Perturbation theory, KAM theory and Celestial Mechanics
5. An example and resonant perturbation theory

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Outline

1. Example of classical perturbation theory

2. Resonant perturbation theory

3. Example of resonant perturbation theory

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Example of classical perturbation theory

• We apply classical perturbation theory to the two–dimensional Hamiltonian function

$$\mathcal{H}(I_1, I_2, \varphi_1, \varphi_2) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon \Big[\cos(\varphi_1 + \varphi_2) + 2\cos(\varphi_1 - \varphi_2) \Big] \ ,$$

which can be shortly written as

$$\mathcal{H}(I_1, I_2, \varphi_1, \varphi_2) = h(I_1, I_2) + \varepsilon f(\varphi_1, \varphi_2), \qquad (1)$$

where

$$h(I_1, I_2) = \frac{I_1^2}{2} + \frac{I_2^2}{2}$$

and

$$f(\varphi_1, \varphi_2) = \cos(\varphi_1 + \varphi_2) + 2\cos(\varphi_1 - \varphi_2).$$

• We implement constructively the proof of the Theorem: look for a generating function, expand in Taylor series, solve the normal form equation, expand in Fourier series, determine the solution.

• Look for a change of coordinates with unknown Φ:

$$
\begin{aligned}
I_1 &= I_1' + \varepsilon \frac{\partial \Phi}{\partial \varphi_1}(I_1', I_2', \varphi_1, \varphi_2) \\
I_2 &= I_2' + \varepsilon \frac{\partial \Phi}{\partial \varphi_2}(I_1', I_2', \varphi_1, \varphi_2) \\
\varphi_1' &= \varphi_1 + \varepsilon \frac{\partial \Phi}{\partial I_1'}(I_1', I_2', \varphi_1, \varphi_2) \\
\varphi_2' &= \varphi_2 + \varepsilon \frac{\partial \Phi}{\partial I_2'}(I_1', I_2', \varphi_1, \varphi_2) \, .
\end{aligned}
$$

• Expanding the Hamiltonian (1) in Taylor series up to the second order:

$$
\begin{aligned}
& h(I_1'+\varepsilon\frac{\partial\Phi}{\partial\varphi_1},I_2'+\varepsilon\frac{\partial\Phi}{\partial\varphi_2})+\varepsilon f(\varphi_1,\varphi_2) \\
= \quad & h(I_1',I_2')+\varepsilon\frac{\partial h}{\partial I_1}(I_1',I_2')\frac{\partial\Phi}{\partial\varphi_1}+\varepsilon\frac{\partial h}{\partial I_2}(I_1',I_2')\frac{\partial\Phi}{\partial\varphi_2}+\varepsilon f(\varphi_1,\varphi_2)+O(\varepsilon^2)\ ,
\end{aligned}
$$

where

$$
\frac{\partial h}{\partial I_1}(I_1',I_2')=I_1'\equiv\omega_1(I_1')\ , \qquad \frac{\partial h}{\partial I_2}(I_1',I_2')=I_2'\equiv\omega_2(I_2')\ ,
$$

which means that the frequencies coincide with the actions.

• Note that in our case the average of f is zero and therefore:

$$
\widetilde{f}(I_1',I_2',\varphi_1,\varphi_2)=f(\varphi_1,\varphi_2)\ .
$$

• The first order terms in ε must be zero; this yields the generating function as the solution of the equation

$$\omega_1 \frac{\partial \Phi}{\partial \varphi_1} + \omega_2 \frac{\partial \Phi}{\partial \varphi_2} = -f(\varphi_1, \varphi_2) \ .$$

• Expand Φ in Fourier series:

$$\Phi(I_1', I_2', \varphi_1, \varphi_2) = \sum_{m,n} \Phi_{m,n}(I_1', I_2') e^{i(m\varphi_1 + n\varphi_2)} \ .$$

