Perturbation theory, KAM theory and Celestial Mechanics
6. Two models of Celestial Mechanics

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Outline

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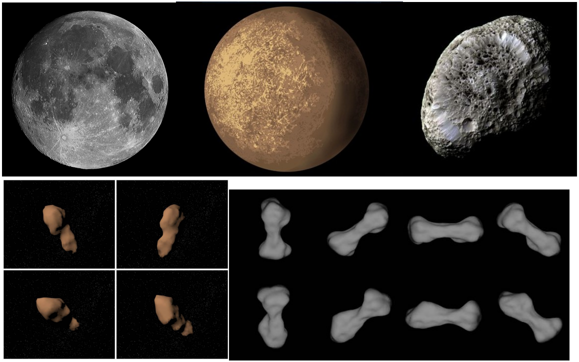
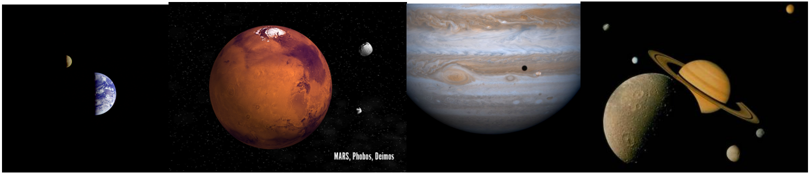
Rotational dynamics

• Rotational dynamics: different shapes

From round bodies (Moon, Mercury), to irregular bodies (Hyperion), to dumbbell satellite (4179 Toutatis, 216 Kleopatra)



• The **Moon** always points the same face to the Earth. All evolved satellites of the Solar System always point the same hemisphere to the host planet.
• **Mars:** Phobos, Deimos. **Jupiter:** Io, Europa, Ganimede, Callisto. **Saturn:** Titan, Rhea, Enceladus, Dione. **Uranus:** Ariel, Umbriel, Titania. **Neptune:** Triton, Proteus. **Pluto:** Charon.
• Only exception: **Mercury** in a 3:2 spin–orbit resonance.
Moon: 1:1 (1 rotation = 1 revolution), Mercury 3:2 spin–orbit resonance (3 rotations = 2 revolutions).



Rotational dynamics: consequences of its study

• Moon: physical librations due to earth tides, study of the internal composition (SMART 1)
• Mercury: study of the gravitational field, the variation of obliquity and libration provide constraints on the internal structure of the planet, such as the existence of a solid surface and a liquid core, thus provoking a dynamo effect responsible of Mercury's magnetic field (BepiColombo)
• Europa: mass distribution, rotation eventually compatible with a liquid ocean which could explain the tectonics (Voyager - Galileo)
• Enceladus: resonance conditions can be responsible of the heat excess and surface geysers
• Hyperion: example of chaotic rotation (in orbital resonance with Titan)
• Titan: an anomalous obliquity might be due to an internal ocean (Cassini–Huygens)

Outline

Conservative/Dissipative spin–orbit problem

• Model: satellite $\mathcal{S}$, ellipsoid rotating about an internal spin–axis and revolving around a central body $\mathcal{P}$:
(i) S moves on a Keplerian orbit;
(ii) the spin–axis coincides with the smallest physical axis (principal rotation);
(iii) the spin–axis is perpendicular to the orbital plane (zero obliquity);
(iv) dissipative forces: tidal torque $\mathcal{T}$ depending linearly on the angular velocity of rotation.

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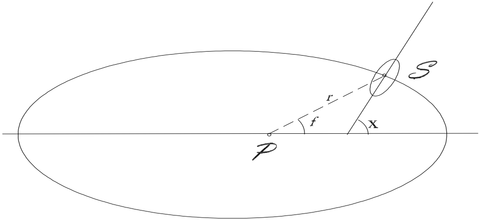
• Notation:
$A < B < C$ principal moments of inertia; $n = \frac{2\pi}{T_{rev}} \equiv 1$ mean motion; a semimajor axis; e eccentricity; r orbital radius; f true anomaly; x angle between pericenter line and major axis of the ellipsoid.

