Perturbation theory, KAM theory and Celestial Mechanics
2. Basics of Hamiltonian dynamics

Alessandra Celletti

Department of Mathematics
University of Roma "Tor Vergata"

Sevilla, 25-27 January 2016

Outline

Outline

Hamiltonian formalism

Mechanical system with n degrees of freedom[1]; for $\underline{\dot{q}} \in \mathbb{R}^n$, $\underline{q} \in \mathbb{R}^n$:
- $T = T(\underline{\dot{q}})$ kinetic energy,
- $V = V(\underline{q})$ potential energy.
- Lagrangian function defined as

$$\mathcal{L}(\underline{\dot{q}}, \underline{q}) \equiv T(\underline{\dot{q}}) - V(\underline{q}) \ .$$

- Introduce the *momenta* conjugated to the coordinates through:

$$\underline{p} \equiv \frac{\partial \mathcal{L}(\underline{\dot{q}}, \underline{q})}{\partial \underline{\dot{q}}} \ . \tag{1}$$

- From Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}(\underline{\dot{q}}, \underline{q})}{\partial \underline{\dot{q}}} = \frac{\partial \mathcal{L}(\underline{\dot{q}}, \underline{q})}{\partial \underline{q}} \qquad \Rightarrow \qquad \underline{\dot{p}} = \frac{\partial \mathcal{L}(\underline{\dot{q}}, \underline{q})}{\partial \underline{q}} \ .$$

[1] i.e., the minimum number of independent coordinates necessary to describe the mechanical system.

• It follows that

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \underline{\dot{q}}} d\underline{\dot{q}} + \frac{\partial \mathcal{L}}{\partial \underline{q}} d\underline{q} = \underline{p} d\underline{\dot{q}} + \underline{\dot{p}} d\underline{q} = d(\underline{p}\ \underline{\dot{q}}) - \underline{\dot{q}} d\underline{p} + \underline{\dot{p}} d\underline{q}\ ,$$

namely

$$d(\underline{p}\ \underline{\dot{q}} - \mathcal{L}) = -\underline{\dot{p}} d\underline{q} + \underline{\dot{q}} d\underline{p}\ . \qquad (2)$$

• Introduce the *Hamiltonian function* as

$$\mathcal{H}(\underline{p}, \underline{q}) \equiv \underline{p}\ \underline{\dot{q}} - \mathcal{L}(\underline{\dot{q}}, \underline{q})\ ,$$

where $\underline{\dot{q}}$ must be expressed in terms of $\underline{p}$ and $\underline{q}$ by inverting (1) (Legendre transformation). From (2) one obtains:

$$d\mathcal{H}(\underline{p}, \underline{q}) = -\underline{\dot{p}} d\underline{q} + \underline{\dot{q}} d\underline{p}\ ;$$

being

$$d\mathcal{H}(\underline{p}, \underline{q}) = \frac{\partial \mathcal{H}}{\partial \underline{p}} d\underline{p} + \frac{\partial \mathcal{H}}{\partial \underline{q}} d\underline{q}\ .$$

• Equating, one finds the *Hamilton's equations:*

$$\begin{aligned} \dot{\underline{q}} &= \frac{\partial \mathcal{H}(\underline{p}, \underline{q})}{\partial \underline{p}} \\ \dot{\underline{p}} &= -\frac{\partial \mathcal{H}(\underline{p}, \underline{q})}{\partial \underline{q}} \, . \end{aligned} \tag{3}$$

• In the Lagrangian case one needs to solve a differential equation of the second order; in the Hamiltonian case one needs to find the solution of two differential equations of the first order.

• In terms of the components of $\underline{p}$ and $\underline{q}$, Hamilton's equations are:

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}(\underline{p}, \underline{q})}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}(\underline{p}, \underline{q})}{\partial q_i} \, , \qquad i = 1, ..., n \, . \end{aligned}$$

Example.

Given the Lagrangian function

$$\mathcal{L}(\dot{q},q)=\frac{1}{2}\dot{q}^2+q\dot{q}+3q^2\ ,$$

the corresponding Hamiltonian function and the solution of Hamilton's equations are found as follows.