Therefore:

$$\frac{\partial \Phi(I_1', I_2', \varphi_1, \varphi_2)}{\partial \varphi_1} = \sum_{m,n} i\, m\, \Phi_{m,n}(I_1', I_2') e^{i(m\varphi_1 + n\varphi_2)} \ ,$$

and

$$\frac{\partial \Phi(I_1', I_2', \varphi_1, \varphi_2)}{\partial \varphi_2} = \sum_{m,n} i\, n\, \Phi_{m,n}(I_1', I_2') e^{i(m\varphi_1 + n\varphi_2)} \ .$$

• Take into account the explicit form of the perturbation:

$$\sum_{m,n} i(\omega_1 m+\omega_2 n)\Phi_{m,n}(I_1',I_2')e^{i(m\varphi_1+n\varphi_2)} = -\Big[\cos(\varphi_1+\varphi_2)+2\cos(\varphi_1-\varphi_2)\Big]\,.$$

• Use the relations

$$\begin{aligned}\cos(\varphi_1+\varphi_2) &= \frac{1}{2}\Big(e^{i(\varphi_1+\varphi_2)}+e^{-i(\varphi_1+\varphi_2)}\Big)\\ \cos(\varphi_1-\varphi_2) &= \frac{1}{2}\Big(e^{i(\varphi_1-\varphi_2)}+e^{-i(\varphi_1-\varphi_2)}\Big)\,,\end{aligned}$$

so that we have:

$$\begin{aligned}&\sum_{m,n} i(\omega_1 m+\omega_2 n)\Phi_{m,n}(I_1',I_2')e^{i(m\varphi_1+n\varphi_2)} =\\ &= -\Big[\frac{1}{2}(e^{i(\varphi_1+\varphi_2)}+e^{-i(\varphi_1+\varphi_2)})+e^{i(\varphi_1-\varphi_2)}+e^{-i(\varphi_1-\varphi_2)}\Big]\,.\end{aligned}$$

• Again:

$$\sum_{m,n} i(\omega_1 m + \omega_2 n)\Phi_{m,n}(I_1', I_2')e^{i(m\varphi_1+n\varphi_2)} =$$

$$= \quad -\Big[\frac{1}{2}(e^{i(\varphi_1+\varphi_2)} + e^{-i(\varphi_1+\varphi_2)}) + e^{i(\varphi_1-\varphi_2)} + e^{-i(\varphi_1-\varphi_2)}\Big] \,.$$

• Equating the coefficients with the same Fourier indexes, one gets:

$$\Phi_{1,1} = -\frac{1}{2i(\omega_1+\omega_2)}\,, \qquad \Phi_{-1,-1} = \frac{1}{2i(\omega_1+\omega_2)}\,,$$

$$\Phi_{1,-1} = -\frac{1}{i(\omega_1-\omega_2)}\,, \qquad \Phi_{-1,1} = -\frac{1}{i(-\omega_1+\omega_2)}\,.$$

• Casting together the above terms, the generating function is given by

$$\begin{aligned}\Phi(I_1', I_2', \varphi_1, \varphi_2) &= -\frac{1}{\omega_1+\omega_2}\left(\frac{e^{i(\varphi_1+\varphi_2)}-e^{-i(\varphi_1+\varphi_2)}}{2i}\right)\\ &- \frac{2}{\omega_1-\omega_2}\left(\frac{e^{i(\varphi_1-\varphi_2)}-e^{-i(\varphi_1-\varphi_2)}}{2i}\right),\end{aligned}$$

namely

$$\Phi(I_1', I_2', \varphi_1, \varphi_2) = -\frac{1}{\omega_1+\omega_2}\sin(\varphi_1+\varphi_2) - \frac{2}{\omega_1-\omega_2}\sin(\varphi_1-\varphi_2)\ .$$

• The generating function is not defined when there appear these zero divisors:

$$\omega_1 \pm \omega_2 = 0\ , \qquad \text{namely} \qquad I_1' = \pm I_2'\ .$$

• The new unperturbed Hamiltonian coincides with the old one (in the new set of variables), since the average of the perturbing function is zero:

$$h'(I_1', I_2') = \frac{I_1'^2}{2} + \frac{I_2'^2}{2}\ .$$

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Resonant perturbation theory

• Consider the following Hamiltonian system with n degrees of freedom

$$\mathcal{H}(\underline{I},\underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I},\underline{\varphi}) \ , \qquad \underline{I} \in \mathbb{R}^n \ , \ \ \underline{\varphi} \in \mathbb{T}^n$$

and let $\underline{\omega}(\underline{I}) = \frac{\partial h(\underline{I})}{\partial \underline{I}}$ be the frequency vector of the motion.