Conservative spin-orbit problem



- Neglecting the dissipation:

$$\ddot{x} + \frac{3}{2} \frac{B-A}{C} \left(\frac{a}{r}\right)^3 \sin(2x - 2f) = 0 \,. \tag{1}$$

$$\ddot{x} + \frac{3}{2}\frac{B-A}{C}(\frac{a}{r})^3 \sin(2x-2f) = 0 .$$

(i) $\varepsilon \equiv \frac{3}{2}\frac{B-A}{C}$ perturbing parameter; Moon–Mercury: $\varepsilon \simeq 10^{-4}$; if $\varepsilon = 0$ the system is integrable.

(ii) r and f are known Keplerian functions of the time:

$$\begin{aligned} r &= a(1-e\cos u) \\ f &= 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right) . \end{aligned}$$

(iii) r, f depend on e and for $e = 0$ one has $r = a, f = t + t_0$ for a suitable constant t_0; hence, for circular orbits one gets the integrable equation $\ddot{x} + \varepsilon\sin(2x - 2t - 2t_0) = 0$.

(iv) Considering the lift of the angle x on $\mathbf{R}$, a $p : q$ spin–orbit resonance for $p, q \in \mathbf{Z}$ with $q > 0$ is a periodic solution for the conservative equation, say $t \in \mathbf{R} \to x = x(t) \in \mathbf{R}$, such that

$$x(t+2\pi q) = x(t) + 2\pi p \qquad \text{for any } \ t \in \mathbf{R} .$$

• Expanding in power series of e and Fourier series, the spin-orbit eq. is

$$\ddot{x} \ + \ \varepsilon \sum_{m\neq 0, m=-\infty}^{+\infty} W(\frac{m}{2}, e) \ \sin(2x - mt) \ = \ 0 \ , \tag{2}$$

where the coefficients $W(\frac{m}{2}, e)$ decay as power series of e.

• Up to the order 4 in e, one obtains

$$\begin{aligned} \ddot{x} \quad &+ \quad \varepsilon \Big[\frac{e^4}{24}\sin(2x+2t) + \frac{e^3}{48}\sin(2x+t) + (-\frac{e}{2} + \frac{e^3}{16}) \ \sin(2x-t) \ + \\ &+ \quad (1 - \frac{5}{2}e^2 + \frac{13}{16}e^4) \ \sin(2x-2t) \ + \ (\frac{7}{2}e - \frac{123}{16}e^3) \ \sin(2x-3t) \ + \\ &+ \quad (\frac{17}{2}e^2 - \frac{115}{6}e^4) \ \sin(2x-4t) \ + \ \frac{845}{48}e^3 \ \sin(2x-5t) \ + \\ &+ \quad \frac{533}{16}e^4 \ \sin(2x-6t) \Big] \ = \ 0 \ . \end{aligned}$$

• The previous equation can be written in compact form as

$$\ddot{x} + \varepsilon V_x(x,t) = 0 \ ,$$

for a suitable periodic function $V = V(x,t)$. Such equation corresponds to that of a pendulum subject to a forcing term, depending periodically upon time.

• In Hamiltonian form it is:

$$\mathcal{H}(y,x,t) = \frac{1}{2} y^2 + \varepsilon V(x,t) \ .$$

The Hamiltonian is integrable for $\varepsilon = 0$, nearly-integrable for $\varepsilon \neq 0$.

