First Lecture

1. Dyn. Systems : exciting and very active area nowadays. Involves tools and techniques from many other areas as analyses, geometry, number theory and has applications in many fields as physics, astronomy, biology, economics

2. Adjective : dynamical refers to the fact that the systems we are interested in is in evolving in time.
Applied dynamics : systems studied could be, for example: box containing molecules of gas in physics, a species population in biology, the financial market in economics, the wind currents in metereology.
Pure mathematics : a dynamical system can be obtained by iterating a function or letting evolve in time the solution of an equation.

Discrete dynamical systems are systems for which the time evolves in discrete units. For example, we could record the number of individuals of a population every year and analyze the growth year by year. The time is parametized by a discrete variable n, $n \in \mathbb{N}$. or $n \in \mathbb{Z}$.
Continuous dynamical system : the time variable changes continuously and it is given a real number $t \in \mathbb{R}$.

Main examples : of discrete dynamics are obtained by iterating a map f, $f : X \to X$, where X is a space. We can think as f as the map which gives the time evolution of the points of X. If $x \in X$, the iterates of x are: $x, f(x), f(f(x)), \dots$

Notation : $f^n(x)$ denotes the n^{th} iterate of f at x, ie,

$$f^n(x) = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}(x)$$

Convention: $f^0 = id_X$

Definition : We denote by $\mathcal{O}_f^+(x)$ the forward orbit of x under f

$$\mathcal{O}_f^+(x) =: \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\} = \{f^n(x),\ n \in \mathbb{N}\}.$$

When f has an inverse, we denote f^{-1} its inverse and write

$$\mathcal{O}_f(x) =: \{f^n(x),\ n \in \mathbb{Z}\} \text{ : the full orbit of } x \text{ under } f.$$

Remark : Even if the rule of evolution is deterministic, The long term behavior of The system is often chaotic, meaning that even if two points x, y are very close, there exists a large n such that $f^n(x)$ and $f^n(y)$ are far apart. This is known as sensitive dependence on initial data. There are various definitions of chaos, but all of them include sensitive initial data. Different branches of dynamical systems, for instance topological and ergodic theory, provide tools to quantify how chaotic is a system and to predict the asymptotic behavior. Often even if one cannot predict the behavior of each single orbit, one can predict the average behavior. Main objetive in dynamical systems : to understand The behavior of all (or almost all) The orbits. Orbits can be fairly complicated even if the map is quite simple.

Rotations of the circle. Let S^1 be a circle of unit radius.

$$S^1 = \{ z \in \mathbb{C} ;\ |z| = 1 \} = \{ e^{2\pi i \theta} ;\ 0 \leq \theta < 1 \}.$$

Let $\pi : \mathbb{R} \to S^1$ defined by $\pi(x) = e^{2\pi i x} = \cos 2\pi x + i \operatorname{sen} 2\pi x$

- π is a local homeomorfism and given $f : S^1 \circlearrowleft$ there is $F : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$. F: lift of f

- π is a cover map of S^1 and induces a continuous homeomorfism $\pi : \underbrace{\mathbb{R}/\mathbb{Z}}_{\text{equivalent classes}} \to S^1$, $\mathbb{R}/\mathbb{Z} = I = [0,1]/\sim \ \underset{\pi}{\equiv} S^1$.
  $S^1 = [0,1]/\sim$, $0 \stackrel{\sim}{\equiv} 1$. $\rightsquigarrow$ "The interval with glued endpoints" equivalent classes : $x + \mathbb{Z}$.
  $x, y \in$ same class $\Leftrightarrow \exists k \in \mathbb{Z};\ x = y + k$.

- distance between points in S^1 :

(a) arclenght distance :

$$d(e^{2\pi i \theta_2}, e^{2\pi i \theta_1}) = \begin{cases} \theta_2 - \theta_1 & \text{if } 0 \leq \theta_1 < \theta_2 \text{ and } \theta_2 - \theta_1 < 2\pi \\ 1 - (\theta_2 - \theta_1) & \text{if } 0 \leq \theta_1 < \theta_2 \text{ and } \theta_2 - \theta_1 \geq 2\pi \end{cases}$$

(a) distance in $S^1 = [0,1]/\sim \ = \mathbb{R}/\mathbb{Z}$:

$$d(x, y) = \min \{ |x - y|, 1 - |x - y| \}.$$

ROTATION (counterclockwise) of angle $2\pi\alpha > 0$:

$$R_\alpha (e^{2\pi i \theta}) = e^{2\pi i (\theta + \alpha)} = e^{2\pi i \theta} \cdot e^{2\pi i \alpha}$$

multiplicative notation : because corresponds to multiplying the complex number $e^{2\pi i \theta}$ by $e^{2\pi i \alpha}$.

Under the identification $S^1 = [0,1]/\sim = \mathbb{R}/\mathbb{Z}$,

the rotation R_α becomes the map

$R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $R_\alpha(x) = x + \alpha \cdot \underline{\underline{\text{mod}}}$

$\downarrow$

α : rotation number of R_α. additive notation

Rotations of the circle display a very different behavior according if the rotation number $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{R}\setminus\mathbb{Q}$

<u>Def</u>. The orbit $\mathcal{O}^+_{R_\alpha}(z)$ is dense if for all $w \in S^1$ and for all $\varepsilon > 0$, $\exists$ $n > 0$ such that $d(R^n_\alpha(z), w) < \omega$.

<u>Theorem</u>. (Dichotomy for Rotations) Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rotation.

(a) if $\alpha = p/q \in \mathbb{Q} \iff$ all orbits are periodic ~~of~~ period q

(b) if $\alpha \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow \mathcal{O}^+_{R_\alpha}(z)$ is dense, $\forall z \in S^1$.

<u>Lemma 1</u>. If z_0 is a periodic point of R_α then all z is periodic of same period as z_0.

<u>Proof</u>. Let (z_0, z) be the oriented arc joing z and z

As R_α is an isommetry,

$$|(z_0, z)| = |R^n_\alpha(z_0, z)| = |(R^n_\alpha(z_0), R^n_\alpha(z))| =$$

$$= |(z_0, R^n_\alpha(z))| \Rightarrow R^n_\alpha(z) = z.$$

Lemma 2. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then R_α does not have periodic points.

Proof In additive notation,

$R_\alpha(x) = x + \alpha$ Assume $\exists\ z \in \mathbb{R}/\mathbb{Z}$;

$R_\alpha^n(z) = z$. Then $R_\alpha^n(z) = z + n\alpha = z + k$, $k \in \mathbb{N}$

$\Rightarrow \alpha = \frac{k}{n} \in \mathbb{Q}$, contradiction. $\blacksquare$

Lemma 3. If $\mathcal{P}(R_\alpha) = \emptyset$ then $\mathcal{O}^*(R_\alpha)(z)$ is dense $\forall z \in S^1 = \mathbb{R}/\mathbb{Z}$.

Proof: Given $z \in S^1$, $\mathcal{O}^*_{R_\alpha}(z) = \{R_\alpha^n(z), n \in \mathbb{Z}\}$.

Since $\mathcal{P}(R_\alpha) = \emptyset$, $R_\alpha^n(z) \neq R_\alpha^m(z)$ for $n \neq m$.

As S^1 is compact, we can assume without loss, that $(R_\alpha^n(z))_{n\in\mathbb{Z}}$ converges. Thus, given $\varepsilon > 0$, $\exists\ n > m$;

$|R_\alpha^n(z) - R_\alpha^m(z)| < \varepsilon$.

Let $k = n - m$. Then

$|R_\alpha^k(z) - z| = |R_\alpha^m(R_\alpha^k(z)) - R_\alpha^m(z)| = |R_\alpha^n(z) - R_\alpha^m(z)| < \varepsilon$

$\Rightarrow |R_\alpha^{2k}(z) - R_\alpha^k(z)| < \varepsilon$ (isometry) $\Rightarrow$

$z, R_\alpha^k(z), R_\alpha^{2k}(z), R_\alpha^{3k}(z), \dots$ split S^1 into points ~~with~~ such that the distance between two consecutive points is less than $\varepsilon \Longrightarrow$

the orbit of z enters the ball $B(\omega, \varepsilon)$ for every $\omega \in S^1$ and so proves the result.

Another proof

Let $\tilde{A} = S^1 \setminus \overline{\mathcal{O}_{R_\alpha}(z)}$. Then $\tilde{A}$ is open and invariant by R_α. Let $A \subset \tilde{A}$ be a connected component of $\tilde{A}$. Then, for all n, $R_\alpha^n(A)$ is also a connected component of $\tilde{A}$. Since by hypothesis $\mathcal{P}(R_\alpha) = \emptyset$, the arcs $\{R_\alpha^n(A) \, ; \, n \in \mathbb{Z}\}$ are two-by-two disjoint. As all of them has the same lenght, ~~th~~ we arrive to a contradiction if $\tilde{A} \neq \emptyset$. Thus, $\tilde{A} = \emptyset$, finishing the proof. $\blacksquare$

Homeomorphisms of S^1.

Def. ~~Dados~~ Given $f, g : S^1 \circlearrowleft$ homeo, f and g are semi-conjugated if there is $h : S^1 \to S^1$ continuous and surjective satisfying $f \circ h = h \circ g$.

Theorem (Poincaré dichotomy for homeomorphisms of S^1)

Let $f : S^1 \to S^1$ be a homeomorphism oriented preserved.

(a) $\mathcal{P}(f) \neq \emptyset \Rightarrow$ any orbit is asymptotic to a periodic orbit and two any periodic orbit have the same period.

(b) $\mathcal{P}(f) = \emptyset \Rightarrow$ f is semi-conjugated to a rotation R_α, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

$f : S^1 \circlearrowleft$ homeo (or map)

<u>Def</u> $\omega(x) = \omega$-limite set of $x = \{ y ;\ \exists\ f^n(x) \underset{n\to\infty}{\longrightarrow} y \}$

$\alpha(x) = \alpha$-limité set of $x = \{ y \in S^1 ;\ \exists\ f^n(x) \underset{n\to-\infty}{\longrightarrow} y \}$

<u>Lemma 1</u>. Let $J \subset \mathbb{R}$, closed interval and $f : J \to J$ ~~homeo~~ invertible, preserves orientation. Then $\mathcal{O}_f(x)$, for every $x \in J$, is asymptotic to a fix point of f.

<u>Proof</u>. if $x \in J$ and $f(x) > x \Rightarrow f^2(x) > f(x)$ because f preserves orientation.

By induction, $f^n(x) = f(f^{n-1}(x)) > f^{n-1}(x) \Rightarrow (f^n(x))_n$ is monotonically encreasing and so

$$\lim_{n\to\infty} f^n(x) = y = \sup_n \{ f^n(x) \} \underset{\text{continuous}}{\overset{f}{\Longrightarrow}} f(y) = f(\lim f^n(x)) = \lim f^{n+1}(x) = y.$$

Thus, $\omega(x) = y$, $f(y) = y$. ∎

Now <u>we prove</u> (a) of Theorem (Poincaré).

Let y be such that $f^k(y) = y$, k: period of y.
Let $g = f^k$. Then y is a fixed point of g.

If f preserves orientation, puting $J = S^1 \setminus \{y\}$, $g/_J$ is increasing and monotonous. (note $g(J) \subset J$!)

