Perturbation theory, KAM theory and Celestial Mechanics
3. Conservative and dissipative standard maps

Alessandra Celletti

Department of Mathematics
University of Roma "Tor Vergata"

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Outline

1. Conservative Standard Map

2. Dissipative Standard Map

3. 4-dimensional standard map

4. Non-twist standard map

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Conservative Standard Map

It is described by the equations (discrete analogue of the spin-orbit problem)

$$\begin{aligned} y' &= y + \varepsilon f(x) \qquad\qquad y \in \mathbb{R}\,,\ x \in \mathbb{T} \\ x' &= x + y'\,, \end{aligned}$$

with $\varepsilon > 0$ *perturbing parameter,* $f = f(x)$ analytic function.

• Classical (Chirikov) standard map: $f(x) = \sin x$.

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• Equivalent notation:

$$\begin{aligned} y_{j+1} &= y_j + \varepsilon \sin(x_j) \\ x_{j+1} &= x_j + y_{j+1} = x_j + y_j + \varepsilon \sin(x_j) \qquad \text{for } j \geq 0\,. \end{aligned}$$

• PROPERTIES:

$A)$ SM is integrable for $\varepsilon = 0$, non–integrable for $\varepsilon \neq 0$:

$$
\begin{aligned}
y_{j+1} &= y_j = y_0 \\
x_{j+1} &= x_j + y_{j+1} = x_j + y_j = x_0 + j y_0 \qquad \text{for } j \geq 0\ ,
\end{aligned} \tag{1}
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namely y_j is constant and x_j increases by y_0.

$A1$) Case $y_0 = 2\pi\frac{p}{q}$ with p, q integers ($q \neq 0$). Then, on the line $y = y_0$:

$$x_1 = x_0 + 2\pi\frac{p}{q}, \quad x_2 = x_0 + 4\pi\frac{p}{q}, \ \ldots, \ x_q = x_0 + 2\pi p = x_0 \ \ !!!$$

Therefore, the orbit is PERIODIC with period $2\pi q$ and the interval $[0, 2\pi)$ is spanned p times.

$A2$) Case $y_0 = 2\pi$-irrational. Then, on the line $y = y_0$, the iterates of x_0 fill densely the line $y = y_0 \rightarrow$ QUASI-PERIODIC MOTIONS (KAM theory): the iterates never come back to the initial condition, but close as you wish after a sufficient number of iterations.

$B)$ The mapping (1) is conservative, since the determinant of the corresponding Jacobian is equal to one; in fact, setting $f_x(x_j) \equiv \frac{\partial f(x_j)}{\partial x}$, the determinant of the Jacobian (1) is equal to

$$\det \begin{pmatrix} 1 & \varepsilon f_x(x_j) \\ 1 & 1 + \varepsilon f_x(x_j) \end{pmatrix} = 1 \ . \tag{2}$$

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C) Fixed points are obtained by solving the equations

$$\begin{aligned} y_{j+1} &= y_j \\ x_{j+1} &= x_j \ ; \end{aligned}$$

$\diamondsuit$ from the first equation $y_{j+1} = y_j + \varepsilon f(x_j) \Rightarrow f(x_j) = 0$;
$\diamondsuit$ from the second equation $x_{j+1} = x_j + y_{j+1} \Rightarrow y_{j+1} = 0 = y_0$;
$\diamondsuit$ if $f(x) = \sin x$, fixed points are $(y_0, x_0) = (0, 0)$ and $(y_0, x_0) = (0, \pi)$.

D) Linear stability is investigated by computing the first variation:

$$\begin{pmatrix} \delta y_{j+1} \\ \delta x_{j+1} \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon f_x(x_0) \\ 1 & 1+\varepsilon f_x(x_0) \end{pmatrix} \begin{pmatrix} \delta y_j \\ \delta x_j \end{pmatrix} .$$

The eigenvalues of the linearized system are determined by solving the characteristic equation ($f = \sin x$):

$$\lambda^2 - (2 \pm \varepsilon)\lambda + 1 = 0 ,$$

with + for $(0,0)$ and - for $(0,\pi)$.

$\diamondsuit$ One eigenvalue associated to $(0,0)$ is greater than one $\Rightarrow$ the fixed point is unstable.

$\diamondsuit$ For $\varepsilon < 4$ the eigenvalues associated to $(0,\pi)$ are *complex conjugate with real part less than one* $\Rightarrow$ $(0,\pi)$ is stable.

