

Numerical techniques for large dimensional dynamical systems

Bosco García-Archilla

Universidad de Sevilla
(Spain)

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Program.

- Remarks on large-dimensional dynamical systems.
- Basic numerical continuation techniques.
- Krylov Methods.
 - GMRES.
 - Remarks on Implicitly Restarted Arnoldi methods.
- Basics of spectral discretization.
- Computations on a convection problem.
- Continuation of periodic orbits.

References

- [1] E. J. Doedel, *Lecture Notes on Numerical Analysis of Bifurcation Problems*, (indy.cs.concorida.ca/auto/notes.pdf)
- [2] W. J. F. Govaerts, *Numerical Methods for Bifurcations of Dynamical Equilibria*, SIAM, Philadelphia, 2000.
- [3] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory (2nd Ed.)*, Springer, New York, 1998.
- [4] K. A. Cliffe, A. Spence & S. J. Tavener, The numerical analysis of bifurcation problems with application to fluid mechanics. *Acta numerica*, 2000, 39-131, *Acta Numer.*, 9, Cambridge Univ. Press, Cambridge, 2000.

Further References

- [5] C. T. Kelley *Iterative methods for linear and nonlinear equations*, SIAM, Philadelphia, 1995.
- [6] C. T. Kelley *Solving nonlinear equations with Newton's method*, SIAM, Philadelphia, 2003.
- [7] R. Seidel, *Practical Bifurcation and Stability Analysis. From Equilibrium to Chaos. (2nd Ed)*, Springer, New York, 1994.
- [8] R. B. Lehoucq, D. C. Sorensen & C. Yang, *ARPACK users' guide. Solution of large-scale eigenvalue problems with implicitly restarted Arnoldi methods*. SIAM, Philadelphia, PA, 1998.

SOFTWARE

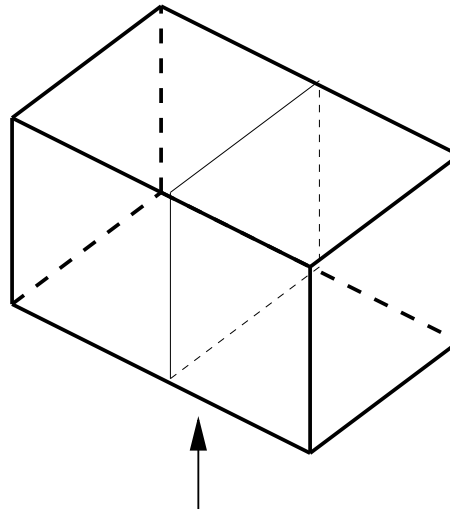
- 1) **AUTO**, by E. S. Doedel *et al.* (software for continuation and bifurcation problems in ordinary differential equations, Fortran/C. Available for UNIX/Linux-based computers)
<http://indy.cs.concordia.ca/auto/>
- 2) **MatCont**, by W. Govaerts and Y. A. Kuznetsov (MATLAB) toolbox for numerical continuation and bifurcation analysis.
<http://www.matcont.ugent.be/>
- 3) **PLTMG**, by R. E. Bank (Software package for solving elliptic partial differential equations by the finite-element method on UNIX system (C/Fortran)).
<http://www.netlib.org/pltmg/>
- 4) **LOCA** by A. Salinger, L. Romero, R. Pawlowski, E. Wilkes, R. Lehoucq, B Burroughs & N. Bou-Rabee. (package for robustly solving and analyzing large-scale systems of nonlinear equations).
<http://trilinos.org/packages/nox-and-loc/>

BASIC CHANGES WITH LARGE DIMENSIONS

- 1) System of linear equations are solved by iterative methods.
- 2) Stability: only part of the spectrum is computed.
- 3) Periodic orbits, invariant tori, etc: Poincaré sections.

