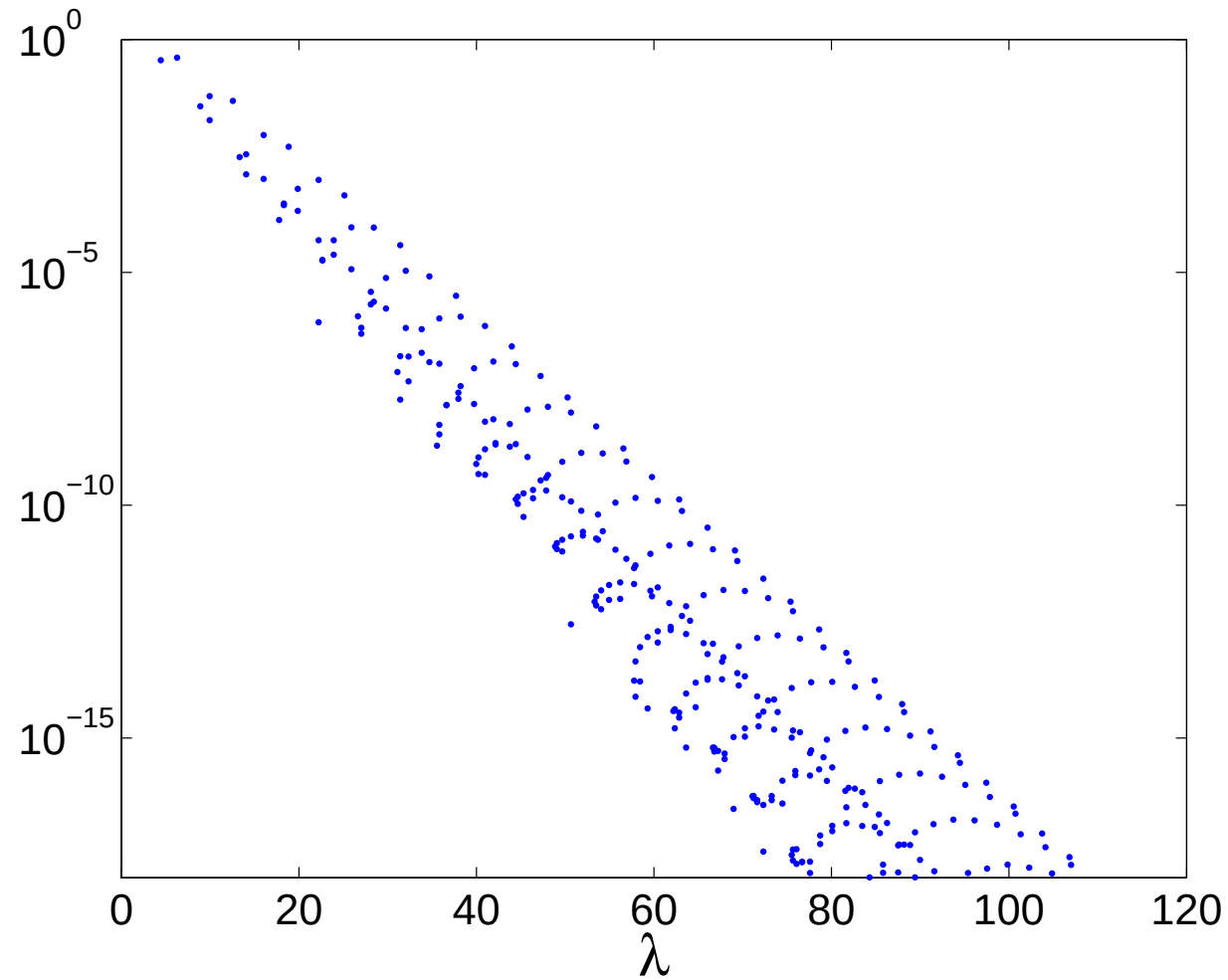


Temperature contour lines (left) and streamlines (right) for $Ra = 100$ on first primary branch.