• We assume that the frequency vector satisfies ℓ resonance relations, with $\ell < n$, of the form

$$\underline{\omega} \cdot \underline{m}_k = 0 \qquad \text{for } k = 1, ..., \ell \ ,$$

for some vectors $\underline{m}_1, ..., \underline{m}_\ell \in \mathbb{Z}^n$.

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• Example: $\underline{\omega} = (1, 2, 3)$, then we have:

$$\begin{aligned} \underline{\omega} \cdot (11, -4, -1) &= 11 - 8 - 3 = 0 \\ \underline{\omega} \cdot (3, 0, -1) &= 3 - 3 = 0 \\ \underline{\omega} \cdot (16, -5, -2) &= 16 - 10 - 6 = 0\ . \end{aligned}$$

• A *resonant perturbation theory* can be implemented to eliminate the non–resonant terms.

• The aim is to construct a change of variables $\mathcal{C} : (\underline{I}, \underline{\varphi}) \to (\underline{I}', \underline{\varphi}')$, such that the new Hamiltonian takes the form

$$\mathcal{H}'(\underline{I}', \underline{\varphi}') = h'(\underline{I}', \underline{m}_1 \cdot \underline{\varphi}', ..., \underline{m}_\ell \cdot \underline{\varphi}') + \varepsilon^2 f'(\underline{I}', \underline{\varphi}') \ , \qquad (2)$$

where h' depends on $\underline{\varphi}'$ only through the combinations $\underline{m}_k \cdot \underline{\varphi}'$ with $k = 1, ..., \ell$.

• Notice that the combinations $\underline{m}_k \cdot \underline{\varphi}$ are slow:

$$\frac{d}{dt}(\underline{m}_k \cdot \underline{\varphi}) = \underline{m}_k \cdot \underline{\omega} = 0$$

and the rate of variation of $\underline{m}_k \cdot \underline{\varphi}'$ will be small.

• Let us first define the angles

$$\begin{aligned} \vartheta_j &= \underline{m}_j \cdot \underline{\varphi}\,, \qquad j = 1, ..., \ell \\ \vartheta_{j'} &= \underline{m}_{j'} \cdot \underline{\varphi}\,, \qquad j' = \ell + 1, ..., n\,, \end{aligned}$$

where the first ℓ angle variables are the resonant angles, while the latter $n - \ell$ angles are defined as arbitrary linear combinations with integer coefficients $\underline{m}_{j'}$.

• The corresponding actions are defined as

$$\begin{aligned} I_j &= \underline{m}_j \cdot \underline{J}\,, \qquad j = 1, ..., \ell \\ I_{j'} &= \underline{m}_{j'} \cdot \underline{J}\,, \qquad j' = \ell + 1, ..., n\,. \end{aligned}$$

• Next we construct a canonical transformation which removes (to higher orders) the dependence on the short–period angles $(\vartheta_{\ell+1}, ..., \vartheta_n)$, while the lowest order Hamiltonian will necessarily depend upon the resonant angles.

• To this end, let us first decompose the perturbation, expressed in terms of the variables $(\underline{J}, \underline{\vartheta})$, as

$$f(\underline{J}, \underline{\vartheta}) = \bar{f}(\underline{J}) + f_r(\underline{J}, \vartheta_1, ..., \vartheta_\ell) + f_n(\underline{J}, \underline{\vartheta}) \ , \tag{3}$$

$\bar{f}(\underline{J})$ is the average of the perturbation over the angles,
$f_r(\underline{J}, \vartheta_1, ..., \vartheta_\ell)$ is the part depending on the resonant angles,
$f_n(\underline{J}, \underline{\vartheta})$ is the non–resonant part.