EXAMPLE OF A LARGE SYSTEM

box with fluid-saturated
porous material

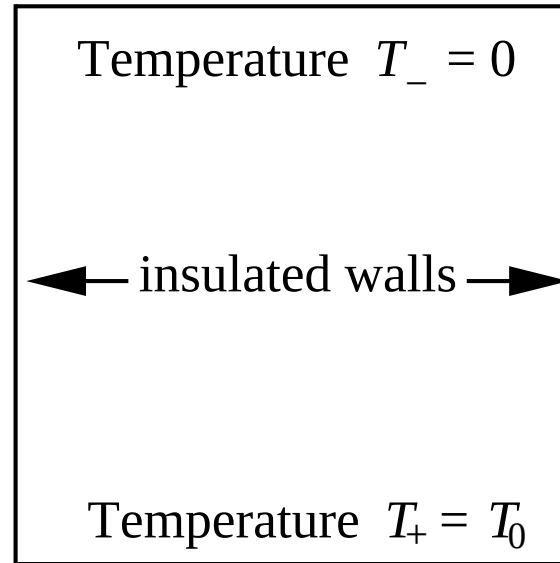


heat from below

Unknowns: Temperature $T = T(t, x, y, z)$,

Fluid velocity $\mathbf{v} = \mathbf{v}(t, x, y, z) = \begin{bmatrix} v_1(t, x, y, z) \\ v_2(t, x, y, z) \\ v_3(t, x, y, z) \end{bmatrix} .$

EXAMPLE OF A LARGE SYSTEM (Continued)



After adimensionalization, Unknowns: u s.t.

Temperature $T(t, x, y) = (1 - y) + u(t, x, y),$

Fluid velocity $\mathbf{v} = \mathbf{v}(t, x, y) = \begin{bmatrix} v_1(t, x, y) \\ v_2(t, x, y) \end{bmatrix}.$

EXAMPLE OF A LARGE SYSTEM (Continued)

$$\Omega = [0, 1] \times [0, 1],$$

$$u_t + \sqrt{\mu} \mathbf{v} \cdot (\nabla u - \mathbf{e}_2) = \Delta u,$$

$$-\nabla p - \mathbf{v} + \sqrt{\mu} u \mathbf{e}_2 = 0,$$

$$\nabla \cdot \mathbf{v} = 0,$$

$$\nabla u \cdot n = 0, \quad x = 0, 1, \quad y \in [0, 1],$$

$$u = 0, \quad x \in [0, 1], \quad y = 0, 1,$$

$$\mathbf{v} \cdot n = 0, \quad (y, z) \in \partial\Omega,$$

$p = p(t, x, y)$ is the pressure

μ is the Rayleigh number, $\mu = \text{Ra} = \frac{g\beta}{\nu\alpha}T_0l^3$

- $g =$ acceleration due to gravity,
- $\beta =$ thermal expansion coefficient,
- $\nu =$ Fluid's kinematic viscosity,
- $\alpha =$ Thermal diffusivity¹,
- $T_0 =$ Temperature jump between top and bottom walls,
- $l =$ Characteristic length of the cavity.