The momentum conjugated to q is

$$p=\frac{\partial\mathcal{L}}{\partial\dot{q}}=\dot{q}+q\ ,$$

which yields

$$\dot{q}=p-q\ .$$

Therefore:

$$\begin{aligned}\mathcal{H}(p,q)\ &=\ p\dot{q}-\mathcal{L}\\ &=\ \frac{1}{2}p^2-pq-\frac{5}{2}q^2\ .\end{aligned}$$

The corresponding Hamilton's equations are

$$\begin{aligned} \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = p + 5q \\ \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = p - q \ . \end{aligned}$$

Differentiating the second equation with respect to time one has

$$\ddot{q} = \dot{p} - \dot{q} = 6q \ ,$$

namely

$$\ddot{q} - 6q = 0 \ ,$$

whose solution is given by

$$q(t) = A_1 e^{\sqrt{6}t} + A_2 e^{-\sqrt{6}t} \ ,$$

where A_1 and A_2 are arbitrary constants depending on the initial data. From $p = q + \dot{q}$ one finds the solution for the momentum:

$$p(t) = \left(A_1 + \sqrt{6}A_1\right) e^{\sqrt{6}t} + \left(A_2 - \sqrt{6}A_2\right) e^{-\sqrt{6}t} \ .$$

Canonical transformations

• Given $\mathcal{H} = \mathcal{H}(\underline{p}, \underline{q})$ with n d.o.f. ($\underline{p} \in \mathbb{R}^n$, $\underline{q} \in \mathbb{R}^n$), consider the coordinate transformation

$$\begin{aligned} \underline{P} &= \underline{P}(\underline{p}, \underline{q}) \\ \underline{Q} &= \underline{Q}(\underline{p}, \underline{q}) \ , \end{aligned} \tag{4}$$

where $\underline{P} \in \mathbb{R}^n$, $\underline{Q} \in \mathbb{R}^n$. The coordinate change (4) is said to be *canonical*, if the equations of motion in the variables $(\underline{P}, \underline{Q})$ keep the Hamiltonian structure, namely the transformed variables satisfy Hamilton's equations with respect to a new Hamiltonian, say $\mathcal{H}_1 = \mathcal{H}_1(\underline{P}, \underline{Q})$.

• Let us derive the conditions under which the transformation (4) is canonical. Introduce the notation

$$\underline{x} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix} \ , \qquad \underline{z} = \begin{pmatrix} \underline{Q} \\ \underline{P} \end{pmatrix}$$

and let $\underline{z} = \underline{z}(\underline{x})$ be the transformation (4).

• Set

$$J \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} ,$$

where I_n is the n–dimensional identity matrix; Hamilton's equations can be written as

$$\dot{\underline{x}} = J \, \frac{\partial \mathcal{H}(\underline{x})}{\partial \underline{x}} .$$

• Let $M = \frac{\partial \underline{z}}{\partial \underline{x}}$; then, the transformed equations are

$$\dot{\underline{z}} = \frac{\partial \underline{z}}{\partial \underline{x}} \dot{\underline{x}} = M\dot{\underline{x}} = MJ \frac{\partial \mathcal{H}(\underline{x})}{\partial \underline{x}} = MJ \frac{\partial \mathcal{H}(\underline{x})}{\partial \underline{z}} \frac{\partial \underline{z}}{\partial \underline{x}} = MJM^T \, \frac{\partial \mathcal{H}(\underline{x})}{\partial \underline{z}} .$$

• The canonicity condition is equivalent to require that

$$MJM^T \; = \; J \; ; \tag{5}$$

equation (5) implies that the matrix M is symplectic, in which case we have Hamilton's equations w.r.t. $\underline{z}$, provided the new Hamiltonian is $\mathcal{H}_1(\underline{z}) = \mathcal{H}(\underline{x}(\underline{z}))$.

• A canonicity criterion is obtained through the *Poisson brackets*, which, for functions $f = f(\underline{p}, \underline{q})$, $g = g(\underline{p}, \underline{q})$, are defined as

$$\{f, g\} \ = \ \sum_{k=1}^{n} \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \ .$$

• A direct computation shows that $MJM^T = J$ is equivalent to say that a transformation is canonical if

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \ , \qquad \{Q_i, P_j\} = \delta_{ij} \ , \qquad i, j = 1, ..., n \ .$$

• In the one–dimensional case ($n = 1$) it suffices to verify that

$$\{Q, P\} \ = \ 1 \ ,$$

since $\{Q, Q\}$ and $\{P, P\}$ are identically zero.