Dissipative spin-orbit problem

• Tidal torque $\mathcal{T}$ due to internal non–rigidity: as in [Correia–Laskar] average over one orbital period:

$$\langle \mathcal{T} \rangle = -\mu(e, K) \Big[\dot{x} - \eta(e) \Big] ,$$

with

$$\mu(e, K) = K \frac{1 + 3e^2 + \frac{3}{8} e^4}{(1 - e^2)^{9/2}} , \quad \eta(e) = \frac{1 + \frac{15}{2} e^2 + \frac{45}{8} e^4 + \frac{5}{16} e^6}{(1 + 3e^2 + \frac{3}{8} e^4)(1 - e^2)^{3/2}} .$$

• The quantity $K \equiv 3n \, \frac{k_2}{\xi Q} (\frac{R_e}{a})^3 \frac{M}{m}$, where $n =$ mean motion, $k_2 =$ *Love number* (depending on the structure of the body), $Q =$ *quality factor* (which compares the frequency of oscillation of the system to the rate of dissipation of energy), ξ is a structure constant such that $I_3 = \xi m R_e^2$, $R_e =$ equatorial radius, $M =$ mass of the central body, $m =$ mass of the satellite.
• $K \simeq 10^{-8}$ for Moon–Mercury depending on the physical and orbital characteristics)

• We are led to consider the following equation of motion for the dissipative spin–orbit problem:

$$\ddot{x} \ + \ \frac{3}{2} \ \frac{B-A}{C} (\frac{a}{r})^3 \ \sin(2x-2f) = -\mu[\dot{x}-\eta]$$

or

$$\ddot{x} \ + \varepsilon V_x(x,t) = -\mu[\dot{x}-\eta] \ . \tag{3}$$

• The tidal torque vanishes provided

$$\dot{x} \equiv \eta(e) = \frac{1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6}{(1-e^2)^{\frac{3}{2}}(1+3e^2+\frac{3}{8}e^4)} \ .$$

• It is readily shown that for circular orbits the angular velocity of rotation corresponds to the synchronous resonance, being $\dot{x}=1$. For Mercury's eccentricity $e=0.2056$, it turns out that $\dot{x}=1.256$.

• Poincaré sections in the plane (x, y), conservative and dissipative settings, different values of the eccentricity.

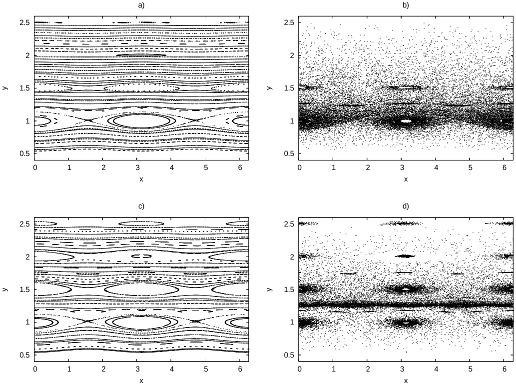


Figure: (a) $e = 0.0549$, $\varepsilon = 10^{-3}$, $K = 0$; (b) $e = 0.0549$, $\varepsilon = 10^{-3}$, $K = 10^{-3}$; (c) $e = 0.2056$, $\varepsilon = 10^{-3}$, $K = 0$; (d) $e = 0.2056$, $\varepsilon = 10^{-3}$, $K = 10^{-3}$.

• SM corresponds to the Poincaré map at times 2π, obtained integrating the conservative spin–orbit problem with a leap–frog method.
• DSM corresponds to the Poincaré map at times 2π, obtained integrating the dissipative spin–orbit problem with a leap–frog method.

$$\ddot{x} + \varepsilon V_x(x,t) = -\mu[\dot{x} - \eta] .$$

is equivalent to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\varepsilon V_x(x,t) - \mu[y-\eta] , \end{aligned}$$

which can be integrated through a leap–frog method with time-step T as

$$\begin{aligned} y' &= (1-\mu T)y + \mu\eta T - \varepsilon V_x(x,t)\, T \\ x' &= x + y' . \end{aligned}$$

Outline

Delaunay action–angle variables

• Action–angle variables for the two–body problem $\mathcal{P}_1 - \mathcal{P}_2$ are known as Delaunay variables.