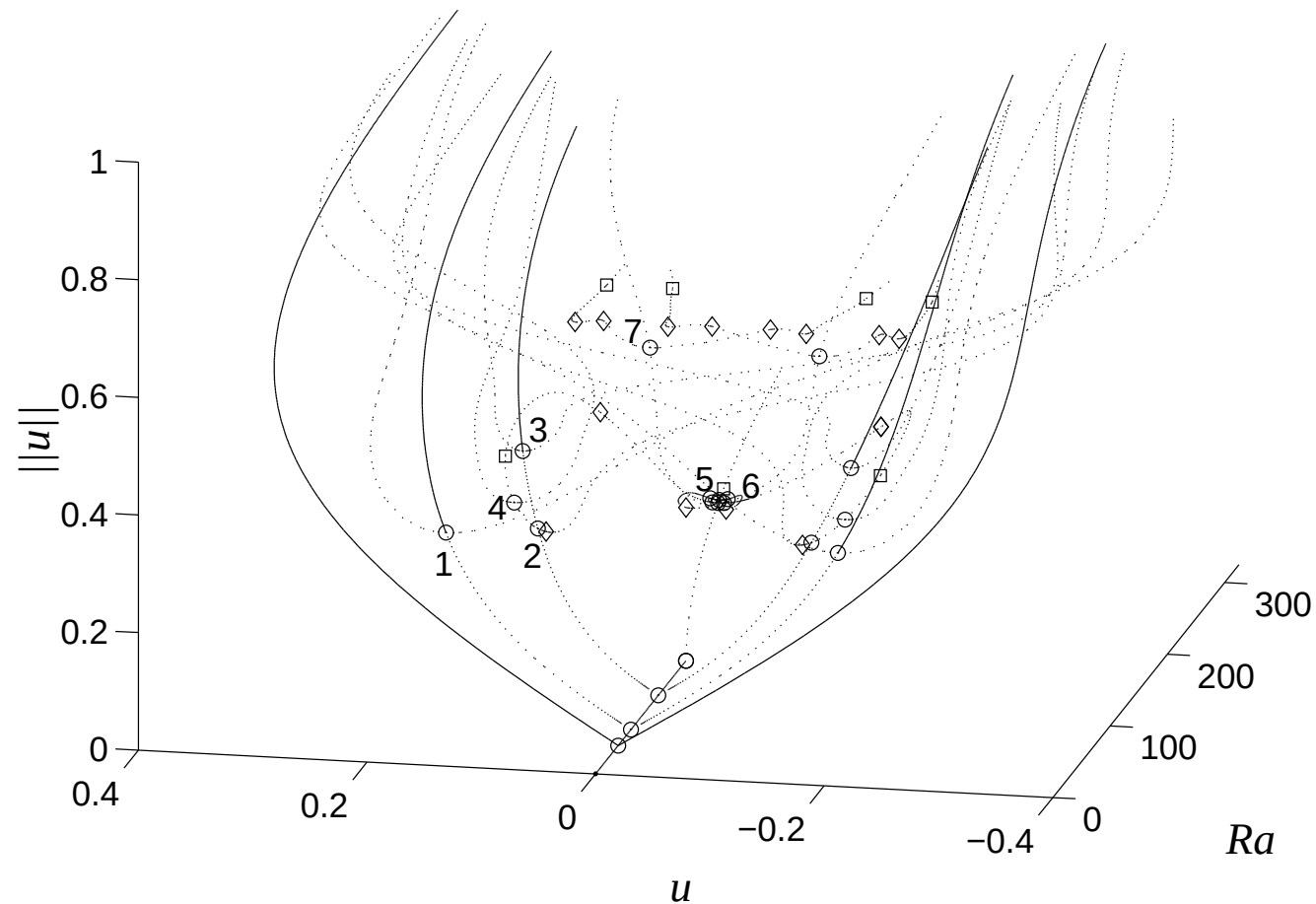
¹Equal to thermal conductivity over the product of density and specific heat capacity.

Steady state solutions: $\mathbf{v} = \mathbf{v}(u)$,

$$f(u, \mu) \equiv -\Delta u + \sqrt{\mu} \mathbf{v}(u) \cdot (\nabla u - \mathbf{e}_2) = 0, \quad \text{in } \Omega$$