By Lemma 1 above, $g^n(z) \to w$, $g(w) = w$, all $z \in S^1$, proving the result.

If f invert orientation, f^2 preserves, and we get $\omega_f(x)$ is either a fixed or a periodic point of period 2. ∎

When $\mathcal{P}(f) \neq \emptyset$:

We introduce the rotation number of f.

1. Given $f : S^1 \to S^1$ homeo, let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f with $F(0) \in (0,1)$. Then $\pi \circ F = f \circ \pi$, where $\pi : \mathbb{R} \to S^1$, $\pi(x) = e^{2\pi i x}$. Observe that F is increasing and ~~$F(x+1) = F(x) + 1 \ \forall x \in \mathbb{R}$~~.

$F(x+1) = F(x) + 1 \Rightarrow F(x+k) = F(x) + k \quad \forall k \in \mathbb{Z}$.

We also have $F(x+1) - (x+1) = F(x) - x \Rightarrow$ $F - id$ is a periodic map of period 1. Similarly $F^n - id$ is periodic of period 1.

Def. Let $f : S^1 \to S^1$ difeomorphism, preserves orien and let F be a lift of f. Define

$$\rho_0(F) = \lim_{n\to\infty} \frac{|F^n(x)|}{n}.$$

Note: if the limite above exists, it does not depend on x:

$$|F^n(x) - F^n(y)| \leq |F^n(x) - x + F^n(y) - y + x + y| \leq$$

$$\leq |(F^n(x) - x) - (F^n(y) - y)| + |x-y| \leq 1 + |x-y| \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} \frac{|F^n(x) - F^n(y)|}{n} = 0.$$

Remark: $|x-y| < 1$, F = lift of $f \Rightarrow$

$$|F^n(x) - F^n(y)| < 1.$$

Def (Rotation number of f) $f : S^1 \to S^1$, $F : \mathbb{R} \to \mathbb{R}$ lift.

$\rho(f) = [\rho_0(F)]$ = fractionary part of $\rho_0(F)$.

Teorem. $f: S^1 \to S^1$ difeo, preserve orientation and $F: \mathbb{R} \to \mathbb{R}$ a lift of f. Then $\rho_0(F) = \lim\limits_{n\to\infty} \dfrac{|F^n(x)|}{n}$ exists and does not depend on x neither on F.

Conclusion: $\rho(f)$ is well defined.

Lemma. $f: S^1 \to S^1$ homeo, pres. or, F lift, $F(0) \in (0,1)$.
$A = \{n\rho + m;\ n, m \in \mathbb{Z}\}$ and $B = \{F^n(0) + m;\ n, m \in \mathbb{Z}\}$
$\rho = \rho(f) \in \mathbb{R} \setminus \mathbb{Q}$.
Let $h_0: A \to B$, $h_0(n\rho + m) = F^n(0) + m$. Then h_0 is a bijection monotonous.

Proof. It is enough to prove that
~~(a) Claim 1~~. $n\rho + m < k\rho + l \iff F^n(0) + m < F^k(0) + l$.

1st Case: $p, q, r \in \mathbb{Z}$, $p < F^q(0) < r \Rightarrow p < q\rho < r$.

Indeded:

$$p < F^q(0) < r \Rightarrow F^q(0) + p = F^q(p) < F^{2q}(0) < F^q(r) = F^q(0) + r$$

$$\Downarrow$$

$$F^q(0) + p < F^{2q}(0) < F^q(0) + r$$

$$\Downarrow \text{ repeating:}$$

$$F^q(0) + (k-1)p < F^{kq}(0) < F^q(0) + (k-1)r \Rightarrow$$

$$\frac{1}{k} F^q(0) + \left(1 - \frac{1}{k}\right)p < q\,\frac{F^{kq}(0)}{qk} < \frac{1}{k} F^q(0) + \left(1 - \frac{1}{k}\right)r$$

$$\Rightarrow p \le q \lim \frac{F^{kq}(0)}{qk} \le r \Rightarrow \left.\begin{array}{l} p \le q\rho \le r \\ \rho \in \mathbb{R}\setminus\mathbb{Q} \end{array}\right\} \Rightarrow p < q\rho < r.$$

Now we prove $F^n(0)+m < F^k(0)+l \Longleftrightarrow n\rho+m<k\rho+l$.

$F^n(0)+m < F^k(0)+l \Leftrightarrow F^{n-k}(0) < l-m \underset{\substack{\text{Claim 1}\\ \text{1st case}}}{\Longleftrightarrow}$

$(n-k)\rho < l-m \Leftrightarrow n\rho+m < k\rho+l$.

Teorem (Poincaré). $f: S^1 \to S^1$ homeo, p.or., $\underbrace{\rho(f)}_{=\rho} \in \mathbb{R}\setminus\mathbb{Q}$
Then $f \overset{\text{semi-conjugated}}{\approx} R_\rho$.

Proof. Let $r: A \to A$, $r(y) = y+\rho$ and
$h_0 : A \to B$, $h_0(n\rho+m) = F^n(0)+m$.

$A = \{n\rho+m;\ n, m \in \mathbb{Z}\}$ $\qquad B = \{F^n(0)+m;\ n, m\in\mathbb{Z}\}$.

We have:

(a) $h_0(x+1) = h_0(x)+1 \quad \forall\, x \in A$.

(b) $h_0(r(y)) = h_0(y+\rho) = h_0(n\rho+m+\rho) = h_0((n+1)\rho+m) =$
$= F^{n+1}(0)+m = F(F^n(0))+m = F(F^m(0)+m) =$
$= F(h_0(y))$.

(c) $\rho\in\mathbb{R}\setminus\mathbb{Q} \Rightarrow A$ is dense $\Rightarrow \exists$ extension $\tilde{h}: \mathbb{R}\to\mathbb{R}$
of h_0, $\tilde{h}(x_0) = \lim\limits_{\substack{x\to x_0\\ x\in A}} h_0(x)$.

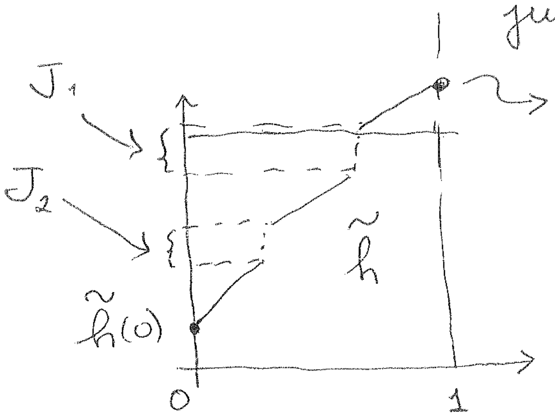
$\tilde{h}$: well defined because h_0 is monotonous.

Since $\tilde{h}$ is periodic with period 1 at x, that is, $\tilde{h}(x+1) = \tilde{h}(x) + 1$, it is $\tilde{\tilde{h}} : S^1 \to S^1$ such that $\tilde{h}$ is ~~the~~ a lift of $\tilde{\tilde{h}}$.

As $\pi(y) = y + \rho$ is a lift of $R_{2\pi\rho}$, we have that $h_0 \circ \pi = f \circ \pi \Rightarrow \tilde{\tilde{h}} \circ R_{2\pi\rho} = F \circ \tilde{\tilde{h}}$.

We have: $\tilde{h}(x+1) = \tilde{h}(x) + 1$.

$\tilde{h}$ monotonous $\Rightarrow \#$ discontinuity points is ~~dense~~ enumerable and discontinuity of jump type.



$\tilde{h}(1) = \tilde{h}(0) + 1$

Then we can define $\tilde{h}^{-1}$, ~~setting~~ puting $\tilde{h}^{-1}(J_i) = \text{constant}$, for each J_i corresponding to a jump of $\tilde{h}$. We can define $\tilde{h}^{-1}$ in such away that it is continuous and holds:

$$\pi \circ \tilde{h}^{-1} = \tilde{h}^{-1} \circ F.$$

Take $h : S^1 \to S^1$ such $\tilde{h}^{-1}$ is a lift of h.

Then h is continuous and it holds

$$R_\rho \circ h = h \circ f.$$

$\boxtimes$

Comments.

Note that h defined above is 1-1 or colapses intervals to points. If J is one of these intervals, then the iterates $f^j(J)$ behave as $R_\rho^j(h(J))$. These intervals are called "wandering intervals" and it holds:

$$f^j(J) \cap J = \emptyset \quad \forall j \neq 0.$$

Poincaré also formulated the question of existence of wandering intervals for diffeomorphisms of S^1.

In 1931, Denjoy proved that C^2-diffeos of S^1 with irrational rotation number don't exhibit wandering intervals.

In particular, if $f: S^1 \to S^1$ is C^2 ~~while~~ without periodic points then f is conjugated to an irrational rotation.

Denjoy also exihbited a C^1 diffeo of S^1 with irrational rotation number presenting a wandering interval.

The construction goes as follows.

To each $n \in \mathbb{Z}$, let $l_n \in \mathbb{R}^+$ such that

$$\sum_{-\infty}^{\infty} l_n = 1 \quad \text{and} \quad \frac{l_{n+1}}{l_n} \xrightarrow[|n| \to \infty]{} 1$$

Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and consider the rotation R_α.
Pick a point $x \in S^1$ and consider its orbit

$$\mathcal{O}_{R_\alpha}(x) = \{ x_n \, , \; x_n = R_\alpha^n(x) \, , \; n \in \mathbb{Z} \}.$$

There is a family of open intervals $\{ I_n \, ; \; n \in \mathbb{Z} \}$;
$|I_n| = l_n$, $I_j \cap I_k = \phi$ if $j \neq k$ and
they are in the same order that of x_n.

Let $A = \bigcup_{n \in \mathbb{Z}} I_n$. Then A is an open set,
$|A| = \sum_{n \in \mathbb{Z}} l_n = 1$. This implies that A is also dense in S^1.

Since $\frac{l_{n+1}}{l_n} \xrightarrow[|n| \to \infty]{} 1$, $\exists$ a C^∞ diffeo $f_n : I_n \to I_{n+1}$

such that $f_n'(a_n) = f_n'(b_n) = 1$, where $I_n = (a_n, b_n)$

and $\max_{x \in I_n} \{ f_n'(x) \} \to 1$, $\min_{x \in I_n} \{ f_n'(x) \} \to 1$.

Now define $f : A \to A$ by $f(x) = f_n(x)$ for $x \in I_n$

and $h : A \to S^1$, $h(I_n) = x_n$.

We have: $h \circ f = R_\alpha \circ f$ by construction

h and f are monotonous.

Since A and $\{ x_n \, ; \; n \in \mathbb{Z} \}$ are dense in S^1,
both h and f extend continuously to S^1.