• Let (r, ϑ) be the polar coordinates and let (p_r, p_ϑ) be the conjugated momenta. It is readily seen that $p_\vartheta = h$, h being the angular momentum.

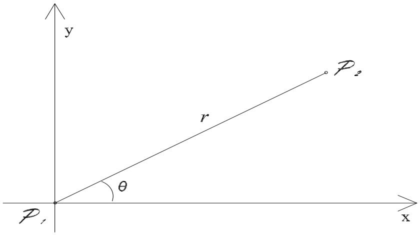


Figure: Geometrical configuration of Kepler's problem.

• The Hamiltonian function governing the two–body motion is given by ($\kappa = G(m_1 + m_2)$)

$$\mathcal{H}(p_r, p_\vartheta, r, \vartheta) = \frac{1}{2}(p_r^2 + \frac{p_\vartheta^2}{r^2}) - \frac{\kappa}{r} \ .$$

• Being ϑ a cyclic variable, we introduce the effective potential (see Figure 3) as

$$V_e(r) = \frac{p_\vartheta^2}{2r^2} - \frac{\kappa}{r} \ . \tag{4}$$

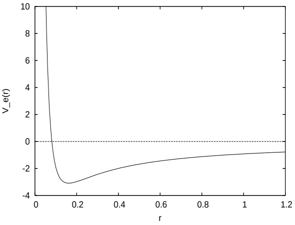


Figure: Graph of the effective potential $V_e(r)$ given in (4) for $p_\vartheta = 0.4025$ and $\kappa = 1$.

• The Hamiltonian can be written as the one–dim. Hamiltonian:

$$\mathcal{H}(p_r, r) = \frac{p_r^2}{2} + V_e(r) \ .$$

• Taking into account that ϑ is cyclic, let us define the Delaunay action variables L_0, G_0 as

$$\begin{aligned} L_0 &\equiv \sqrt{\kappa a} \\ G_0 &\equiv p_\vartheta = h = \sqrt{\kappa a(1-e^2)} = L_0\sqrt{1-e^2} \ . \end{aligned}$$

• Notice that one can express the elliptic elements a, e in terms of the Delaunay action variables as

$$a = \frac{L_0^2}{\kappa} \ , \qquad e = \sqrt{1 - \frac{G_0^2}{L_0^2}} \ .$$

• The Hamiltonian function expressed in terms of the action variables becomes

$$\mathcal{H} = \mathcal{H}(L_0) = -\frac{\kappa^2}{2L_0^2} \ . \tag{5}$$

• The Delaunay angle variables are the mean anomaly

$$\ell_0 \equiv n(t - t_0) = \frac{2\pi}{T}(t - t_0)$$

and the argument of perihelion g_0.

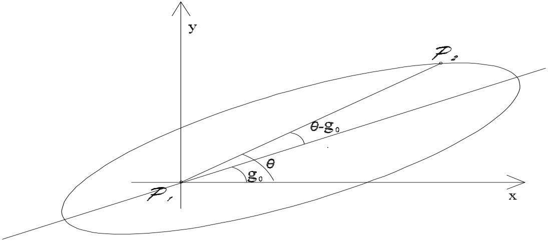


Figure: The argument of perihelion g_0.

Outline

Mean, eccentric anomaly and Kepler's equation

- We introduce as follows a quantity u called the *eccentric anomaly:*

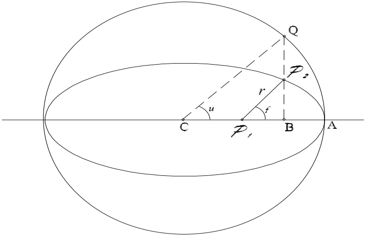


Figure: The eccentric anomaly u.

- It follows that

$$\begin{aligned} r &= a(1 - e\cos u) \\ \tan\frac{f}{2} &= \sqrt{\frac{1+\mathrm{e}}{1-\mathrm{e}}}\,\tan\frac{u}{2} \\ \ell_0 &= u - e\sin u\,, \end{aligned}$$

the latter known as *Kepler's equation.*

• Solve this equation to get u as a function of the time, being $\ell_0 = n(t - t_0)$ as well as $u = u(t)$; insert it in the previous relations to obtain $r = r(t), f = f(t)$.