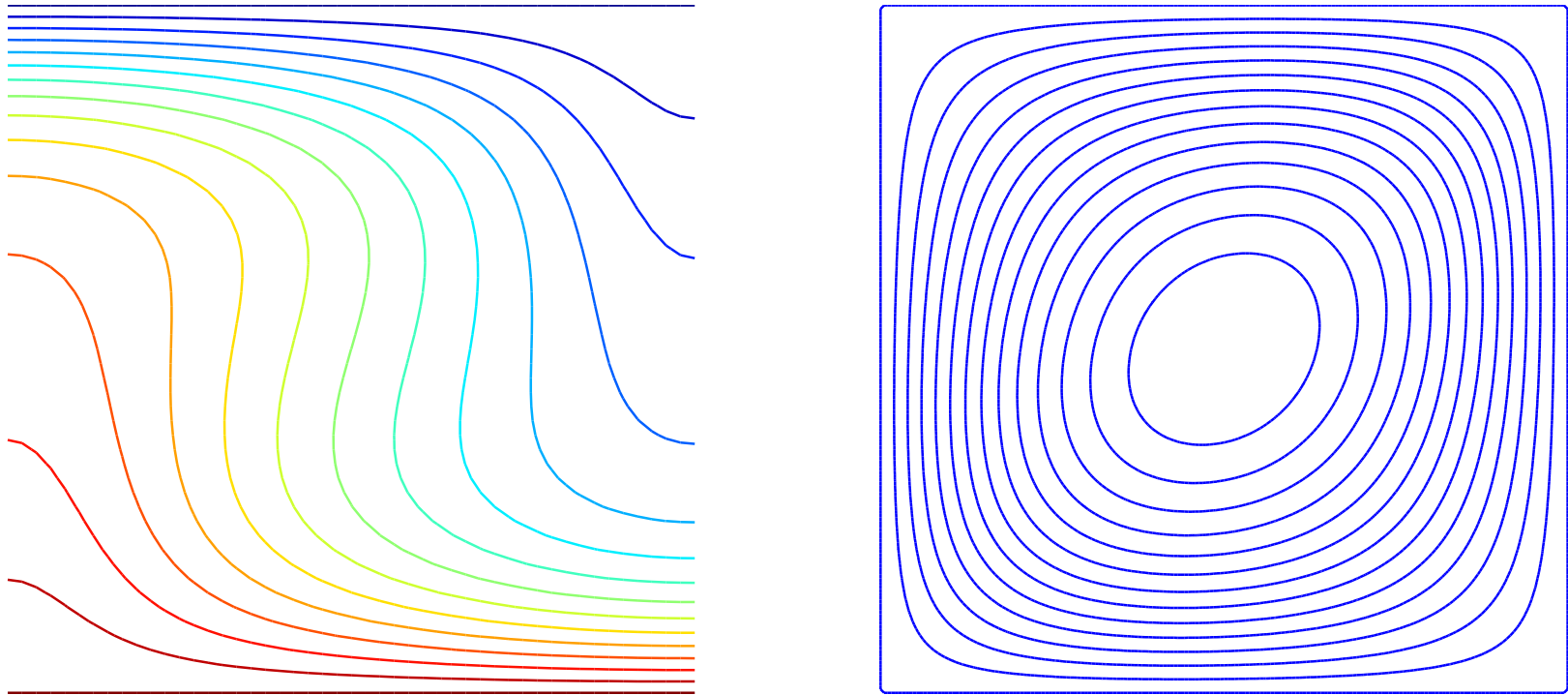
$$\begin{aligned} \nabla u \cdot n &= 0, & x = 0, 1, \quad y \in [0, 1], \\ u &= 0, & x \in [0, 1], \quad y = 0, 1, \end{aligned}$$

EXAMPLE OF A LARGE SYSTEM (Continued)



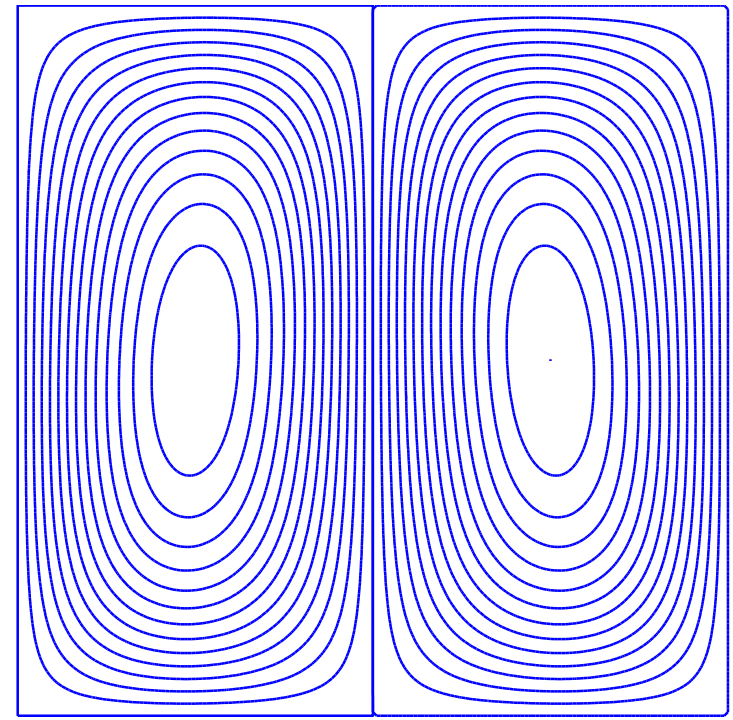
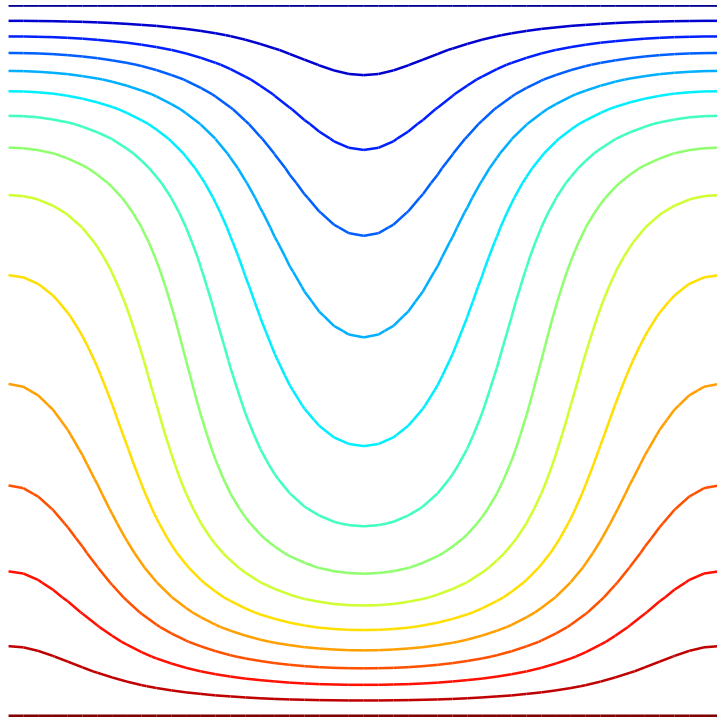
Bifurcation diagram of steady-state solutions.