E) Twist property:

$$\frac{\partial x'}{\partial y} = 1 > 0$$

F) The standard map is generated by $F(x,x') = \frac{1}{2}(x'-x)^2 + \varepsilon \cos x$, so that

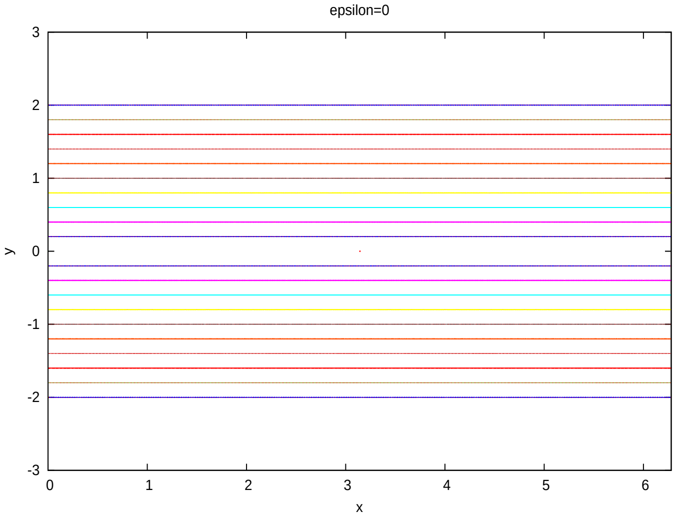
$$y = -\frac{\partial F}{\partial x}\ , \qquad y' = \frac{\partial F}{\partial x'}\ .$$

G) The standard map can be obtained from a discrete Lagrangian variational principle. For any configuration sequence $\{..., x_{s-1}, x_s, x_{s+1}, ...\}$ define the discrete action as

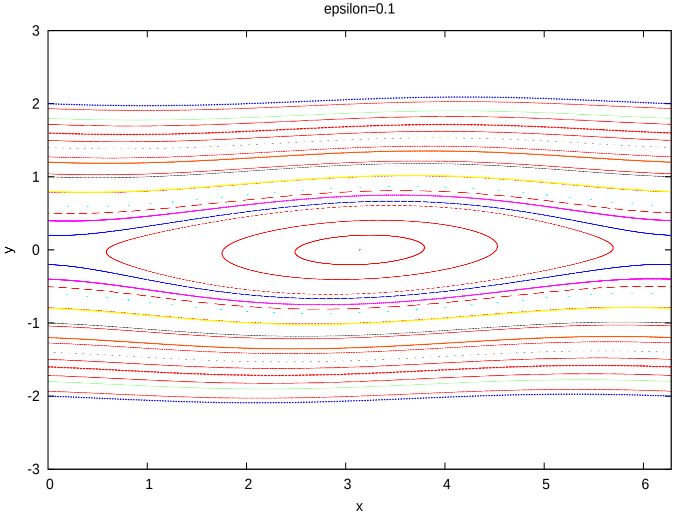
$$\mathcal{A}[..., x_{s-1}, x_s, x_{s+1}, ...] = \sum_s F(x_s, x_{s+1})\ .$$

An orbit is a sequence which is a critical point of $\mathcal{A}$, yielding the discrete Euler-Lagrange equation:

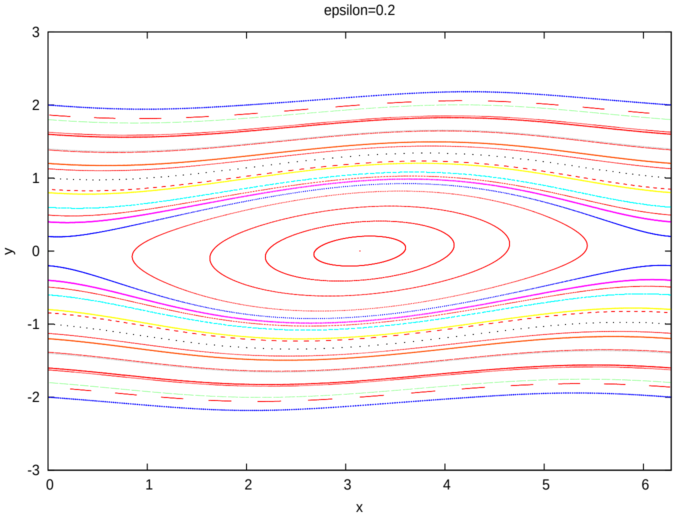
$$x_{s+1} - 2x_s + x_{s-1} = \varepsilon \sin x\ .$$