• The *generating function* of a canonical transformation is introduced as follows. Consider a time–dependent canonical transformation

$$\begin{aligned} \underline{Q} &= \underline{Q}(\underline{q}, \underline{p}, t) \\ \underline{P} &= \underline{P}(\underline{q}, \underline{p}, t) \ . \end{aligned} \tag{6}$$

The generating function is a function of the form

$$F = F(\underline{q}, \underline{Q}, t) \ ,$$

such that the following transformation rules hold:

$$\begin{aligned} \underline{p} &= \frac{\partial F}{\partial \underline{q}} \\ \underline{P} &= -\frac{\partial F}{\partial \underline{Q}} \ . \end{aligned}$$

• If $\mathcal{H}_1 = \mathcal{H}_1(\underline{P}, \underline{Q}, t)$ is the Hamiltonian in the new set of variables, then

$$\mathcal{H}_1(\underline{P}, \underline{Q}, t) = \mathcal{H}(\underline{p}, \underline{q}, t) + \frac{\partial F}{\partial t} \ .$$

• Equivalent forms of the generating functions are the following:

$i)$ $F = F(\underline{q}, \underline{P}, t)$ with transformation rules:

$$\begin{aligned} \underline{p} &= \frac{\partial F}{\partial \underline{q}} \\ \underline{Q} &= \frac{\partial F}{\partial \underline{P}} \ ; \end{aligned}$$

$ii)$ $F = F(\underline{p}, \underline{Q}, t)$ with transformation rules:

$$\begin{aligned} \underline{q} &= -\frac{\partial F}{\partial \underline{p}} \\ \underline{P} &= -\frac{\partial F}{\partial \underline{Q}} \ ; \end{aligned} \tag{7}$$

$iii)$ $F = F(\underline{p}, \underline{P}, t)$ with transformation rules:

$$\underline{q} = -\frac{\partial F}{\partial \underline{p}}$$

$$\underline{Q} = \frac{\partial F}{\partial \underline{P}} \ .$$

Example

Compute α and β for which the following transformation is canonical:

$$\begin{aligned} P &= \alpha p e^{\beta q} \\ Q &= \frac{1}{\alpha} e^{-\beta q} \ ; \end{aligned}$$

for such values find the corresponding generating function.

Use Poisson brackets to check canonicity in the one–dimensional case:

$$\{Q, P\} \equiv \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \ = \ 1 \ .$$

Therefore one has:

$$-\frac{\beta}{\alpha} e^{-\beta q} \cdot \alpha e^{\beta q} \ = \ 1 \ ,$$

which is satisfied for $\beta = -1$ and for any $\alpha \neq 0$.

In this case the transformation becomes:

$$\begin{aligned} P &= \alpha p e^{-q} \\ Q &= \frac{1}{\alpha} e^q \, . \end{aligned} \tag{8}$$

Let us look for a generating function $F = F(q, P)$, whose transformation rules are given by

$$\begin{aligned} p &= \frac{\partial F}{\partial q} \\ Q &= \frac{\partial F}{\partial P} \, . \end{aligned}$$

Inverting the first of (8) one has:

$$\begin{aligned} p &= \frac{P}{\alpha} e^q \\ Q &= \frac{1}{\alpha} e^q \, . \end{aligned}$$

• Therefore it should be

$$\frac{\partial F}{\partial q} = \frac{P}{\alpha} e^q \ , \tag{9}$$

namely $F(q,P) = \frac{P}{\alpha} e^q + f(P)$, where $f(P)$ is a total function of P.

• Analogously, from the relation

$$\frac{\partial F}{\partial P} = \frac{1}{\alpha} e^q \ , \tag{10}$$

one finds $F(q,P) = \frac{P}{\alpha} e^q + g(q)$, where $g(q)$ depends only on the variable q.