EXAMPLE OF A LARGE SYSTEM (Continued)



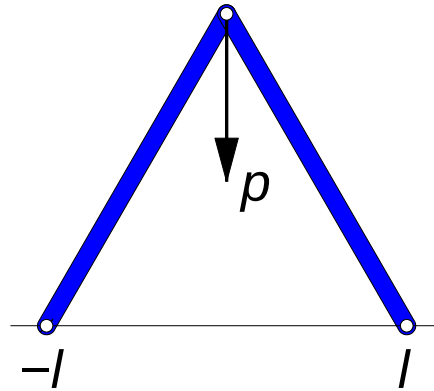
Temperature contour lines (left) and streamlines (right) for $Ra = 100$ on first primary branch.

EXAMPLE OF A LARGE SYSTEM (Continued)



Temperature contour lines (left) and streamlines (right) for $Ra = 100$ on second primary branch.

BUT WE START WITH A SMALL SYSTEM



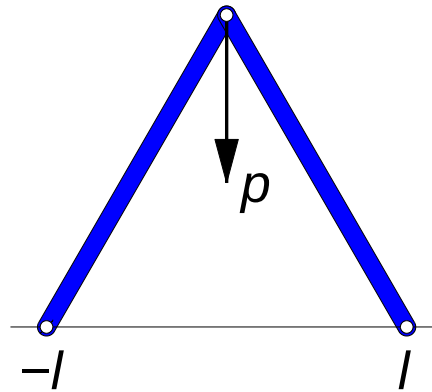
$$kl_0 \left(\frac{x+l}{r_1} + \frac{x-l}{r_2} \right) - 2kx = 0,$$

$$kl_0 \left(\frac{y}{r_1} + \frac{y}{r_2} \right) - 2ky - p = 0,$$

where

$$r_1 = \sqrt{(x+l)^2 + y^2}, \quad y \quad r_2 = \sqrt{(x-l)^2 + y^2},$$

BUT WE START WITH A SMALL SYSTEM



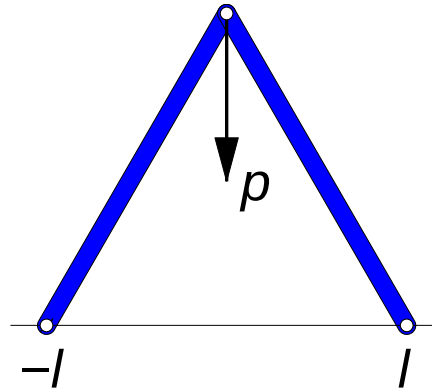
Dimensionless variables and parameters:

$$u_1 = x/l, \quad u_2 = y/l, \quad \lambda = p/(kl), \quad \text{and} \quad \mu = l_0/l$$

where

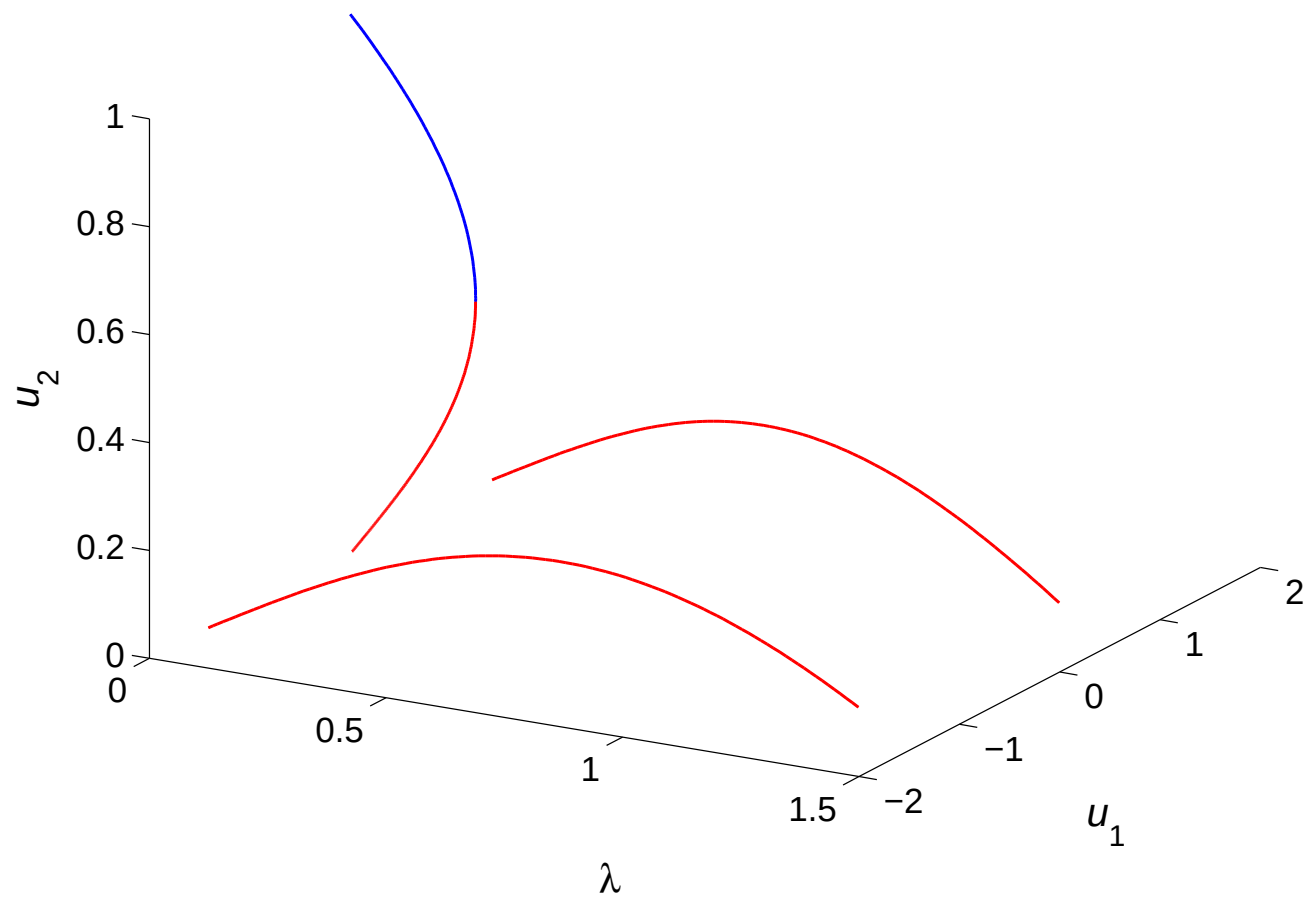
$$f(u, \lambda) = 0,$$
$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