• Example: assume that

$$f(\underline{J}, \underline{\vartheta}) = J_1^2 + J_2^3 + J_1 \cos(\vartheta_1 - \vartheta_2) + J_1 J_2^2 \cos(2\vartheta_1 + 3\vartheta_2) \ .$$

Assume that the resonance relation is:

$$\omega_1 - \omega_2 = 0 \ .$$

Then, we have:

$$\begin{aligned} \bar{f}(\underline{J}) &= J_1^2 + J_2^3 \\ f_r(J_1, J_2, \vartheta_1 \vartheta_2) &= J_1 \cos(\vartheta_1 - \vartheta_2) \\ f_n(J_1, J_2, \vartheta_1 \vartheta_2) &= J_1 J_2^2 \cos(2\vartheta_1 + 3\vartheta_2) \ . \end{aligned}$$

• In analogy to classical perturbation theory, we implement a canonical transformation of the form

$$\begin{aligned} \underline{J} &= \underline{J}' + \varepsilon \frac{\partial \Phi(\underline{J}', \underline{\vartheta})}{\partial \underline{\vartheta}} \\ \underline{\vartheta}' &= \underline{\vartheta} + \varepsilon \frac{\partial \Phi(\underline{J}', \underline{\vartheta})}{\partial \underline{J}'} \ , \end{aligned} \tag{4}$$

such that the new Hamiltonian is

$$\mathcal{H}'(\underline{J}', \underline{\vartheta}') = h'(\underline{J}', \underline{m}_1 \cdot \underline{\vartheta}', ..., \underline{m}_\ell \cdot \underline{\vartheta}') + \varepsilon^2 f'(\underline{J}', \underline{\vartheta}') \ .$$

• Using the decomposition of f and expanding up to 2^{nd} order in ε, one obtains:

$$\begin{aligned} & h(\underline{J}' + \varepsilon \frac{\partial \Phi}{\partial \underline{\vartheta}}) + \varepsilon f(\underline{J}', \underline{\vartheta}) + O(\varepsilon^2) \\ = \ & h(\underline{J}') + \varepsilon \sum_{k=1}^{n} \frac{\partial h}{\partial J_k} \frac{\partial \Phi}{\partial \vartheta_k} + \varepsilon \bar{f}(\underline{J}') + \varepsilon f_r(\underline{J}', \vartheta_1, ..., \vartheta_\ell) + \varepsilon f_n(\underline{J}', \underline{\vartheta}) + O(\varepsilon^2) \ . \end{aligned}$$

• We obtain the desired results

$$h'(\underline{J}', \vartheta_1, ..., \vartheta_\ell) = h(\underline{J}') + \varepsilon\bar{f}(\underline{J}') + \varepsilon f_r(\underline{J}', \vartheta_1, ..., \vartheta_\ell) \ , \tag{5}$$

provided

$$\sum_{k=1}^{n} \Omega_k \frac{\partial \Phi}{\partial \vartheta_k} = -f_n(\underline{J}', \underline{\vartheta}) \ , \tag{6}$$

where $\Omega_k = \Omega_k(\underline{J}') \equiv \frac{\partial h(\underline{J}')}{\partial J_k}$.

• The solution of (6) gives the generating function allowing to reduce the Hamiltonian to the required form; moreover, the conjugated action variables, say $J'_{\ell+1}$, ..., J'_n, are constants of the motion up to 2^{nd} order in ε, since (5) does not depend on $\vartheta_{\ell+1}, ..., \vartheta_n$.

• Notice that using the new frequencies Ω_k, the resonant relations take the form $\Omega_k = 0$ for $k = 1, ..., \ell$.