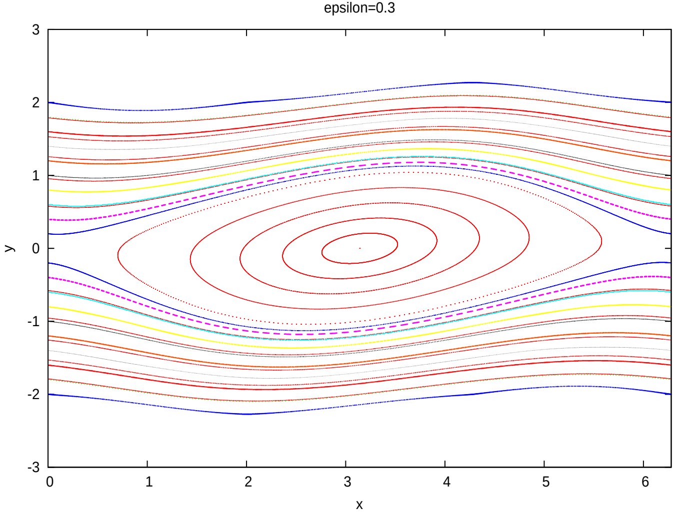
$\varepsilon = 0$: the system is integrable, only quasi–periodic curves (lines), a stable equilibrium point at $(0, \pi)$ and an unstable at $(0, 0)$.



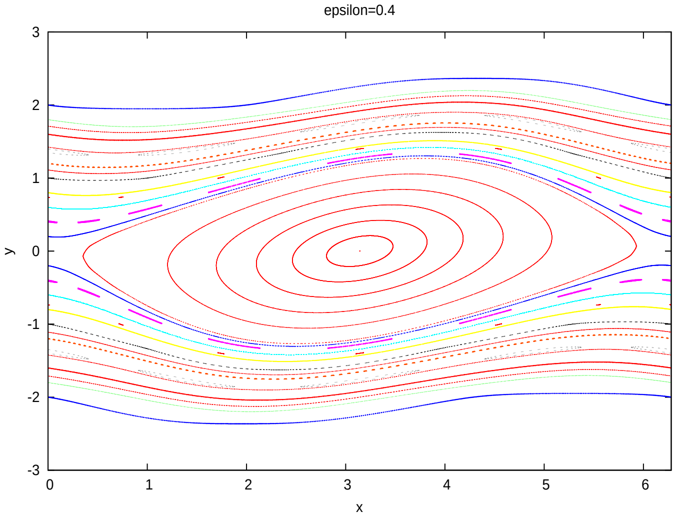
$\varepsilon = 0.1$: switch on the perturbation, the system is non–integrable, the quasi–periodic (KAM) curves are distorted, the stable point $(0, \pi)$ is surrounded by elliptic islands.



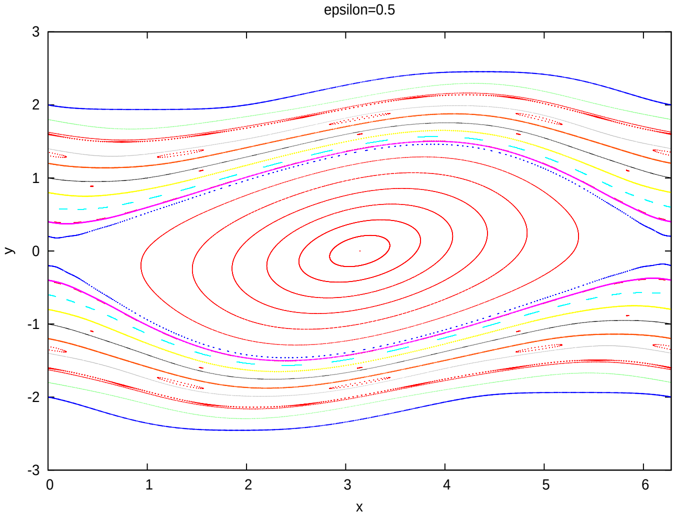
$\varepsilon = 0.2$: increasing the perturbation, the amplitude of the islands increases.