By constuction, the extension of h is a semi-conjugacy between $\underbrace{f}$ and R_α

$\downarrow$

The extension of f to S^1

We are left to prove that f is C^1.
To do so, it is enough to prove that f is diferentiable at each $y \in S^1 \setminus A$. and that $f'(y) = 1$.
Let then $[y, z]$ the smallest arc joing y to z in S^1.
To prove that f is differentiable at y and that its derivative at y is equal to 1, we have to prove that

$$(*) \quad \lim_{z \to y} \frac{|f([y,z])|}{|[y,z]|} = 1 = \lim_{z \to y} \frac{|f(y) - f(z)|}{|y - z|}$$

Remark: f monotone $\Rightarrow |f[y,z]| = |f(y) - f(z)|$

Since $f_n'(a_n) = f_n'(b_n) = 1$ (the derivative is 1 at the extremum points of I_n), we have $(*)$ holds always when $(y,z) \subset I_n$ for some n (note that in this case, y is at the boundary of I_n, and so, $y = a_n$ or $y = b_n$).

Assume then (y,z) is not contained at I_n, $\forall n$.

The size of $f([y,z]) = |f(y) - f(z)|$ can be estimated adding the size of all $I_n \subset f([y,z])$ plus the size of one I_n that contains z if such interval exists.

Analogously for the size of $[y,z]$.

But since $I_n \subset [y,z] \iff I_{n+1} \subset f([y,z])$,

and by hypothesis $\frac{l_{n+1}}{l_n} \to 1$, we obtain $\circledast$ ▨

Denjoy's Theorem

$f : S^1 \to S^1$, diffeo C^2, $Per(f) = \emptyset$.
Then f does not have wandering intervals.

Consequence : f is conjugate to an irrational rotation.

Proof. The proof goes by contradiction. Assume the existence of wandering intervals and let J be a maximal wandering interval for f.
Recall : $f^n(J) \cap f^m(J) = \emptyset$ for $n \neq m$.

Claim 1 Let $x_0, y_0 \in J$. Then $\exists$ $k > 0$ such that $\frac{1}{k} \leq \left| \frac{Df^n(x_0)}{Df^n(y_0)} \right| \leq k$

Proof of the claim: Given $x_0, y_0 \in J$, we have

$$\log \frac{|Df^n(x_0)|}{|Df^n(y_0)|} = \log \prod_{i=0}^{n-1} \frac{|f'(x_i)|}{|f'(y_i)|} \leq$$

$$\leq \sum_{i=0}^{n-1} \log|f'(x_i)| - \log|f'(y_i)| \leq$$

$$\leq \sum_{i=0}^{n-1} \frac{|f''(z_i)|\,|x_i - y_i|}{|f'(z_i)|}, \qquad z_i \in (x_i, y_i), \quad x_i = f^i(x_0), \quad y_i = f^i(y_0)$$

As f is a diffeomorphism, $f'(z_i) \neq 0$ $\forall i$ and as f is C^2 and S^1 is compact, $\exists$ $C_0 > 0$ such that

$$\frac{|f''(z)|}{|f'(z)|} < C_0 \quad \forall z \in S^1.$$

This implies $\log\left|\frac{Df^n(x_0)}{Df^n(y_0)}\right| \leq C_0 \sum_{i=0}^{n-1} |x_i - y_i| \leq$

$$\leq C_0 \sum_{i=0}^{\infty} |f^i(J)| < C_0$$

because $J, f(J), \ldots, f^n(J), \ldots$ are pairwise disjoint and so $\sum_{i=0}^{n-1} |x_i - y_i| \leq \sum_{i=0}^{\infty} |f^i(J)| < 1$.

Thus, $\log\left|\frac{Df^n(x_0)}{Df^n(y_0)}\right| \leq C_0 \Rightarrow$

$$\frac{1}{k} = e^{-C_0} \leq \left|\frac{Df^n(x_0)}{Df^n(y_0)}\right| \leq e^{C_0} = k, \quad \blacksquare$$

By continuity, Claim 1 holds for x_0 a extremum of J.

Claim 2 For all $x_0 \in J$, $\sum_{n=0}^{\infty} |Df^n(x_0)| < \infty$.

Proof: First note that claim 1 implies

$$|f^n(J)| = |Df^n(z)|\,|J| \geq \frac{1}{k} |Df^n(x_0)| \quad \forall n.$$

Then $\sum_{n=0}^{\infty} |Df^n(x_0)| \leq k_2 \sum_{n=0}^{\infty} |f^n(J)| < k$, proving the claim 2 ▨

Claim 3. $\exists\ \lambda > 1$ and $\delta > 0$ such that

$$|y - x_0| < \delta \implies |Df^n(y)| \leq \lambda |Df^n(x_0)| \quad \forall n.$$

Proof Since f is C^2, f' is continuous and so $\exists$ $\lambda > 1$ and $\delta_0 > 0$ such that

$$|Df(y)| \leq \lambda |Df(x_0)|.$$

Let δ_1 be so that $e^{C_0 \delta_1 \lambda A} < \lambda$ where $A = \sum_{n=0}^{\infty} |Df^n(x_0)|$ and let $\boxed{\delta = \min\{\delta_0, \delta_1\}.}$

Assume, by induction, that

$$|y - x_0| < \delta \implies |Df^j(y)| \leq \lambda |Df^j(x_0)| \text{ for } j = 1, \dots, n-1.$$

Then, $\log \left| \frac{Df^n(y)}{Df^n(x_0)} \right| \leq \sum_{i=0}^{n-1} C_0 |f^i(y) - f^i(x_0)| =$

$$\underset{z_i \in (y, x_0)}{=} C_0 \sum_{i=0}^{n-1} |Df^i(z_i)| |y - x_0| = C_0 |y - x_0| \sum_{i=0}^{n-1} |Df^i(z_i)| \underset{\text{induction hypothesis}}{\leq}$$

$$\leq C_0 |y - x_0| \lambda \sum |Df^i(x_0)| =$$

$$= C_0 \lambda A |y - x_0| < C_0 \lambda A \delta \implies$$

$$\frac{|Df^n(y)|}{|Df^n(x_0)|} < e^{C_0 \lambda A \delta} < \lambda$$ and we finish the proof.

Finally we prove Denjoy's Theorem.

Let J be a maximal wandering interval.

Let I be an adjacent interval to J satisfying

$\dfrac{|I|}{|J|} = \varepsilon$, $\varepsilon > 0$, small, and $|I| < \delta$, where

δ is as in Claim 3.

Let n big enough such that $f^n(J) \subset I$ (continuity)

~~Let $x_0 = I \cap J$.~~

~~Claim 1 $\Rightarrow$ $|Df^n(z)| \leq k\,|Df^n(x_0)|$~~

We have $\dfrac{|f^n(I)|}{|f^n(J)|} = \dfrac{|Df^n(z)|\,|I|}{|Df^n(y)|\,|J|}$, $z \in I$, $y \in J$

Let $x_0 = I \cap J$. Claim 1 $\Rightarrow \begin{cases} |Df^n(z)| \leq k\,|Df^n(x_0)| \\ \text{and} \\ |Df^n(y)| \geq \frac{1}{k}\,|Df^n(x_0)| \end{cases}$

Thus, $|f^n(I)| \leq \tilde{K}\,|f^n(J)|$.

Now, we can take n arbitrarily big so that $|f^n(J)|$ is arbitrarily small and so that

$$\underbrace{f^n(J \cup I) \subset}_{\downarrow \; J \text{ is maximal wandering}} I \subset J \cup I \Rightarrow \exists \text{ periodic point, contradiction}$$

$\blacksquare$

Minimal sets

Let $f: S^1 \to S^1$ be a diffeomorphism.

$K \subset S^1$ is a minimal set for f if

(a) K is closed, $K \neq \emptyset$

(b) $f(K) \subset K$

(c) if $J \subset K$ satisfy (a) and (b) then $J = K$.

In another words, K does not contain proper subsets satisfying (a) and (b).

If K is a periodic orbit then K is a trivial minimal set.

Def An orbit γ of f is recurrent if $\gamma \subset \overline{\gamma}$.

Note: $\left.\begin{array}{l} K \text{ minimal} \\ \gamma \text{ orbit} \subset K \end{array}\right\} \Rightarrow \gamma$ is recurrent.

Indeed, ω-limit set of γ, $\omega(\gamma)$, is closed, $\omega(\gamma) \neq \emptyset$, $f(\omega(\gamma)) \subset \omega(\gamma)$ and $\omega(\gamma) \subset K$.

Then $\omega(\gamma) = K \supset \gamma$.

Similarly we prove $\omega(\gamma) \supset \gamma$.

Consequence of Denjoy's Theorem

$f: S^1 \to S^1$ diffeo C^2 and K a minimal set for f.

Then K is a periodic orbit or $K = S^1$.

Proof: If $Per(f) = \emptyset \Rightarrow f$ conjugated to an irrational rotation $\Rightarrow K = S^1$

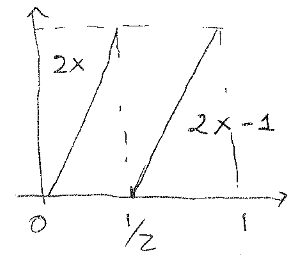
If $\mathcal{P}(f) \neq \emptyset \Rightarrow K$ is a periodic orbit because otherwise, $\exists\ \gamma \subset K$, γ non periodic. But by a Lemma proved before, γ is asymptotic to a periodic orbit $\Gamma \underset{K \text{ is closed}}{\Longrightarrow} \Gamma \subset K \underset{K \text{ minimal}}{\Longrightarrow} K = \Gamma$.

∎

Second Lecture

The doubling map: is defined as:

$$f(x) = 2x \mod 1 = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2x-1 & 1/2 \leq x < 1 \end{cases}$$



The map is also well defined on $\mathbb{R}/\mathbb{Z} = [0,1]/\sim$ because $f(0) = f(1) = 0$.

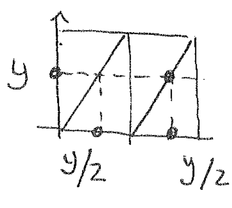
So, we can think of f as a map on $\mathbb{R}/\mathbb{Z} = [0,1]/\sim = S^1$.

Since we saw that $\mathbb{R}/\mathbb{Z}$ is identificated with S^1, via the map $x \longmapsto e^{2\pi i x}$, we can see f in multiplicative coordinates as a map from $S^1 \to S^1$ given by

$$f(e^{2\pi i x}) = e^{2\pi i 2x} = \left(e^{2\pi i x}\right)^2.$$

Thus, the angles are doubled and this explains the name "doubling map".

- f is continuous and not invertible: each y has two pre-images:



$$f^{-1}(y) = \{ y/2,\ y/2 + 1/2 \}.$$

- f expand distances: if $d(x,y) < 1/4 \Rightarrow d(f(x), f(y)) = 2\, d(x,y)$.

Goal: answer the questions:

Questions:

- Are there periodic points?
- Are there points with a dense orbit?

To answer these questions, let us introduce a tool: conjugacy and coding

Conjugacy: Let X, Y be two topological spaces, $f: X \circlearrowleft$, $g: Y \circlearrowleft$ be two maps.

Def A conjugacy between f and g is an invertible map

$h : Y \to X$ so that $h \circ g = f \circ h$.

$$\begin{tikzcd} X \arrow[r, "f"] & X \\ Y \arrow[u, "h"] \arrow[r, "g"] \arrow[ur, phantom, "\circlearrowleft"] & Y \arrow[u, "h"'] \end{tikzcd}$$

If h is only surjective, is a semi-conjugacy.

<u>Lemma 1</u> $f \overset{h}{\sim} g$ $\sim$: conjugated or semi-conjugated.