• Comparing the solutions of (9) and (10) one obtains $f(P) = g(q) = 0$, thus yielding

$$F(q,P) = \frac{P}{\alpha} e^q \ .$$

Integrable systems

• A Hamiltonian system with n d.o.f. is said to be *integrable*, if there exist n integrals, U_1, ..., U_n, which satisfy the following assumptions:
1) the integrals are in involution: $\{U_j, U_k\} = 0$ for any $j, k = 1, ..., n$;
2) the integrals are independent, i.e. the following matrix has rank n:

$$\begin{pmatrix} \frac{\partial U_1}{\partial p_1} & \dots & \frac{\partial U_1}{\partial p_n} & \frac{\partial U_1}{\partial q_1} & \dots & \frac{\partial U_1}{\partial q_n} \\ \vdots & & & & & \\ \frac{\partial U_n}{\partial p_1} & \dots & \frac{\partial U_n}{\partial p_n} & \frac{\partial U_n}{\partial q_1} & \dots & \frac{\partial U_n}{\partial q_n} \end{pmatrix} ;$$

3) in place of 2) one can require the non–singularity condition:

$$\det \begin{pmatrix} \frac{\partial U_1}{\partial p_1} & \dots & \frac{\partial U_1}{\partial p_n} \\ \vdots & & \\ \frac{\partial U_n}{\partial p_1} & \dots & \frac{\partial U_n}{\partial p_n} \end{pmatrix} \neq 0 ;$$

notice that this condition is stronger than the independence of item 2).

- Having fixed a point $(\underline{p}_0, \underline{q}_0)$, let $\underline{\alpha}_0 = \underline{U}(\underline{p}_0, \underline{q}_0)$, where $\underline{U} \equiv (U_1, ..., U_n)$.
- For $\underline{\alpha} \in \mathbb{R}^n$ define the manifold $M_{\underline{\alpha}}$ as

$$M_{\underline{\alpha}} = \{(\underline{p}, \underline{q}) \in \mathbb{R}^{2n} \ : \ U_1(\underline{p}, \underline{q}) = \alpha_1, ..., U_n(\underline{p}, \underline{q}) = \alpha_n\} \ .$$

The integrability of a Hamiltonian system can be obtained through the following Liouville–Arnold theorem.

Theorem

Suppose that the Hamiltonian $\mathcal{H}(\underline{p}, \underline{q})$, $\underline{p}$, $\underline{q} \in \mathbb{R}^n$, admits n integrals U_1, ..., U_n, satisfying the above conditions of involution and non–singularity. Assume that the manifold $M_{\underline{\alpha}}$ is compact in a suitable neighborhood of $\underline{\alpha}_0$. Then, there exists a transformation of coordinates from $(\underline{p}, \underline{q})$ to $(\underline{I}, \underline{\varphi})$ with $\underline{I} \in \mathbb{R}^n$, $\underline{\varphi} \in \mathbb{T}^n$, such that the new Hamiltonian $\mathcal{H}_1$ takes the form

$$\mathcal{H}_1(\underline{I}, \underline{\varphi}) \ \equiv \ h(\underline{I}) \ ,$$

for a suitable function $h = h(\underline{I})$.

Action–angle variables

• Consider the mechanical system described by $\mathcal{H}(\underline{p}, \underline{q})$, where $\underline{p} \in \mathbb{R}^n$, $\underline{q} \in \mathbb{R}^n$. When dealing with integrable systems one can introduce a canonical transformation $\mathcal{C} : (\underline{p}, \underline{q}) \in \mathbb{R}^{2n} \to (\underline{I}, \underline{\varphi}) \in \mathbb{R}^n \times \mathbb{T}^n$, such that the transformed Hamiltonian depends only on the action variables $\underline{I}$:

$$\mathcal{H} \circ \mathcal{C}(\underline{I}, \underline{\varphi}) = h(\underline{I}) = h(I_1, ..., I_n) \ ,$$

for some function $h = h(\underline{I})$. The coordinates $(\underline{I}, \underline{\varphi})$ are known as *action–angle variables.*

• Liouville–Arnold theorem provides an explicit algorithm to construct the action–angle variables: introduce as transformed momenta the actions $(I_1, ..., I_n)$ defined through the relation

$$I_j \ = \ \oint p_j \, dq_j \ ,$$

where the integral is computed over a full cycle of motion.