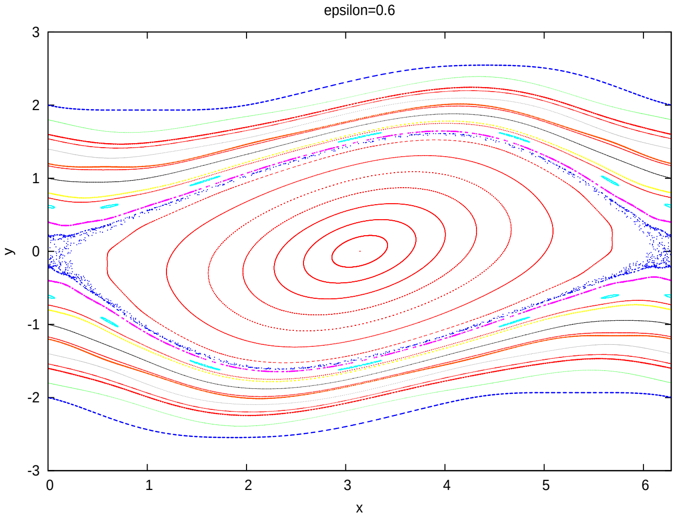
$\varepsilon = 0.3$: The amplitude of the islands increases more.



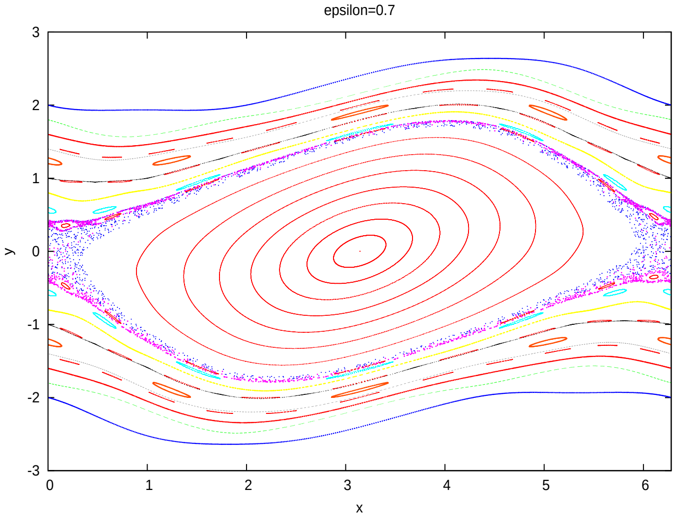
$\varepsilon = 0.4$: ... and more... minor resonances appear.



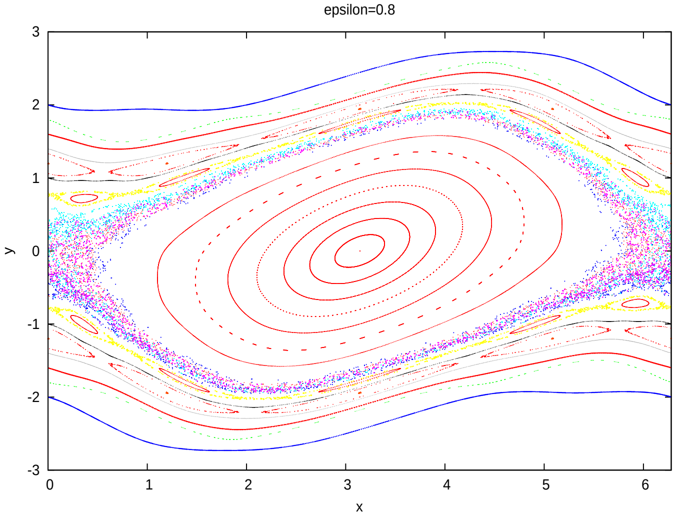
$\varepsilon = 0.5$: ... other minor resonances.



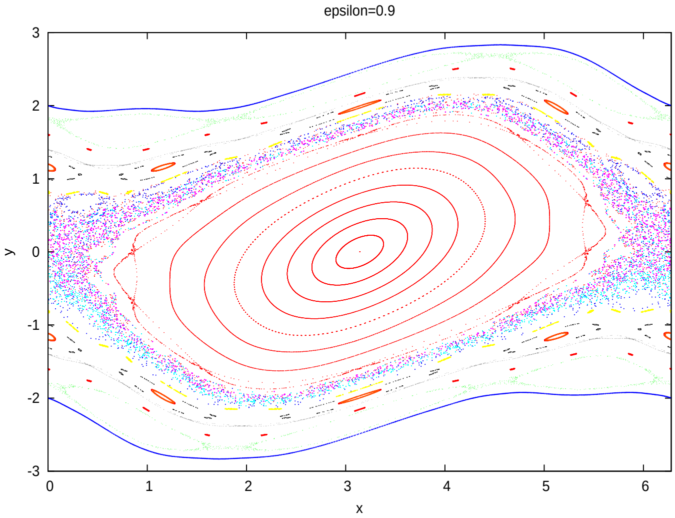
$\varepsilon = 0.6$: A marked chaotic region around the unstable point.