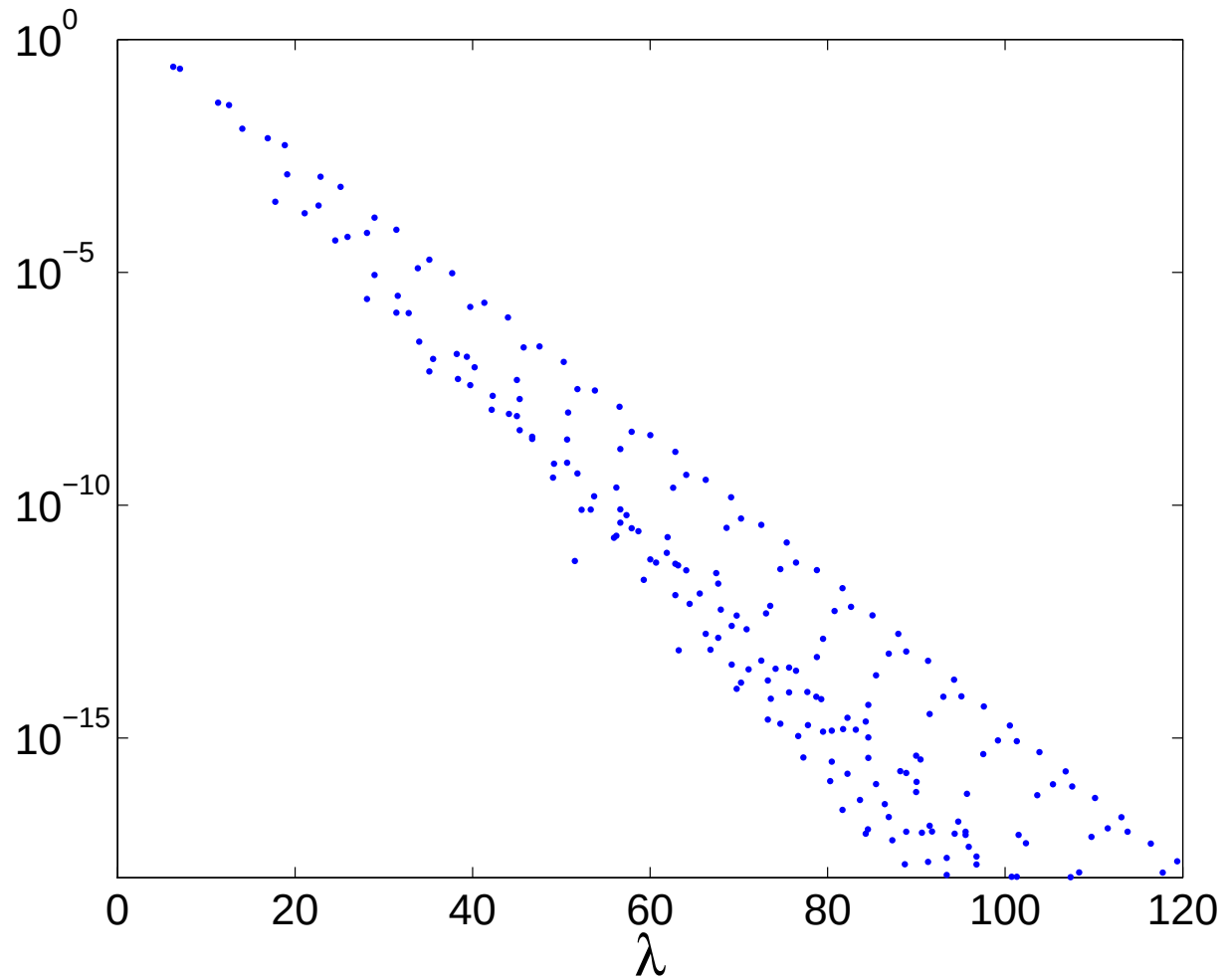
EXAMPLE OF A LARGE SYSTEM (Continued)



Temperature contour lines (left) and streamlines (right) for $Ra = 100$ on second primary branch.



Value of $|\hat{u}_{j,k}|$ against wave number $\lambda = \pi\sqrt{j^2 + k^2}$
for solutions at $\mu = 100$ (first primary branch).



Value of $|\hat{u}_{j,k}|$ against wave number $\lambda = \pi\sqrt{j^2 + k^2}$
for solutions at $\mu = 100$ (second primary branch).

Step 1 Obtain the Fourier coefficients of $\partial_x U_N$ and $\partial_y U_N$

$$-j\pi\hat{U}_{j,k}, \quad \text{and} \quad k\pi\hat{U}_{j,k}, \quad j = 0, \dots, N, \quad k = 1, \dots, N-1.$$

Step 2 Obtain the vectors of nodal values $\underline{\partial_x U_N}$ and $\underline{\partial_y U_N}$ on the grid $\mathcal{G}_{2N}$

Step 3 Obtain the Fourier coefficients

$$\sqrt{\mu} \frac{-jk}{j^2 + k^2} \hat{U}_{j,k}, \quad \text{and} \quad \sqrt{\mu} \frac{j^2}{j^2 + k^2} \hat{U}_{j,k}, \quad j, k = 0, \dots, N,$$

of the two components $V_N^{(x)}$ and $V_N^{(y)}$, respectively, of the velocity $\mathbf{V}_N$.

Step 4 Obtain the vectors of nodal values $\underline{V_N^{(x)}}$ and $\underline{V_N^{(y)}}$ on the grid $\mathcal{G}_{2N}$.

Step 5 Obtain the vector of nodal values $\underline{\mathbf{V}_N \cdot (\nabla U_N - \mathbf{e}_2)}$ on the grid $\mathcal{G}_{2N}$.

Step 6 Use $\hat{g}_{j,k} = (\underline{p}_{j,k}, \underline{g})_N / \|\underline{p}_{j,k}\|^2$ to obtain the $4N^2 - 1$ Fourier coefficients of $\mathbf{V}_N \cdot (\nabla U_N - \mathbf{e}_2)$, and discard those corresponding to $j > N$ or $k \geq N$.

Remark. Steps 2, 4 and 6 are standard operations by means of discrete sine and cosine transforms, typically implemented with the help of the *fast Fourier transform* (FFT). FFT algorithms allow to obtain the $N^2 - 1$ coefficients

$$\hat{g}_{j,k} = \frac{(\underline{p}_{j,k}, \underline{g})_N}{\|\underline{p}_{j,k}\|^2}, \quad j = 0, \dots, N, \quad k = 1, \dots, N - 1,$$

in $O(N^2 \log(N))$ operations, rather than $O(N^4)$.