• The canonical variables conjugated to $(I_1, ..., I_n)$ are named *angle variables;* they will be denoted as $(\varphi_1, ..., \varphi_n)$.
• Hamilton's equations become integrable; indeed, let us define the *frequency* or *rotation vector* as

$$\underline{\omega} = \underline{\omega}(\underline{I}) = \frac{\partial h(\underline{I})}{\partial \underline{I}} \ ;$$

then, one has:

$$\begin{aligned} \dot{\underline{I}} &= -\frac{\partial h(\underline{I})}{\partial \underline{\varphi}} = \underline{0} \\ \dot{\underline{\varphi}} &= \frac{\partial h(\underline{I})}{\partial \underline{I}} = \underline{\omega}(\underline{I}) \ . \end{aligned}$$

• The *action* $\underline{I}$ is constant along the motion, $\underline{I} = \underline{I}_0$, while the *angle* $\underline{\varphi}$ varies as $\underline{\varphi} = \underline{\omega}(\underline{I}_0)t + \underline{\varphi}_0$, where $(\underline{I}_0, \underline{\varphi}_0)$ denote the initial conditions.

Example.

Action–angle variables for the harmonic oscillator:

$$\mathcal{H}(p,q) = \frac{1}{2m}(p^2 + \omega^2 q^2) \ .$$

Setting $\mathcal{H}(p,q) = E$, one has

$$p^2 = 2mE - \omega^2 q^2$$

and the corresponding action variable is:

$$I = \oint p dq = \oint \sqrt{2mE - \omega^2 q^2}\, dq \ .$$

Let $q = \sqrt{\frac{2mE}{\omega^2}} \sin\vartheta$; then, one has:

$$\begin{aligned} I &= \int_0^{2\pi} \sqrt{2mE - 2mE\sin^2\vartheta}\, \sqrt{\frac{2mE}{\omega^2}} \cos\vartheta\, d\vartheta \\ &= \frac{2mE}{\omega} \int_0^{2\pi} \cos^2\vartheta d\vartheta = \frac{2\pi mE}{\omega} \ . \end{aligned}$$

The Hamiltonian in action–angle variables becomes:

$$E = \mathcal{H}(I) = \frac{\omega}{2\pi m}\, I \ .$$

The associated Hamilton's equations are

$$\begin{aligned} \dot I &= 0 \\ \dot\varphi &= \frac{\omega}{2\pi m} \ , \end{aligned}$$

whose solution is found to be

$$\begin{aligned} I(t) &= I(0) \\ \varphi(t) &= \frac{\omega}{2\pi m} t + \varphi(0) \ . \end{aligned}$$

Nearly–integrable systems

Nearly–integrable systems of the form

$$\mathcal{H}(I,\varphi) = h(I) + \varepsilon f(I,\varphi)\ ,$$

where $I \in \mathbb{R}^n$ (actions), $\varphi \in \mathbb{T}^n$ (angles), $\varepsilon > 0$ is a small parameter.

• In the *integrable* approximation $\varepsilon = 0$ Hamilton's equations are solved as

$$\begin{aligned} \dot I &= -\frac{\partial h(I)}{\partial \varphi} = 0 \qquad \Rightarrow \qquad I(t) = I(0) = const. \\ \dot\varphi &= \frac{\partial h(I)}{\partial I} \equiv \omega(y) \qquad \Rightarrow \qquad \varphi(t) = \omega\,(I(0))\,t + \varphi(0)\ , \end{aligned}$$

where $(I(0), \varphi(0))$ are the initial conditions.