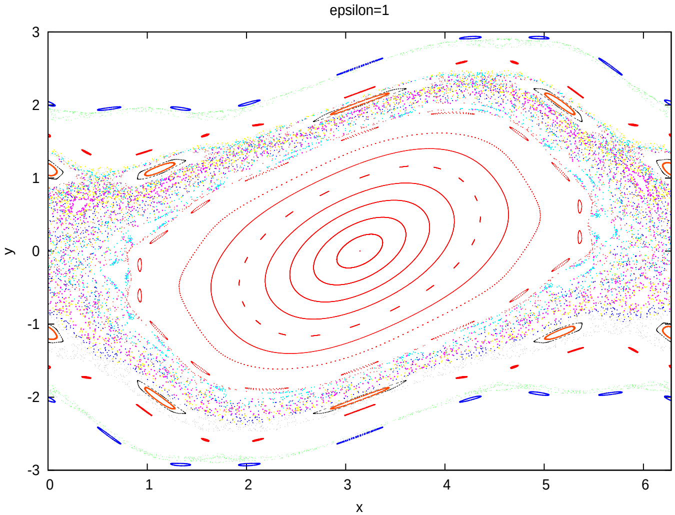
$\varepsilon = 0.7$: the chaotic region increases in size...



$\varepsilon = 0.8$: less and less rotational tori survive.



$\varepsilon = 0.9$: for a large perturbation, a lot of chaos, a few quasi–periodic curves, islands around higher–order periodic orbits.



$\varepsilon = 1$: very large perturbation, no more quasi–periodic curves.

Figure: Conservative Chirikov standard map as ε varies.

Summary

$\diamond$ For $\varepsilon = 0$ one gets an *integrable* mapping, since the dynamics can be exactly solved: all motions are periodic or quasi–periodic. A *non–integrable* system occurs when $\varepsilon \neq 0$.

$\diamond$ For $\varepsilon \neq 0$ but sufficiently small, the quasi–periodic invariant curves are slightly displaced and deformed w.r.t. the integrable case. Periodic orbits are surrounded by *librational curves.*

$\diamond$ As ε increases the rotational curves are more and more deformed and distorted, while the librational curves increase their amplitude; chaotic motions start to appear and they fill an increasing region as ε grows. Close to criticality invariant tori leave place to *cantori,* which are still invariant sets, but they are graphs of a Cantor set.

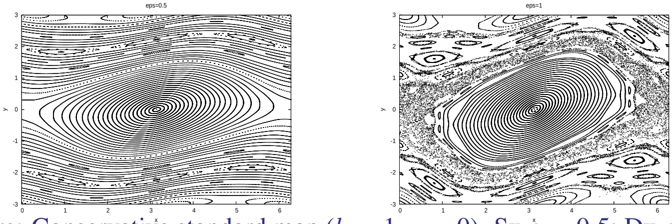


Figure: Conservative standard map ($b = 1$, $c = 0$). Sx $\varepsilon = 0.5$; Dx $\varepsilon = 1$.

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Dissipative Standard Map:

It is described by the equations (discrete analogue of the spin-orbit problem with tidal torque)

$$\begin{aligned} y' &= \lambda y + \mu + \varepsilon\, g(x) && y \in \mathbb{R}\,,\ x \in \mathbb{T} \\ x' &= x + y'\,, && \lambda, \mu, \varepsilon \in \mathbb{R}\,, \quad \varepsilon \geq 0\,, \end{aligned}$$

$0 < \lambda < 1$ dissipative parameter, μ = drift parameter.

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$0 < \lambda < 1$ dissipative parameter, μ = drift parameter.