Then y is a periodic point of period n for g if and only if $h(y)$ is a periodic point of period n for f.

~~Proof~~ $\cancel{g^n(y) = y \Rightarrow f^n(h(y)) = h(y)}$.

~~The proof goes by induction:~~

~~Assume~~ $\cancel{g(y) = y}$. ~~Then,~~ $\cancel{\underbrace{h(g(y))}_{h(y)} = f(h(y)) \Rightarrow \text{OK}}$.

~~Assume OK for~~ $\cancel{n-1}$ ~~and let~~ $\cancel{y;\ g^n(y) = y}$.

$\cancel{f^n(h(y)) = f(h(h^{n-1}(y))) = h\,g(h^{n-1}(y))}$

$\cancel{= f(f^{n-1}(h(y))) =}$

<u>Claim</u>: $f \overset{h}{\sim} g \Rightarrow f^n \overset{h}{\sim} g^n$

Proof: by induction; assume $\to$ $h \circ g^{n-1} = f^{n-1} \circ h$. Then,

$$h \circ g^n = h \circ g^{n-1} \circ g = f^{n-1} \circ h \circ g = f^{n-1} \circ f \circ h = f^n \circ h.$$

$\square$

Now assume $g^n(y) = y$. Then, $f^n(h(y)) = f^n \circ h(y) \underset{\text{Claim}}{=} h \circ g^n(y) = h(y)$. $\boxtimes$

Thus, many times in dynamics, ~~we~~ to understand the behavior of a map, we first conjugate the map with another one that we know better!

We will define a semi-conjugacy between the doubling map and ~~an~~ ~~abstract~~ map defined on an abstract space that will help us understand periodic points ~~and~~ and dense orbits.

To do so, let us recall the binary expansion of $x \in [0,1]$.
Given $x \in [0,1]$, we can write $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$, $x_i \in \{0,1\}$

x_i: digits of the binary expansion of x.

Note that if $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$ then

$$f(x) = 2x \mod 1 = \sum_{i=1}^{\infty} 2 \cdot \frac{x_i}{2^i} \mod 1 =$$

$$= x_0 + \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}} \mod 1 = \sum_{i=2}^{\infty} \frac{x_i}{2^{i-1}} \overset{j=i-1}{=}$$

$$= \sum_{j=1}^{\infty} \frac{x_{j+1}}{2^j} \Rightarrow \text{the digits of}$$

the binary expansion of $f(x)$ are $\boxed{\text{shifted by } 1}$

Let us construct a map on the space of digits of binary expansion which mimic this behavior.

$$\Sigma_2^+ = \{0,1\}^{\mathbb{N}} = \text{set of sequences of digits 0 and 1}$$
$$= \{(a_i)_{i=1}^{\infty} ; a_i \in \{0,1\}\}.$$

~~The shift map:~~ $\cancel{\sigma : \Sigma_2^+ \to \Sigma_2^+}$

Define: given $s, w \in \Sigma_2^+$,

$$d(s,w) = \sum_{i=1}^{\infty} \frac{|s_i - w_i|}{2^i}$$

Exercice : d is a metric.

<u>Definition</u> : The shift map $\sigma : \sum_2^+ \to \sum_2^+$:

$$(a_i)_{i=1}^{\infty} \overset{\sigma}{\longmapsto} (b_i)_{i=1}^{\infty} \; ; \quad b_i = a_{i+1} .$$

OBS • σ is not invertible : $\sigma(a_1 a_2 a_3 \cdots) = \sigma(b_1 a_2 a_3 \cdots)$
$a_1 \neq b_1$.

• $\sigma(a_i)_{i=1}^{\infty}$ is obtained from $(a_i)_{i=1}^{\infty}$ by dropping the first digit a_1 and by shifting all the other digits one place to the left.

• if we know $\sigma(a_i)_{i=1}^{\infty}$ we can not recover $(a_i)_{i=1}^{\infty}$ because we lost the information about the first digit a_1.

Let us define $h : \sum_2^+ \longrightarrow [0,1]$

$$(a_i)_{i=1}^{\infty} \longmapsto \sum_{i=1}^{\infty} \frac{a_i}{2^i} \in [0,1].$$

Properties of h :

• well defined because $\displaystyle\sum_{i=1}^{\infty} \frac{a_i}{2^i} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty$.

• surjective : because all number has a binary expansion.

• not injective ; $\exists$ numbers with two binary expansions for instance: $\frac{1}{2} = \frac{1}{2} + \sum_{i=2}^{\infty} \frac{0}{2^i}$

$$\frac{1}{2} = \sum_{i=2}^{\infty} \frac{1}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} - 1 - \frac{1}{2} = 2 - 1 - 1/2 = \frac{1}{2}$$

$$\boxed{\text{PG}: \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}$$

<u>Proposition</u> $h(a_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$ is a semi-conjugacy between σ and the double map f.

$$\begin{tikzcd} \sum_2^+ \arrow[r, "\sigma"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & \sum_2^+ \arrow[d, "h"] \\ {[0,1]} \arrow[r, "f"] & {[0,1]} \end{tikzcd} \qquad h\circ\sigma = f\circ h.$$

<u>Proof</u>. (1) h is surjective.
Given $x \in [0,1]$, x has at least one binary expansion.
$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$, $a_i \in \{0,1\}$. Thus, the sequence

$(a_i)_{i=1}^{\infty} \in \sum_2^+$ is so that $h(a_i)_{i=1}^{\infty} = x$, by definition.

(2) $h\circ\sigma = f\circ h$

Given $(a_i)_{i=1}^{\infty} \in \sum_2^+$, $\sigma(a_i)_{i=1}^{\infty} = (b_i)_{i=1}^{\infty} = (a_{i+1})_{i=1}^{\infty}$.

Thus, $h(\sigma(a_i)_{i=1}^{\infty}) = h(a_{i+1})_{i=1}^{\infty} = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}$ (*)

Now, let us compute $f \circ h (a_i)_{i=1}^{\infty}$:

$$f\left(h(a_i)_{i=1}^{\infty}\right) = f\left(\sum_{i=1}^{\infty} \frac{a_i}{2^i}\right) = \sum_{i=1}^{\infty} \frac{2a_i}{2^i} = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}$$

$(**)$

And $(*) \equiv (**) \Longrightarrow$ implies the result. $\blacksquare$

Properties of $\sigma : \Sigma_2^+ \longrightarrow \Sigma_2^+$.

(1) σ is continuous

<u>Note</u> ~~that~~ The Ball of center $(a_i)_{i=1}^{\infty}$ and radius $\varepsilon > 0$:

given $\varepsilon > 0$, let $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \varepsilon$.

$(b_i)_{i=1}^{\infty} \in B_{\varepsilon}(a_i)_{i=1}^{\infty} \Longleftrightarrow a_i = b_i$ for $1 \leq i \leq N$.

Now we prove that σ is continuous.
For this, let $\varepsilon > 0$ and $(a_i)_{i=1}^{\infty} \in \Sigma_2^+$.

Let $N \in \mathbb{N}$; $\frac{1}{2^N} < \varepsilon$ and $\delta = \frac{1}{2^{N+2}}$

Then, if $d(a,b) < \delta = \frac{1}{2^{N+2}} \Longrightarrow \underbrace{a_i = b_i, \ 1 \leq i \leq N+2}_{*}$

$\Longrightarrow d(\sigma(a), \sigma(b)) \leq \frac{1}{2^N} < \varepsilon$ because

$* \Longrightarrow a_i = b_i$ for $2 \leq i \leq N+1$.

2. $\# \mathcal{P}_n(\sigma) = 2^n$

In fact, periodic points of period n for σ are all sequences whose digits repeat periodically with period n. Thus, there are 2^n such sequences, since there are 2^n distint blocks of lenght n.

3. $\overline{\mathcal{P}(\sigma)} = \Sigma_2^+$ (density of periodic points).

Given $(a_i)_{i=1}^{\infty} \in \Sigma_2^+$ and $\varepsilon > 0$, fix N; $\frac{1}{2^N} < \varepsilon$.

Then $b = \underbrace{a_1 a_2 \cdots a_N}\ \underbrace{a_1 a_2 \cdots a_N} \cdots$ is so that

$d(a, b) < \varepsilon$ and b is periodic.

4. $\exists$ dense orbit

given $\varepsilon > 0$.

$S^* = \underbrace{\quad}_{\substack{\downarrow \\ \text{possible} \\ 1\text{-blocks}}}\ \underbrace{\quad}\ \cdots\ \underbrace{\quad}_{\substack{\downarrow \\ \text{possible } n\text{-blocks}}}\ \underbrace{\quad}\ - - - -$

Given $(a_i)_{i=1}^{\infty}$, $\exists\, n$ with $\frac{1}{2^n} < \varepsilon$, $\sigma^n(S^*)_i = a_i$, $1 \leq i \leq n$

$$\Rightarrow d(a, \sigma^n(S^*)) < \frac{1}{2^n} < \varepsilon .$$

<u>Theorem</u> The doubling map $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ has $2^n - 1$ periodic points of period n.

Proof: since f is conjugated to σ, $\# \mathcal{P}_n(f) \leq \# \mathcal{P}_n(\sigma)$.

But since the sequences $(0,0,0,\ldots)$ is mapped to 0 and the sequence $(1,1,1,\ldots)$ is " to 1 and $0 \equiv 1$ are the same points in $\mathbb{R}/\mathbb{Z}$, $\Rightarrow$

$\Rightarrow$ there are $2^n - 1$ periodic pts of period n.

Example. Periodic points of period 3 for σ are the periodic sequences obtained repeating the block of digits:

$$000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \quad 110 \quad 111$$

The corresponding binary expansions: recall: $a_{i+3} = a_i$

$$\sum_{i=1}^{\infty} \frac{a_i}{2^i} = \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3}\right) + \left(\frac{a_1}{2^4} + \frac{a_2}{2^5} + \frac{a_3}{2^6}\right) + \dots =$$

$$= \sum_{j=0}^{\infty} \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3}\right) \cdot \left(\frac{1}{2^3}\right)^j$$

For instance, starting from the sequence obtained repeating the block 101 we get

$$\sum_{j=0}^{\infty} \left(\frac{1}{2} + \frac{0}{4} + \frac{1}{8}\right)\left(\frac{1}{8}\right)^j = \frac{5}{8} \sum_{j=0}^{\infty} \frac{1}{8^j} = \frac{5}{8}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{5}{7}$$

Thus we find ~~that~~ the $7 = 2^3 - 1$ periodic points for f:

$$0, \ \frac{1}{7}, \ \frac{2}{7}, \ \frac{3}{7}, \ \frac{4}{7}, \ \frac{5}{7}, \ \frac{6}{7}. \qquad \blacksquare$$

Coding the doubling map.

Here we shall use a notion of itinerary of x by $\overset{\text{a map}}{f}$:

Let $I = [0,1]$, $I_0 = [0, 1/2)$, $I_1 = [1/2, 1)$.