Outline

Example of resonant perturbation theory

• As an application of resonant perturbation theory we consider the three–body problem Hamiltonian

$$\mathcal{H}(L,G,\ell,g) \;=\; -\frac{\mu^2}{2L^2}+\varepsilon R(L,G,\ell,g)\ , \qquad (7)$$

with the perturbing function expanded as

$$\begin{aligned} R &= R_{00}(L,G)+R_{10}(L,G)\cos\ell+R_{11}(L,G)\cos(\ell+g) \\ &+ R_{12}(L,G)\cos(\ell+2g)+R_{22}(L,G)\cos(2\ell+2g) \\ &+ R_{32}(L,G)\cos(3\ell+2g)+R_{33}(L,G)\cos(3\ell+3g) \\ &+ R_{44}(L,G)\cos(4\ell+4g)+R_{55}(L,G)\cos(5\ell+5g)+\dots\ , \qquad (8) \end{aligned}$$

for some coefficients R_{ij}.

• Let $\underline{\omega} \equiv (\omega_\ell, \omega_g)$ be the frequency of motion; we assume that the following resonance relation is satisfied:

$$\omega_\ell + 2\omega_g = 0 \ .$$

• We perform the canonical change of variables

$$\begin{aligned} \vartheta_1 &= \ell + 2g \ , \qquad & J_1 = \frac{1}{2} G \ , \\ \vartheta_2 &= 2\ell \ , \qquad & J_2 = \frac{1}{2} L - \frac{1}{4} G \ . \end{aligned}$$

• In the new coordinates the unperturbed Hamiltonian takes the form

$$h(\underline{J}) \equiv -\frac{\mu^2}{2(J_1 + 2J_2)^2} - 2J_1 \ .$$

• The perturbing function is given by

$$\begin{aligned} R(J_1,J_2,\vartheta_1,\vartheta_2) &\equiv R_{00}(\underline{J})+R_{10}(\underline{J})\cos(\frac{1}{2}\vartheta_2)+R_{11}(\underline{J})\cos(\frac{1}{2}\vartheta_1+\frac{1}{4}\vartheta_2) \\ &+ R_{12}(\underline{J})\cos(\vartheta_1)+R_{22}(\underline{J})\cos(\vartheta_1+\frac{1}{2}\vartheta_2) \\ &+ R_{32}(\underline{J})\cos(\vartheta_1+\vartheta_2)+R_{33}(\underline{J})\cos(\frac{3}{2}\vartheta_1+\frac{3}{4}\vartheta_2) \\ &+ R_{44}(\underline{J})\cos(2\vartheta_1+\vartheta_2)+R_{55}(\underline{J})\cos(\frac{5}{2}\vartheta_1+\frac{5}{4}\vartheta_2)+... \end{aligned}$$

• Let us split the perturbation as

$$R=\overline{R}(\underline{J})+R_r(\underline{J},\vartheta_1)+R_n(\underline{J},\underline{\vartheta}) \ ,$$

where
$\overline{R}(\underline{J})$ is the average over the angles,
$R_r(\underline{J},\vartheta_1)=R_{12}(\underline{J})\cos(\vartheta_1)$ is the resonant part,
R_n contains all the remaining non–resonant terms.

• We look for a change of coordinates close to the identity with generating function $\Phi = \Phi(\underline{J}', \underline{\vartheta})$ such that

$$\underline{\Omega}(\underline{J}') \cdot \frac{\partial \Phi(\underline{J}', \underline{\vartheta})}{\partial \underline{\vartheta}} = -R_n(\underline{J}', \underline{\vartheta}) \ ,$$

being $\underline{\Omega}(\underline{J}') \equiv \frac{\partial h(\underline{J}')}{\partial \underline{J}}$.

• The above expression is well defined since $\underline{\Omega}$ is non–resonant for the Fourier components appearing in R_n. Finally, the new unperturbed Hamiltonian is given by

$$h'(\underline{J}', \vartheta_1) \equiv h(\underline{J}') + \varepsilon R_{00}(\underline{J}') + \varepsilon R_{12}(\underline{J}') \cos(\vartheta_1) \ .$$