Nearly–integrable systems

Nearly–integrable systems of the form

$$\mathcal{H}(I,\varphi) = h(I) + \varepsilon f(I,\varphi) \ ,$$

where $I \in \mathbb{R}^n$ (actions), $\varphi \in \mathbb{T}^n$ (angles), $\varepsilon > 0$ is a small parameter.
• In the *integrable* approximation $\varepsilon = 0$ Hamilton's equations are solved as

$$\begin{aligned} \dot{I} &= -\frac{\partial h(I)}{\partial \varphi} = 0 \qquad \Rightarrow \qquad I(t) = I(0) = const. \\ \dot{\varphi} &= \frac{\partial h(I)}{\partial I} \equiv \omega(y) \qquad \Rightarrow \qquad \varphi(t) = \omega\left(I(0)\right) t + \varphi(0) \ , \end{aligned}$$

where $(I(0), \varphi(0))$ are the initial conditions.
• In the three–body problem, the integrable part coincides with the Keplerian two–body interaction, while the perturbing function provides the gravitational attraction with the third body and the perturbing parameter is the mass ratio of the primaries.

Nearly–integrable dissipative systems

• In many cases it is useful to consider also *nearly–integrable dissipative* systems, like ($\lambda > 0$ dissipative constant, μ drift term):

$$\begin{aligned} \dot{I} &= -\varepsilon \frac{\partial f(I,\varphi)}{\partial \varphi} - \lambda(I-\mu), \\ \dot{\varphi} &= \omega(I) + \varepsilon \frac{\partial f(I,\varphi)}{\partial I} . \end{aligned}$$

Nearly–integrable dissipative systems

• In many cases it is useful to consider also *nearly–integrable dissipative* systems, like ($\lambda > 0$ dissipative constant, μ drift term):

$$
\begin{aligned}
\dot{I} &= -\varepsilon \frac{\partial f(I,\varphi)}{\partial \varphi} - \lambda (I - \mu), \\
\dot{\varphi} &= \omega(I) + \varepsilon \frac{\partial f(I,\varphi)}{\partial I} .
\end{aligned}
$$

• It represents, for example, the spin-orbit model subject to a tidal torque, due to the non-rigidity of the satellite.

Outline

Dynamical behaviors

In a dynamical system we can have:

• Periodic motion: a solution of the equations of motion which retraces its own steps after a given interval of time, called *period.*

• Quasi–periodic motion: a solution of the equations of motion which comes indefinitely close to its initial conditions at regular intervals of time, though ever exactly retracing itself.

• Regular motion: we will refer to periodic or quasi–periodic orbits as *regular* motions.

• Chaotic motion: irregular motion showing an *extreme sensitivity to the choice of the initial conditions.*

♢ The divergence of the orbits will be measured by the *Lyapunov exponents* or by the *FLI.*

♢ Chaotic motions are unpredictable, but not necessarily unstable.

Outline

Poincaré maps

• The Poincaré map reduces the study of a continuous system to that of a discrete mapping.

• Consider the n–dimensional differential system

$$\dot{\underline{z}} = \underline{f}(\underline{z}) , \qquad \underline{z} \in \mathbb{R}^n ,$$

where $\underline{f} = \underline{f}(\underline{z})$ is a generic regular vector field.

• Let $\underline{\Phi}(t; \underline{z}_0)$ be the flow at time t with initial condition $\underline{z}_0$.

• Let Σ be an $(n-1)$–dimensional hypersurface, the *Poincaré section*, transverse to the flow, which means that if $\underline{\nu}(\underline{z})$ denotes the unit normal to Σ at $\underline{z}$, then $\underline{f}(\underline{z}) \cdot \underline{\nu}(\underline{z}) \neq 0$ for any $\underline{z}$ in Σ.

• For a periodic orbit , let $\underline{z}_p$ be the intersection of the periodic orbit with Σ; let U be a neighborhood of $\underline{z}_p$ on Σ. Then, for any $\underline{z} \in U$ we define the Poincaré map as $\underline{\Phi}' = \underline{\Phi}(T; \underline{z})$, where T is the first return time of the flow on Σ.