$\{I_0, I_1\}$: partition of I. $\longrightarrow$ $I_0 \cup I_1 = I$, $I_0 \cap I_1 = \emptyset$

Define: $\Phi : I/\sim \longrightarrow \Sigma_2^+$

$$x \longmapsto \Phi(x) = (a_k)_{k=0}^{\infty}; \quad \begin{cases} a_k = 0 & \text{if } f^k(x) \in I_0 \\ a_k = 1 & \text{if } f^k(x) \in I_1 \end{cases}$$

The sequence $\phi(x) = (a_k)_{k=0}^{\infty}$ is called the itinerary of x with respect to the partition $\{I_0, I_1\}$: it is obtained by iterating $f^k(x)$ and recording which interval, whether I_0 or I_1, is visited at each k.

If $a_0, a_1, a_2, \ldots, a_k, \ldots$ is the itinerary of $\mathcal{O}_f(x)$

then $x \in I_{a_0}$, $f(x) \in I_{a_1}$, $f^2(x) \in I_{a_2}, \ldots, f^k(x) \in I_{a_k}, \ldots$

Itineraries of the doubling map produce the digits of binary expansion

<u>Proposition</u> If $a_0 a_1 \cdots a_n \cdots$ is the itinerary of $x \in [0,1)$ one has

$$x = \frac{a_0}{2} + \frac{a_1}{2^2} + \cdots = \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i}$$

<u>Proof</u>. Let $a_0, a_1, \ldots, a_n, \ldots$ be the itinerary of $\mathcal{O}_f(x)$.
We have to check that it gives a binary expansion for x, that is, we can write

$$x = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots = \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i}$$

<u>Step 1</u>: a_0

If the first digit ~~is~~

- $a_0 = 0 \Rightarrow x \in I_0 \Rightarrow 0 < x < 1/2 \Rightarrow$ the first digit of the binary expansion of x is 0.
- $a_0 = 1 \Rightarrow x \in I_1 \Rightarrow 1/2 \leq x < 1 \Rightarrow$ The first digit of the binary expansion of x is 1.

<u>Step 2</u> We have to prove that the k^{th} entry a_k of the itinerary gives the k^{th} digit of the binary expansion of x.

For this, recall that if $x_1 x_2 \ldots x_k \ldots$ are the digits of the binary expansion of x then the digits of $f(x)$ is $x_2 . x_3 \ldots$ and the digits of the binary expansion of $f^k(x)$ are $x_k, x_{k+1}, \dots$ (the doubling map shifts one position to the left).

By definition of itinerary, the itinerary of $f^k(x)$ is $a_k, a_{k+1}, \dots$

If $a_k = 0 \Rightarrow f^k(x) \in I_0 \Rightarrow 0 \le f^k(x) < 1/2$

If $a_k = 1 \Rightarrow f^k(x) \in I_1 \Rightarrow 1/2 \le f^k(x) < 1$

Then, the first digit of the binary expansion of $f^k(x)$ is 0 or 1, that is, $x_k = a_k$. ∎

Coding

Let $a_0 a_1 \dots a_n$ be a sequence of 0's and 1's.

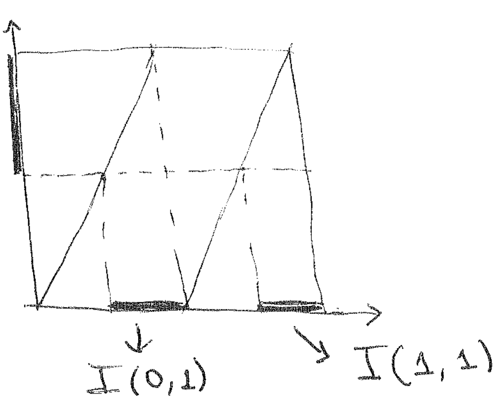
Let $I(a_0, a_1, \dots, a_n) = \{x \in [0,1];\ \Phi(x) = (a_0, \dots, a_n)\} =$

$$= \{x \in [0,1];\ f^k(x) \in I_{a_k},\ 0 \le k \le n\}$$

Recall: Φ is the itinerary map.

We have $I(a_0, a_1, \dots, a_n) = I_{a_0} \cap f^{-1}(I_{a_1}) \cap \dots \cap f^{-n}(I_{a_n})$.

Example:



$I(0,0) = [0, 1/4)$ $\quad$ $I(0,1) = [1/4, 1/2)$

$I(1,0) = [1/2, 3/4)$ $\quad$ $I(1,1) = [3/4, 1)$

<u>Proposition</u> • length of $I(a_0, \dots a_n) = \frac{1}{2^{n+1}}$

• $\bigcup_{(a_1,\dots,a_n) \in \{0,1\}^{n+1}} I(a_0, a_1, \dots, a_n) = [0, 1)$

Proof. Exercice.

<u>Note</u>: a $(n+1)$-tuple $(a_0, a_1, \dots, a_n)$ gives a partition of $[0, 1)$ into 2^n intervals of length $\frac{1}{2^{n+1}}$.

Each interval is of the form

$$\left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}} \right), \quad 0 \leq k < 2^{n+1}.$$

Now we use conjugacy and coding to construct a dense orbit for the doubling map.

<u>Theorem</u>. Let f be the doubling map. There exists a point $\bar{x}$ such that $\mathcal{O}_f(\bar{x})$ is dense.

<u>Proof</u>. Note that to prove that an orbit $\mathcal{O}_f(x)$ is dense, it is enough to show that for each $n \geq 1$, it <u>visits</u> all interval of the form $I(a_0, a_1, \dots, a_n)$.

Indeed, it this is the case, given $y \in [0, 1]$ and $\varepsilon > 0$, take N such that $\frac{1}{2^N} < \varepsilon$. and the interval $I(a_0, a_1, \dots, a_N)$ that <u>contains</u> y.

($\exists$ such interval because they form a partition of $[0, 1]$).

If we assume that $\mathcal{O}_f(x)$ visits all such intervals,

$\exists\, k;\ f^k(x) \in I(a_0, a_1, \dots, a_N)$. Thus,

$y,\ f^k(x) \in I(a_0, a_1, \dots, a_N) \Rightarrow$

$d(y, f^k(x)) \leq |I(a_0, a_1, \dots, a_N)| = \frac{1}{2^{N+1}} < \varepsilon$

and we are done.

To construct an orbit that visits all dyadic interval: list, for each n, all possible sequences $a_0, a_1, \dots, a_n$ of length n (there are 2^{n+1} of them) and create

$s^* = (s_i^*)_i$ just apposing all such <u>blocks</u> for $n = 0$, then $n = 1$, then $n = 2$. and so on:

$0, 1, \underbrace{00\ 01\ 10\ 11}_{2\text{-blocks}}\ \underbrace{000\ 001\ 010\ 011\ 100}_{3\text{-blocks}} \dots$

The orbit $\mathcal{O}_f(s^*)$ visits all intervals of the form $I(a_0, a_1, \dots, a_n)$: in fact, the block $(a_0, \dots, a_n)$ appears inside s^* just taking k so that $f^k(s^*) = a_0, \dots a_n$.

$\blacksquare$

Third Lecture

The quadratic map $f_\mu(x) = \mu x(1-x)$, $0<\mu$

Goal: understand the behavior of f_μ.

Before, let us introduce some definitions.

Def A point q is forward asymptotic to p provide $|f^j(q) - f^j(p)| \xrightarrow[j\to\infty]{} 0$. If p is periodic, of period n, q is asymptotic to p if $|f^{jn}(q) - p| \xrightarrow[j\to\infty]{} 0$.

stable set of $p \doteq W^s(p) = \{q;\ q$ is forward asymptotic to $p\}$

If f is invertible, same notions to backward asymptotic:

$$|f^j(q) - f^j(p)| \xrightarrow[j\to-\infty]{} 0$$

Unstable set of $p \doteq W^u(p) = \{q;\ q$ is backward asymp. to $p\}$.

Def. p is Lyapunov stable (L-stable) if given $\varepsilon>0$, $\exists\ \delta>0$; $(B_\delta(p) = \{x;\ |x-p|<\delta\})$ such that

$$|f^j(x) - f^j(p)| < \varepsilon \quad \forall\ x \in B_\delta(p).$$

This says that for x near enough p, the orbit of x stays near the orbit of p.

p is asymptotically stable provided it is L-stable and $W^s(p)$ contains a neighborhood of p. In another terms, $\underbrace{\exists\ \delta>0;}_{B_\delta(p)\subset W^s(p) \text{ and}}$ $\bigcap_{n\geq 0} f^n(B_\delta(p)) = \{p\}$.

A periodic point asymp. stable is also called an "ATTRACTING Periodic Point"

If p is a periodic point and $W^u(p)$ is a neighborhood of p, it is called a <u>repelling periodic point</u>, i.e.

$$\exists\, \delta>0;\quad \bigcap_n f^{-n}(B_\delta(p)) = \{p\} \qquad (\text{and } B_\delta(p)\subset W^u(p))$$

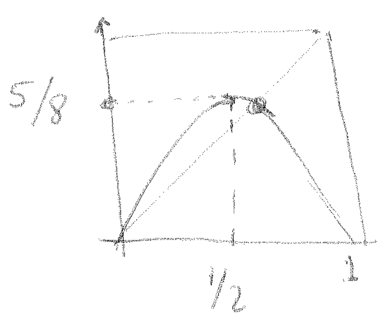
Back to $f_\mu(x) = \mu x(1-x)$.

Note: if $0<\mu\leq 4 \Rightarrow f_\mu[0,1]\subset[0,1]$

if $\mu>4 \Rightarrow \exists\, A_0\subset[0,1];\ |f_\mu(x)|>1\ \forall x\in A_0$.

<u>Example</u> $\mu = 5/2$, $f_\mu(x) = \frac{5}{2}x(1-x)$

fixed points: $f_\mu(x) = x \Leftrightarrow \frac{5}{2}x(1-x) = 0 \Leftrightarrow x=0$ and $x=\frac{3}{5}$



<u>Discuss graphical analyses</u>

note: $\frac{3}{5}$ "attracts" orbits nearby it.

<u>Criterion</u> for a periodic point to be attracting:

<u>Theorem.</u> $f:\mathbb{R}\to\mathbb{R}$ C^1, $p\in\mathbb{R}$, $f(p)=p$

(a) $|f'(p)|<1 \Rightarrow p$ is an attracting fixed point.

(b) $|f'(p)|>1 \Rightarrow p$ is a repelling " ".

<u>Proof</u>. f' continuous and $|f'(p)|<1 \Rightarrow \exists\, \varepsilon>0$ s.t.

for $y\in \underbrace{[p-\varepsilon, p+\varepsilon]}_{= B_\varepsilon(p)} \Rightarrow |f'(y)|<\lambda<1$.

From now on : $\mu > 1$.

Fixed points : $\quad 0 \quad$ and $\quad p_\mu = \dfrac{\mu - 1}{\mu}$

$$f_\mu'(0) = \mu \quad \text{and} \quad f_\mu'(p_\mu) = 2 - \mu$$

Thus: $\begin{cases} 0 \text{ is repelling } \forall \mu > 1. \\ p_\mu \text{ is } \begin{cases} \text{attracting for } 1 < \mu < 3 \\ \text{repelling for } \mu > 3 \\ f_\mu'(p_\mu) = -1 \text{ when } \mu = 3. \end{cases} \end{cases}$

<u>Proposition</u>. For all $x \in \mathbb{R} \setminus [0,1]$, $\lim\limits_{n \to \infty} f_\mu^n(x) = -\infty$

<u>Proof</u>: For $x < 0$, $f_\mu'(x) = \mu - 2\mu x > 1 \implies$

$0 > x > f_\mu(x) > f_\mu^2(x) > \dots > f_\mu^n(x)$ is decreasing.