BUT WE START WITH A SMALL SYSTEM



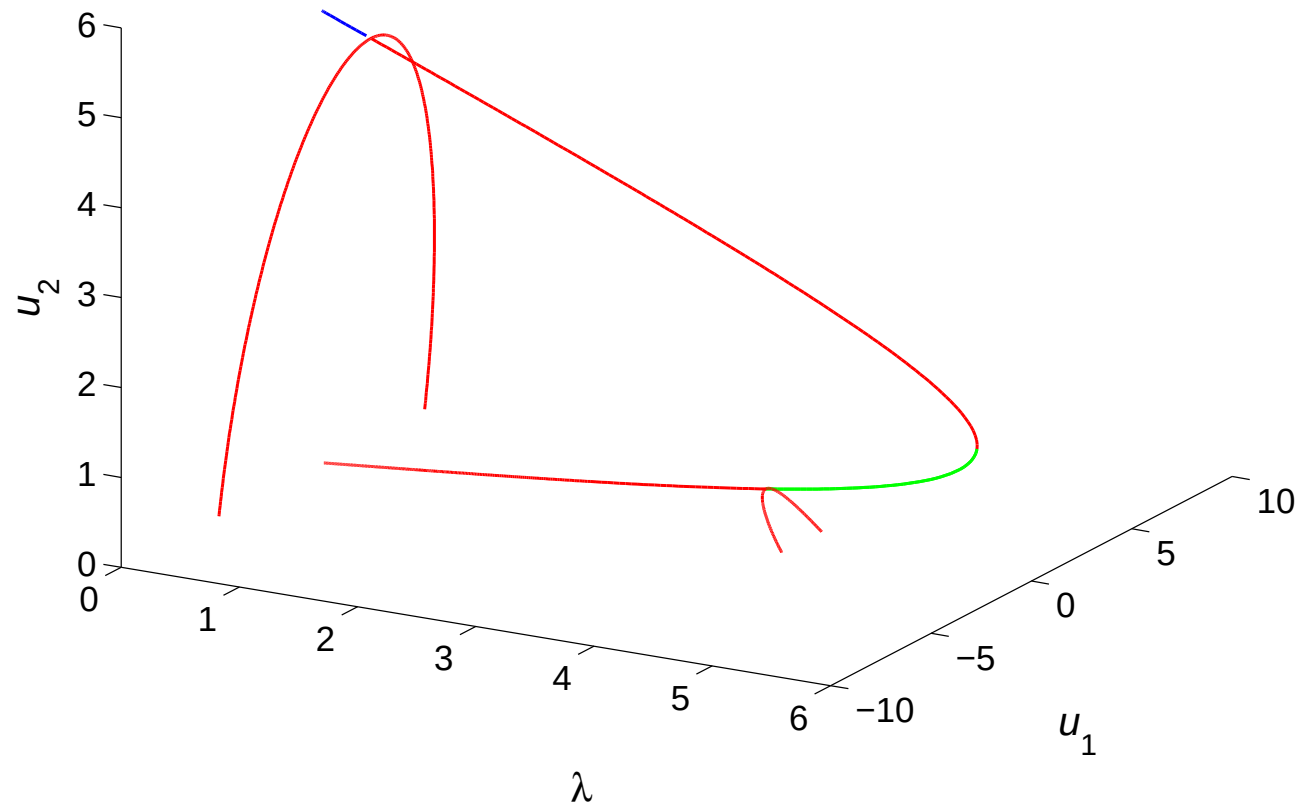
and

$$f(u, \lambda) = g(u, \lambda, \mu) = \begin{bmatrix} \mu \left(\frac{u_1 + 1}{d_1} + \frac{u_1 - 1}{d_2} \right) - 2u_1 \\ \mu \left(\frac{u_2}{d_1} + \frac{u_2}{d_2} \right) - 2u_2 - \lambda \end{bmatrix},$$
$$d_1 = \sqrt{(u_1 + 1)^2 + u_2^2}, \quad \text{and} \quad d_2 = \sqrt{(u_1 - 1)^2 + u_2^2}.$$



Equilibria for $\mu = \sqrt{2}$.

Blue: stable equilibria. Red: unstable equilibria.



Equilibria for $\mu = 1/\cos(7\pi/16)$.
 Blue: stable equilibria. Red: unstable equilibria.