- Example of the Poincaré map of the spin–orbit model:

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -\varepsilon \left(\frac{a}{r}\right)^3 \sin(2x - 2f)
\end{aligned}
$$

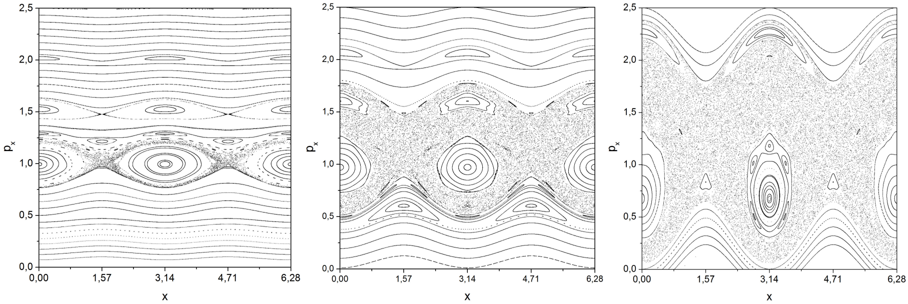
with

$$
\begin{aligned}
r &= a(1 - e\cos u) \\
\tan\frac{f}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2} \\
\ell &= u - e\sin u \\
\ell &= n\,t + \ell_0 \ .
\end{aligned}
$$

- One–dimensional, time–dependent (2π–periodic in time):

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \varepsilon g(x, t) \ .
\end{aligned}
$$

• Poincaré maps of the spin–orbit problem taking the intersections at $t = 2\pi k$, $k \in \mathbb{Z}_+$ for $\varepsilon = 0.024, 0.1, 0.4$.



Lyapunov exponents

• Lyapunov exponents provide the divergence of nearby orbits.
• Quantitatively, two nearby trajectories at initial distance $\delta \underline{z}(0)$ diverge at a rate given by (within the linearized approximation)

$$|\delta \underline{z}(t)| \approx e^{\lambda t} |\delta \underline{z}(0)| \ ,$$

where λ is the *Lyapunov exponent.*
• The rate of separation can be different in different directions $\rightarrow$ there is a spectrum of Lyapunov exponents equal in number to the dimension of the phase space.
• The largest Lyapunov exponent is called *Maximal Lyapunov exponent* (MLE) and a positive value gives an indication of chaos. It can be computed as

$$\lambda = \lim_{t \to \infty} \lim_{\delta \underline{z}(0) \to 0} \frac{1}{t} \ln \frac{|\delta \underline{z}(t)|}{|\delta \underline{z}(0)|} \ .$$

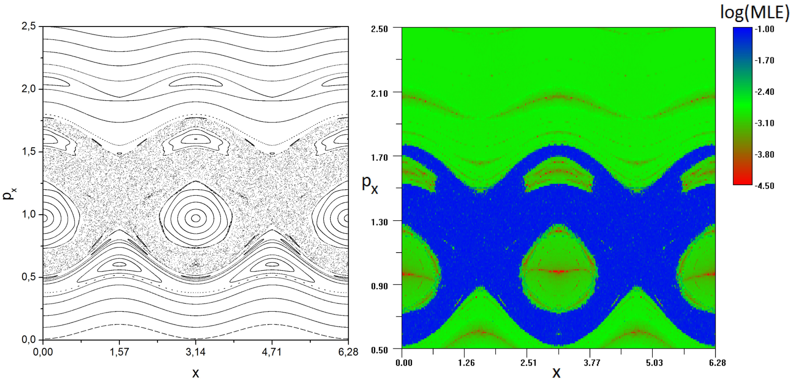
• Fast Lyapunov Indicator (FLI) is obtained as the value of the MLE at a fixed time, say T.
• A comparison of the FLIs as the initial conditions are varied allows one to distinguish between different kinds of motion (regular, resonant or chaotic).
• Consider $\dot{\underline{z}} = \underline{f}(\underline{z})$, $\underline{z} \in \mathbb{R}^n$ and let the variational equations be

$$\dot{\underline{v}} = \left(\frac{\partial \underline{f}(\underline{z})}{\partial \underline{z}}\right) \underline{v} \, .$$

• Definition of the FLI: given the initial conditions $\underline{z}(0) \in \mathbb{R}^n$, $\underline{v}(0) \in \mathbb{R}^n$, the FLI at time $T \geq 0$ is provided by the expression

$$FLI(\underline{z}(0), \underline{v}(0), T) \equiv \sup_{0<t\leq T} \log ||\underline{v}(t)|| \, .$$

• MLE for the spin-orbit problem in the x, $p_x = y$ plane: green/red = regular motions, blue = chaotic dynamics



... and in the parameter space ε versus p_x (with $x_0 = 0$) for Mercury (left) and Moon (right)