If the orbit of x were bounded, it would have to converge to a fixed point, which would be a negative point. Since no such fixed point exists, $f_\mu^n(x)$ goes to minus infinity.

If $x > 1 \implies f_\mu(x) < 0 \implies f_\mu^j(x) = f_\mu^{j-1}(\underbrace{f_\mu(x)}_{<0}) \longrightarrow -\infty$

proved above

Since $f(p) = p$, by the Mean Value Theorem we get $z \in B_\varepsilon(p)$;

$$|f(y) - p| = |f(y) - f(p)| = |f'(z)|\,|y - p| < \lambda\,|y - p| < \lambda\varepsilon < \varepsilon$$

for all $y \in B_\varepsilon(p) \Rightarrow f(B_\varepsilon(p)) \subset B_\varepsilon(p)$.

Let us prove by induction that $|f^n(y) - p| \leq \lambda^n \varepsilon$ for all $y \in B_\varepsilon(p)$.

It is already proved for $n = 1$. Assume it is true for n.

By the Mean Value Theorem,

$$|f^{n+1}(y) - f^{n+1}(p)| = |f(f^n(y)) - p| = |f'(z)|\,|f^n(y) - p| \leq \lambda^{n+1}\varepsilon$$

Thus, $\lim_{n\to\infty} f^n(y) = p$ for all $y \in B_\varepsilon(p)$.

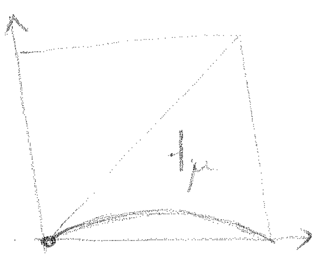
This proves the result.

Analogously we prove (2).

Note: the local dynamics is controled by the derivative.

Fixed points for the quadratic family.

Note: $0 < \mu < 1 \Rightarrow \mathrm{Fix}(f_\mu) = \{0\}$ and it is a global attracting



<u>Proposition</u> $f_\mu(x) = \mu x(1-x)$, $1 < \mu < 3$. Then

(a) p_μ is attracting and 0 is repelling

(b) $W^s(p_\mu) = (0,1)$, that is, $\lim_{n\to\infty} f_\mu^n(x) = p_\mu$, $\forall\, x \in (0,1)$.

<u>Proof</u> First recall $f_\mu'(x) =$ $\mu - 2\mu x$.

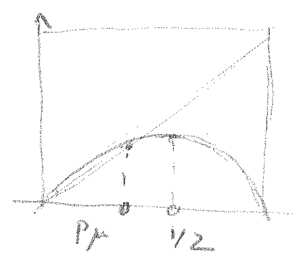
$$\Downarrow$$

$f_\mu'(0) = \mu > 1 \Rightarrow 0$ is repelling

$|f_\mu'(p_\mu)| = |2-\mu| < 1 \Rightarrow$ is attracting

To prove (b) we split in cases.

Case 1. $1 < \mu \leq 2$



$1/2$: point of maximum

Value maximum $\rightarrow$: $f_\mu(1/2) = \mu/4 \leq \frac{1}{2}$ ~~the graphic~~

~~lies~~ below the diagonal and ~~so~~ $p_\mu \leq 1/2$.

f_μ is monotonically increasing on $(0, p_\mu)$ and the graphic lies above the diagonal $\Rightarrow$

$\lim_n f_\mu^n(x) = p_\mu$ for $x \in (0, p_\mu)$.

For ~~$x\in$~~ $(p_\mu, 1/2)$, f_μ is also monotonically increasing and the graphic is below the diagonal. Thus, if $x \in (p_\mu, 1/2]$, $f_\mu^n(x)$ monotonically decreases to p_μ.

Finally, for $x \in (1/2, 1)$, $f_\mu(x) \in (0, 1/2)$ so $f_\mu^n(x) \xrightarrow[n]{} p_\mu$.

<u>Case 2</u> $\quad 2 < \mu < 3 \Rightarrow \frac{1}{2} < p_\mu = \frac{\mu - 1}{\mu} < 1$.

Let $\hat{p}_\mu = 1 - p_\mu = \frac{1}{\mu}$.

Consider the interval $[1/2, p_\mu]$.

<u>Claim</u> $\quad f_\mu^2 [1/2, p_\mu] \subset [1/2, p_\mu]$.

As f_μ^2 is monotone on $[1/2, p_\mu]$, it is enough to ~~show~~ determine the image of the end points and to prove the claim, it is enough to prove $f_\mu^2(1/2) > 1/2$.

We have: $f_\mu(1/2) = \mu/4$ and $f_\mu^2(1/2) = \mu \frac{\mu}{4}(1 - \mu/4)$.

<u>But</u>

$\mu \cdot \mu/4 \,(1 - 1/4) > 1/2 \iff 0 > \mu^3 - 4\mu^2 + 8 = (\mu-2)(\mu^2 - 2\mu - 4)$

Roots of $\mu^2 - 2\mu - 4$: $1 \pm \sqrt{5} \Rightarrow \mu^2 - 2\mu - 4 < 0$ for <u>$\mu < 3$</u>

$\mu - 2 > 0$ for $2 < \mu < 3$

so, $f_\mu^2(1/2) > 1/2$ for $2 < \mu < 3$.

finish The claim

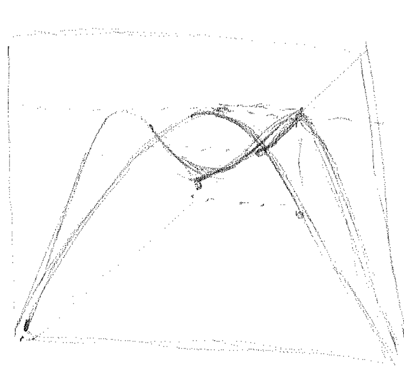
$\boxed{1/2}$

f_μ^2 is monotone on $[1/2, p_\mu]$, $f_\mu^2[1/2, p_\mu] \subset [1/2, p_\mu]$ and $f_\mu^2(1/2) > 1/2 \Longrightarrow$ graphic of f_μ^2 is above the diagonal below $1/2$ on $[1/2, p_\mu]$. and intersects the diagonal once at p_μ. Thus, $f_\mu^n(x) \longrightarrow p_\mu$ for all $x \in [1/2, p_\mu]$. Recall: $|f_\mu'(p_\mu)| < 1$.

- Now, as before, $f_\mu[\hat{p}_\mu, 1/2] = f_\mu[1/2, p_\mu] \Longrightarrow$ $f_\mu^2[\hat{p}_\mu, 1/2] \subset [1/2, p_\mu] \Longrightarrow \forall x \in [\hat{p}_\mu, 1/2]$, $f_\mu^n(x) \longrightarrow p_\mu$.

- f_μ is monotonically increasing on $(0, \hat{p}_\mu)$ and given any $x \in (0, \hat{p}_\mu)$, $\exists\, k \geq 1$ such that $f_\mu^k(x) \in (\hat{p}_\mu, p_\mu) \Longrightarrow f_\mu^n(x) \xrightarrow[n \to \infty]{} p_\mu$.

  Similarly for $x \in (p_\mu, 1)$.
  Combining the cases we have the proposition $\blacksquare$



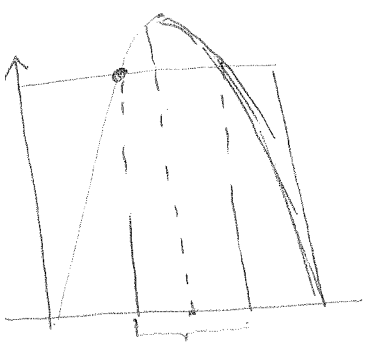
The quadratic map $f_\mu(x) = \mu x(1-x)$, $\mu > 4$.

For $\mu > 4$, the interval $I = [0, 1]$ is no longer invariant under f_μ, i.e, there are points that are mapped outside I.

It is still possible to consider the dynamics of f_μ, but one has to restrict the domain to an invariant subset of $[0, 1]$ of the form:

$$C = \bigcap_{n \in \mathbb{N}} f_\mu^{-n}(I).$$

Now we analyse the behavior of $f_\mu(x) = \mu x(1-x)$ for $\mu > 4$.



$A_0 = \{x \in [0, 1]; \quad f_\mu(x) > 1\}$

$A_1 = \{x \in [0, 1]; \ f_\mu(x) \in A_0\} = f_\mu^{-1}(A_0)$

$A_2 = \{x \in [0, 1]; \ f_\mu^2(x) \in A_0\} = f_\mu^{-2}(A_0)$

$\vdots$

$A_n = \{x \in [0, 1]; \ f_\mu^n(x) \in A_0\} = f_\mu^{-n}(A_0)$

Fact: $x \in A_n \Rightarrow \lim_{n \to \infty} f_\mu^n(x) = -\infty$

Consequence: $f_\mu^n(x) \in I \ \forall n \iff x \in I \setminus \bigcup_{n\geq 0} A_n = \Lambda$

<u>Theorem</u>: Λ is a Cantor set.

Let $I \setminus A_0 = I_0 \cup I_1$, $f(I_0) = f(I_1) = I$

$$f_\mu^{-1}(A_0) = \underbrace{A_{00} \cup A_{01}}_{= A_1} \ ; \ A_0 \subset I_0 \ , \ A_{01} \subset I_1$$

Then, $I \setminus (A_0 \cup A_1)$ is the union of four closed intervals, J_j such that $f_\mu(J_j) = I_0$ or $f_\mu(J_j) = I_1 \implies$

$\implies f_\mu^2(J_j) = I$ and f_μ^2 / J_j is monotone

⋮

$I \setminus (A_0 \cup \ldots \cup A_n)$ is the union of 2^{n+1} closed intervals, and f_μ^{n+1} / J_j is surjective and monotone

The graph of f_μ^{n+1} is alternatively increasing and decreasing on these intervals. Thus, the graph of f_μ^{n+1} crosses the diagonal at least 2^n points. Thus, f_μ has 2^n periodic points in I.

Definition $K \subset \mathbb{R}$ is a Cantor set if it is closed, totally disconnected (does not contain intervals) and perfect (any point of K is accumulated by points of K).

We shall prove here that Λ is a Cantor set for $\mu > 2+\sqrt{5}$. This is because the proof is simplier.

The reason to choose $\mu > 2+\sqrt{5}$ is because for such parameter values it holds that $|f_\mu'(x)| > \lambda > 1$ for all $x \in I \setminus A_0$.

Note that to fix $\mu > 2+\sqrt{5}$ we do the following

$$f_\mu(x) = 1 \iff \mu x(1-x) = 1 \iff X_\pm = \frac{\mu \pm (\mu^2-4\mu)^{1/2}}{2\mu}$$

$$f_\mu'(X_\pm) = (\mu^2-4\mu)^{1/2}$$

Then, $|f_\mu'(X_\pm)| > 1 \iff (\mu^2-4\mu)^{1/2} > 1 \Rightarrow \mu > 2 \pm \sqrt{5}$

<u>Claim 1</u>. For $\mu > 2+\sqrt{5}$, Λ_μ does not contain intervals.