• PROPERTIES:
• $\lambda = 1$, $\mu = 0$ one recovers the conservative SM.
• $\lambda = 0$ one obtains the one–dimensional mapping $x' = x + \mu + \varepsilon g(x)$.
• $\lambda = 0$ and $\varepsilon = 0$ one obtains the circle map $x' = x + \mu$.
• $\lambda \neq 1$, dissipative, since the determinant of the Jacobian amounts to λ.

• The drift μ plays a very important role. In fact, consider $\varepsilon = 0$ and look for an invariant solution, such that

$$y' = y \qquad \Rightarrow \qquad \lambda y + \mu = y \qquad \Rightarrow \qquad y = \frac{\mu}{1-\lambda}\ .$$

If $\mu = 0$, then $y = 0$!

• This shows that for $\varepsilon = 0$ the trajectory $\{y = \frac{\mu}{1-\lambda}\} \times \mathbb{T}$ is invariant.

• The dynamics associated to the DSM admits attracting periodic orbits, invariant curve attractors as well as strange attractors, which have an intricate geometrical structure; introducing a suitable definition of dimension, the strange attractors are shown to have a non–integer dimension (namely a *fractal* dimension).

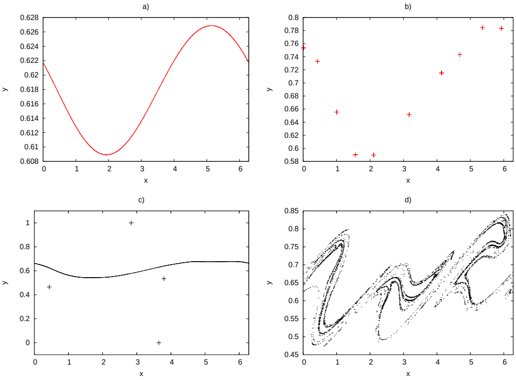

Figure: SMD attractors. a) Invariant attractor; b) periodic of period 10; c) invariant attractor coexisting with 0/1, 1/2, 1/1 periodic orbits; d) strange attractor.

• Basins of attraction for the coexisting case (500×500 random initial conditions with preliminary iterations).

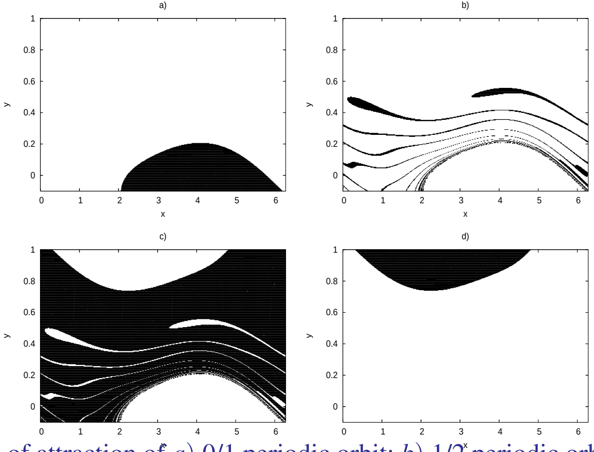

Figure: Basins of attraction of a) 0/1 periodic orbit; b) 1/2 periodic orbit; c) quasi–periodic attractor; d) 1/1 periodic orbit.

Figure: Dissipative standard map as ε varies for $\lambda = 0.8$.

Outline

4-dimensional standard map

• An extension of the standard map to study higher dimensional systems is the 4-dimensional standard map:

$$\begin{aligned} y_1' &= y_1 + \varepsilon\ \left(g_1(x_1) + \eta\, r_1(x_1, x_2)\right) \\ x_1' &= x_1 + y_1' \\ y_2' &= y_2 + \varepsilon\ \left(g_2(x_1) + \eta\, r_2(x_1, x_2)\right) \\ x_2' &= x_2 + y_2'\ . \end{aligned}$$

• When the coupling parameter $\eta = 0$, we have 2 uncoupled standard maps.
• When $\eta \neq 0$, we have coupled equations.

Outline

Non-twist standard map

• An extension of the standard map to non-twist maps was introduced by del-Castillo-Negrete and Morrison

$$\begin{aligned} y' &= y + \varepsilon \sin(x) \\ x' &= x + a(1 - y'^2) \end{aligned}$$

for $a \in \mathbb{R}$. The map is area-preserving, but violates the twist condition:

$$\frac{\partial x'}{\partial y} = -2a(y + \varepsilon \sin x) = 0$$

along the curve $y = -\varepsilon \sin x$.