<u>Proof</u> The proof goes by contradiction.

Assume there is an interval $[x, y] \subset \Lambda_\mu$.

Mean Value Theorem $\Rightarrow |(f_\mu^n)'(x) - (f_\mu^n)'(y)| \geq \lambda^n |x-y| > 1$

(we used $|f_\mu'(z)| > \lambda > 1 \quad \forall z \in [x, y]$).

$\Longrightarrow$ either $f_\mu^n(x) \notin I$ or $f_\mu^n(y) \notin I$, a contradiction.

<u>Claim 2</u> Λ_μ is closed.

Proof. Λ_μ is ~~the sequen~~ intersection of a nested sequence of closed intervals.

<u>Claim 3</u> Λ_μ is perfect

<u>Proof</u>. Observe that the extremum points of A_k, $\forall k$, belong to Λ_μ (because eventually they are sent by f_μ to 0 = zero).

If $\exists\, p$ isolated $\Rightarrow p \in A_k$, some k.

Thus, given $p \in \Lambda$, or p is a extremum of A_k

or $p = \lim_{k\to\infty} a_k$, a_k a extremum of A_k.

Last Lecture

Theorem for $\mu > 4$, f_μ / Λ_μ is conjugated to the shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$.

Proof Define, for $x \in \Lambda_\mu$, the itinerary of x with respect to I_0, I_1. Then,

$$\phi(x) = (a_i)_{i=0}^{\infty}, \quad a_i = \begin{cases} 0 & \text{if } f_\mu^i(x) \in I_0 \\ 1 & \text{if } f_\mu^i(x) \in I_1 \end{cases}$$

Claim 1 $\phi : \Lambda_\mu \to \Sigma_2^+$ is a homeomorphism

Proof (a) ϕ is 1-1

By contradiction, assume there are $x \neq y \in \Lambda_\mu$ with $\phi(x) = \phi(y)$

Then, for each n, $f_\mu^n(x)$ and $f_\mu^n(y)$ are at the same side of $1/2$.

This implies $f_\mu / _{[f_\mu^n(x), f_\mu^n(y)]}$ is monotone

and $f_\mu [f_\mu^n(x), f_\mu^n(y)] \subset I_0 \cup I_1 \quad \forall n$.

Thus, we get that $[x, y] \subset \Lambda_\mu$, a contradiction because Λ_μ does not contain intervals.

(b) Φ is surjective.

Proof. Given $(a_i)_{i=0}^{\infty} \in \sum_2^+$, define

$I(a_0 a_1 \dots a_n) = \{x \in I;\ x \in I_{a_0}, f_\mu(x) \in I_{a_1}, \dots, f_\mu^n(x) \in I_{a_n}\} =$

$= I_{a_0} \cap f_\mu^{-1}(I_{a_1}) \cap \dots \cap f_\mu^{-n}(I_{a_n})$.

Note that for every n, $I(a_0 a_1 \dots a_1)$ is a closed interval.

<u>Claim</u>. $I(a_0 \dots a_n)$ is a nested sequence of intervals.

Proof of the claim: $I(a_0 \dots a_n) = I_{a_0} \cap f_\mu^{-1}(I(a_1 \dots a_n))$.

By induction, assume $I(a_1 \dots a_n)$ is a non empty interval. Then, by definition of f_μ, $f_\mu^{-1}(I(a_1 \dots a_n))$ consists of two intervals, one contained in I_0 and the other contained in I_1.

Then $I_{a_0} \cap f_\mu^{-1}(I(a_1 \dots a_n))$ is a unique interval and it is non empty.

Now, note that
$I(a_0 a_1 \dots a_n) = I(a_0 a_1 \dots a_{n-1}) \cap f_\mu^{-1}(I_{a_n}) \subset$

$\subset I(a_0 a_1 \dots a_{n-1})$

and so they are nested. This proves the claim.

Thus, $\bigcap_{n \geq 0} I(a_0 a_1 \dots a_n) = \{\bar{x}\}$

If $\bar{x} \in \bigcap_{n \geq 0} I(a_0 \dots a_n) \underset{\text{by definition}}{\Longrightarrow}$

$\bar{x} \in I_{a_0}, \; f_\mu(\bar{x}) \in I_{a_1}, \dots, \; f_\mu^n(\bar{x}) \in I_{a_n} \underset{\text{by definition}}{\Longrightarrow}$

$\phi(\bar{x}) = (a_i)_{i=0}^{\infty} \Rightarrow \phi$ is surjective.

Remark: As ϕ is 1-1, $\bigcap_{n \geq 0} I(a_0 \dots a_n) = \{x\} \Rightarrow$

$|I(a_0 a_1 \dots a_n)| \underset{n \to \infty}{\longrightarrow} 0$.

(c) ϕ is continuous.

Proof. Take $x \in \Lambda_\mu$ with $\phi(x) = (a_i)_{i=0}^{\infty}$.

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be so that $\frac{1}{2^N} < \varepsilon$.

Consider all possible $I(b_0 b_1 \dots b_N)$ for all $N+1$-block of 0's and 1's.

These intervals are pairwise disjunt and

$\Lambda_\mu \subset \bigcup_{(b_0 \dots b_N) \in \Sigma_N} I(b_0 b_1 \dots b_N)$.

There are 2^{N+1} of such intervals and $I(a_0 a_1 \dots a_N)$ is one of them. Clearly $x \in I(a_0 a_1 \dots a_N)$.

Now choose $\delta > 0$ such that if $y \in \Lambda_\mu$ and $|y - x| < \delta$ then $y \in I(a_0 . a_1 ... a_N)$.

Then, the itinerary of y coincide (N+1 digits of the)

with the N+1 digits of the itinerary of x.

Then, if $\Phi(y) = (b_i)_{i=0}^{\infty}$, we have

$b_i = a_i$ for $0 \leq i \leq N$ $\Longrightarrow$

$$\Rightarrow \quad d(\Phi(x), \Phi(y)) = d\left((a_i)_{i=0}^{\infty}, (b_i)_{i=0}^{\infty}\right) < \frac{1}{2^N} < \varepsilon .$$

Thus we have the result.
In the same way Φ^{-1} is continuous. ▨

<u>Theorem</u>. $\Phi \circ f_\mu = f_\mu \circ \sigma$

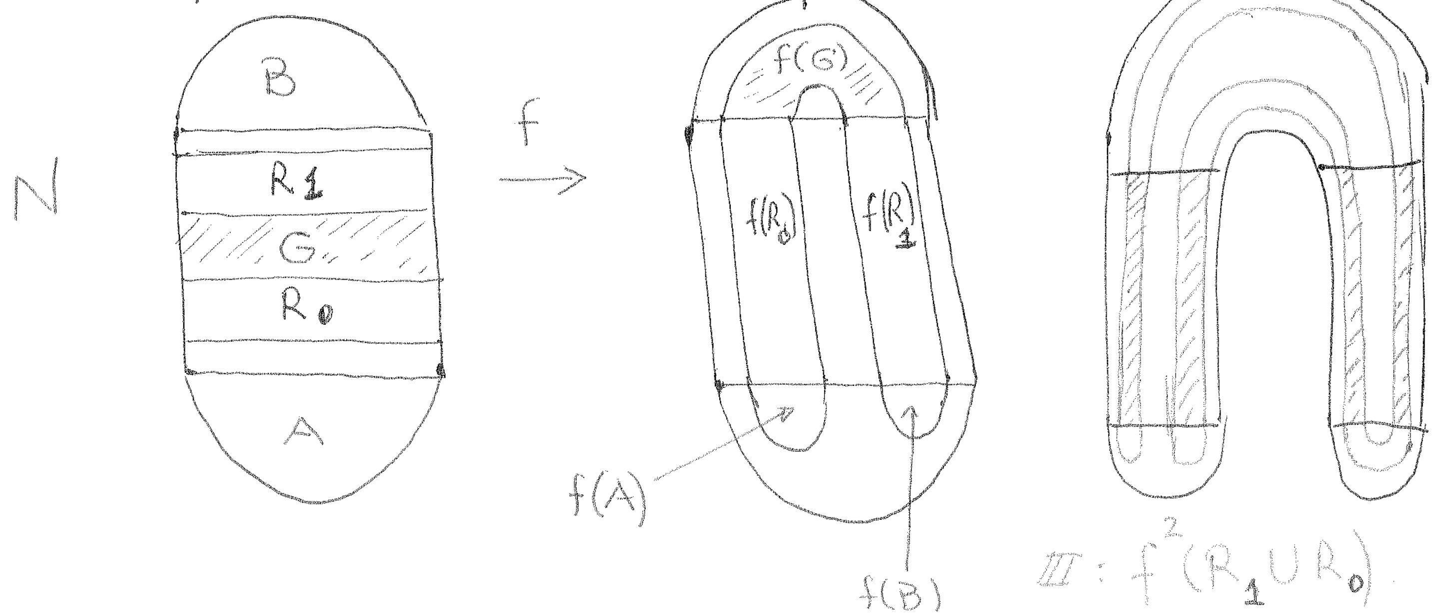
<u>proof</u>. exercice

<u>Consequence</u>.

$$\begin{array}{l} \mu > 4 \\ f_\mu|_{\Lambda_\mu} \end{array} \Longrightarrow \begin{cases} \text{(a)} \ \# \mathcal{P}_n(f_\mu) = 2^n \\ \text{(b)} \ \overline{\mathcal{P}(f_\mu)} = \Lambda_\mu \\ \text{(c)} \ f_\mu \text{ has a dense orbit} \end{cases}$$

▨

Example 1 Horseshoe map.



$$f(N) \subset N \Rightarrow f^2(N) \subset f(N) \subset N.$$

- $\left.\begin{array}{l} f(A) \subset A \\ f \text{ contraction} \end{array}\right\} \Rightarrow \exists\, p_0 \in A,\ f(p_0) = p_0.$

- $f(B) \subset A \Rightarrow$ ω-limit $\omega(x) = p_0 \quad \forall x \in B$

- Let $Q = [0,1] \times [0,1]$.

Let $\mathcal{R}(f)$ be the recurrent set of f.

Let $\Lambda = \bigcap_{j \in \mathbb{Z}} f^j(Q)$. Then

$$\mathcal{R}(f) \subset \Lambda \cup \{p_0\} \cup p_\infty$$

p_∞ : a source in the sphere.

Λ : the horseshoe (introduced by Smale)

Note : $\forall x \in Q$, $W^u(x)$ is the connected component of a vertical line at x.

$W^s(x)$: connected component of the horizontal line at x.

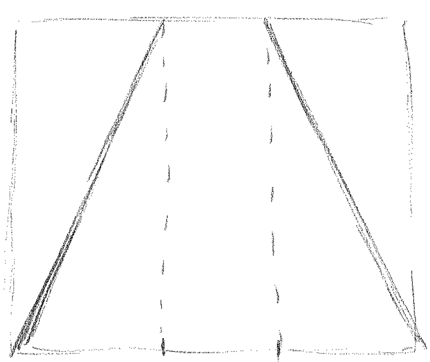
$\mathcal{F}^u$: unstable foliation, $\mathcal{F}^s$: stable foliation

vertical lines horizontal lines.

Let $f^{\vee} : [0, 1] \longrightarrow [0, 1]$ defined by :

given $x \in [0, 1]$, let $l_x \in \mathcal{F}^u$, $x \in l_x$.

then, $f^{\vee}(x) = l_{f(x)} \cap [0, 1]$, $l_{f(x)}$ is the vertical line at $f(x)$

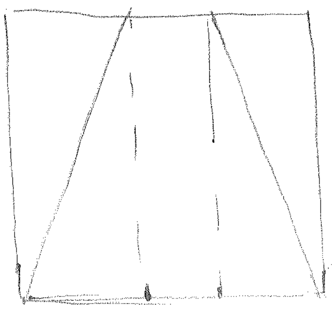


$f^{\vee}$

As before, the maximal invariant set for $f^{\vee}$ is a Cantor set $\Lambda_{\vee}$

Define $f^h : [0,1] \longrightarrow [0,1]$ similarly:

It can be seen that the graph of f^h is similar to the



graph of f^v

And the maximal invariant set for f^h is a Cantor set Λ_h

And the horseshoe $\equiv$ the maximal invariant set for the horseshoe map is equal to

$$\Lambda = \Lambda_v \times [0,1] \cap [0,1] \times \Lambda_h .$$

<u>Coding the horseshoe</u>:

<u>Theorem</u>. Let $\sum_2 = \{0,1\}^{\mathbb{Z}}$ and $\sigma : \sum_2 \to \sum_2$ the full shift map. Let $\Phi : \Lambda \longrightarrow \sum_2$ be the itinerary map, that is, $\Phi(x) = (a_i)_{i=-\infty}^{\infty}$

where $a_i = 0$ if $f^i(x) \in R_0$

$a_i = 1$ if $f^i(x) \in R_1$

Then Φ gives a conjugacy between $f/_\Lambda$ and $\sum_2$.

Example 2 : Hyperbolic Toral Automorphisms

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$L_A : \mathbb{R}^2 \to \mathbb{R}^2$ linear map induced by A.

$\det(A) = 1$, entries $a_{ij} \in \mathbb{Z}$

$\Downarrow$

Induces a map $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ (torus T^2).

Properties of A :

eigenvalues : $\lambda^{\pm} = \dfrac{1 \pm \sqrt{5}}{2} \Rightarrow \begin{cases} |\lambda^+| > 1 \\ |\lambda^-| < 1 \end{cases}$

eigenvectors : $v^s = \begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix}$ and $v^u = \begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix}$.

Properties of f_A :

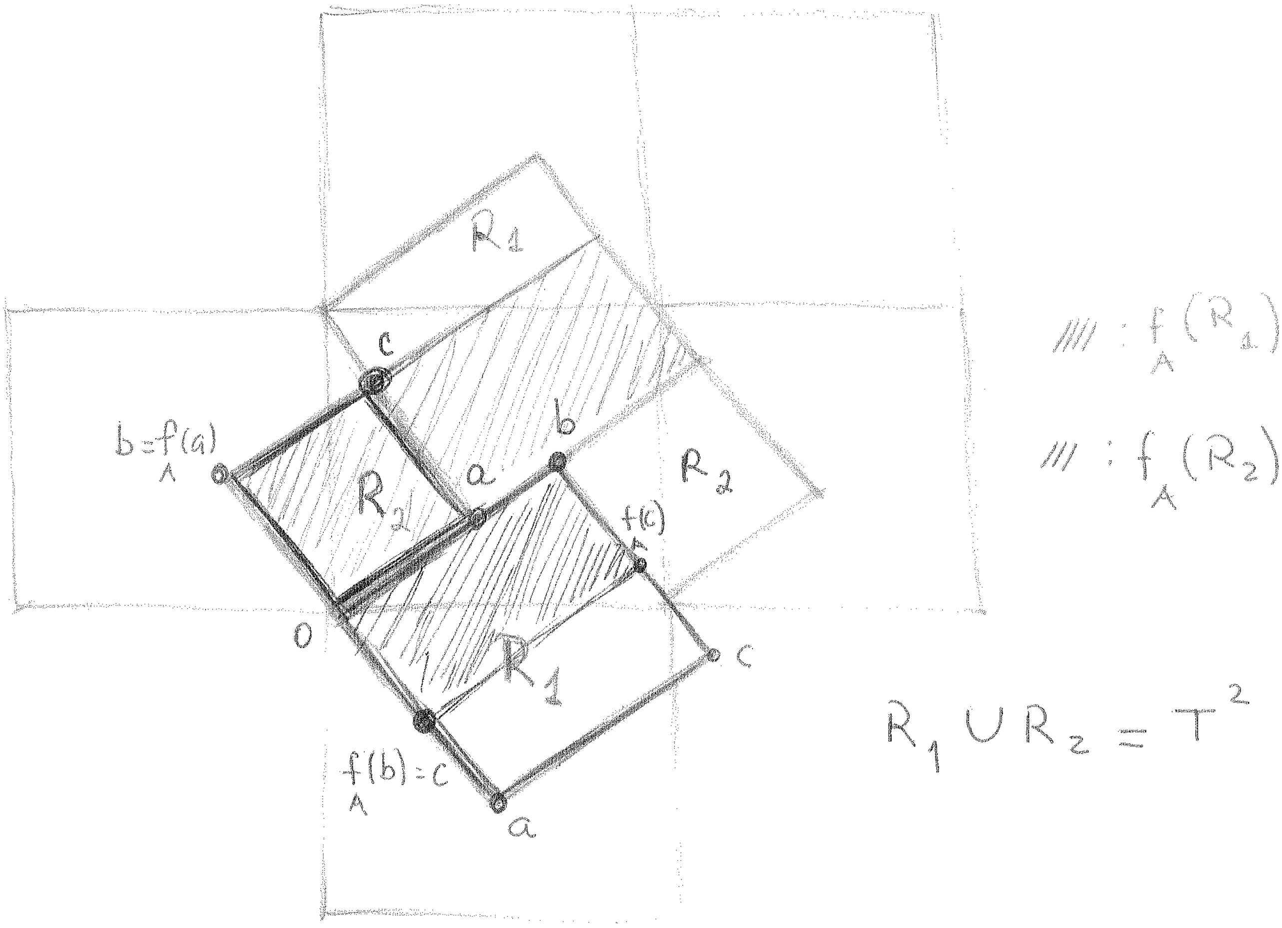
(a) $\overline{\mathcal{P}(f_A)} = T^2$

(b) f_A has a hyperbolic structure

(c) $W^u(p)$ is dense in T^2

(d) $W^s(p)$ is dense in T^2

A Markov partition for f_A, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$



$R_1 \cup R_2 = T^2$

stable boundary of $R_i \subset W^s(0)$

unstable boundary of $R_i \subset W^u(0)$

We define a transition matrix which indicates which itineraries for the orbit of a point is allowable: for a transition from rectangle R_i to R_j to be allowable, it must be possible for an orbit of a point to pass from the interior of R_i to the interior of R_j.

In this example the transition matrix is

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\text{notice}: B = A!)$$

Now, the shif space for B is the two sided subshift of finite type:

$$\sum_B = \{ (s_i)_{i=0}^{\infty} \; ; \; b_{s_i s_{i+1}} = 1 \}$$

Note: $f(R_1) \cap R_1 \neq \emptyset \qquad f(R_2) \cap R_1 \neq \emptyset$

$f(R_1) \cap R_2 \neq \emptyset \qquad f(R_2) \cap R_2 = \emptyset$

To define the symbolic dynamics, we can not get a continuous map (conjugacy or semiconjugacy) h from T^2 to $\sum_B$ because T^2 is connected and $\sum_B$ is totally disconnected. Also, for $p \in \partial R_i$, there are two choices. To bypass this dificulty, we define $h : \sum_B \longrightarrow T^2$.

and prove that $f_A \circ h = h \circ \sigma_B$,

$\sigma_B : \sum_B \circlearrowleft$ the shift, $h(s) = \bigcap_{n=0}^{\infty} \overline{[\cap f_A^{-j}(R_{s_j})]}$.

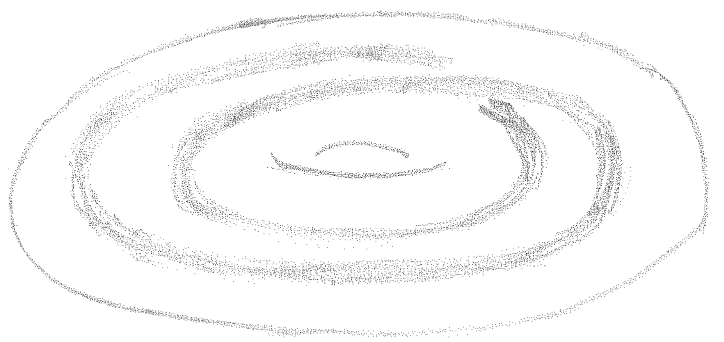
Example 3 The solenoid attractor.

$D^2 = \{ z \in \mathbb{C} ; \ |z| \leq 1$ $\qquad S^1 = [0,1]/\sim$

$N = S^1 \times D^2$

Let $g : S^1 \to S^1$ the double map, $g(t) = 2t \mod 1$.

$$f : N \to N$$
$$(t, z) \longmapsto \left(g(t), \ \frac{1}{4} z + \frac{1}{2} e^{2\pi i t} \right)$$



solid torus N.

Theorem. Let $\Lambda = \bigcap_{k=0}^{\infty} f^k(N)$. Then Λ is a hyperbolic expanding attractor for f.

Proposition For t_0 fixed, $\Lambda \cap D(t_0)$ is Cantor.

Proposition Λ has the properties:

(a) Λ is connected
(b) Λ is not locally connected
(c) Λ is not path connected
(d) the topological dimension of Λ is one

<u>Proposition</u> $f/_\Lambda$ has the properties:

(a) $\overline{\mathcal{P}(f/_\Lambda)} = \Lambda$

(b) $f/_\Lambda$ is topologically transitive

(c) $f/_\Lambda$ has a hyperbolic structure

Symbolic Dynamics for $f/_\Lambda$

Limit inverse:

$$\sum_1 = \{ s \in (S^1)^N \ ; \ g(s_{j+1}) = s_j \}$$

$\sigma : \sum_1 \longrightarrow \sum$

$$\sigma(s) = t \quad \text{if}$$

$$t_j = \begin{cases} s_{j-1} & \text{if } j \geq 1 \\ g(s_0) & \text{if } j = 0 \end{cases}$$

Note: if $s \in \sum_1$, then $g(s_{j+1}) = s_j$ so $s_{j+1} \in g^{-1}(s_j)$ is one of the two pre-images of s_j. The pair $(\sum, \sigma)$ is called the inverse limit of g.

<u>Theorem</u> Let $h : \Lambda \longrightarrow (S^1)^{\mathbb{N}}$ be defined as $h(p) = s$, where $f^{-j}(p) \in D(s_j)$, with $s_j \in S^1$ for $j = 0 \ldots$

Then h is a conjugacy from $f/_\Lambda$ to the inverse limit of g, σ on Σ.

For the proofs: Robinson

An Introduction to Dynamical Systems

Stability, Symbolic Dynamics and Chaos