• An approximate solution can be computed as far as e is small. Indeed, the inversion of Kepler's equation provides u as a function of ℓ_0 as a series of e:

$$\begin{aligned} u &= \ell_0 + e \sin u \\ &= \ell_0 + e \sin(\ell_0 + e \sin u) \\ &= \ell_0 + e \sin(\ell_0 + e \sin(\ell_0 + e \sin u)) \\ &= \ell_0 + (e - \frac{e^3}{8}) \sin \ell_0 + \frac{1}{2} e^2 \sin(2\ell_0) + \frac{3}{8} e^3 \sin(3\ell_0) + O(e^4) \ , \end{aligned}$$

where $O(e^4)$ denotes a quantity of order e^4.

• The complete solution can be expressed as

$$u = \ell_0 + e \sum_{k=1}^{\infty} \frac{1}{k} \Big[J_{k-1}(ke) + J_{k+1}(ke) \Big] \sin(k\ell_0) \ , \tag{6}$$

where $J_k(x)$ are the *Bessel's functions* of order k and argument x; they are defined by the relation

$$J_k(x) \equiv \frac{1}{2\pi} \int_0^{2\pi} \cos(kt - x \sin t) dt \ .$$

• The functions $J_k(x)$ can be developed as follows:

$$\begin{aligned} J_0(x) &\equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \Big(\frac{x}{2}\Big)^{2m} \\ J_k(x) &\equiv \Big(\frac{x}{2}\Big)^k \frac{1}{k!} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \prod_{j=1}^{m}(k+j)} \Big(\frac{x}{2}\Big)^{2m} \ . \end{aligned} \tag{7}$$

Outline

The restricted three–body problem

• Consider a particle (i.e. an asteroid) under the influence of 2 primaries $\mathcal{P}_1$, $\mathcal{P}_2$ with masses m_1, m_2 (i.e. Sun and Jupiter). Assume that

◇ all bodies move on the same plane;

◇ the mass of the particle is so small that it does not influence the primaries;

◇ the primaries move on circular Keplerian orbits.

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◇ all bodies move on the same plane;

◇ the mass of the particle is so small that it does not influence the primaries;

◇ the primaries move on circular Keplerian orbits.

• This problem is named the restricted, circular, planar 3–body problem (RCPTBP) $\to$ described by a 2 d.o.f. Hamiltonian:

$$H(L, G, \ell, g; \varepsilon) = -\frac{1}{2L^2} - G + \varepsilon F_\varepsilon(L, G, \ell, g; \varepsilon)\ .$$

• *Angle variables:* ℓ is the mean-anomaly, $g = g_0 - \psi$ with $g_0 =$ argument of the perihelion, $\psi =$ longitude of $\mathcal{P}_2$, coinciding with time if the common frequency of the primaries is 1 and if $m_1 + m_2 = 1$.

• *Action variables:* $L = \sqrt{\kappa a}$ and $G = L\sqrt{1 - e^2}$.

• Perturbative parameter $\varepsilon = m_2/(m_1 + m_2)$.

- About the perturbation $F_\varepsilon(L, G, \ell, g; \varepsilon)$.

- Setting $x^{(2)}$ the Jupiter–Sun vector, $x^{(A)}$ the asteroid–Sun vector, the perturbation is

$$F_\varepsilon = x^{(A)} \cdot x^{(2)} - \frac{1}{|x^{(A)} - x^{(2)}|} \ ,$$

expressed in terms of the Delaunay variables, with $x^{(2)}$ being the relative circular motion of $\mathcal{P}_1$: $x^{(2)} = \big(\cos(t_0 + t), \sin(t_0 + t)\big)$.

- Expanding in Fourier-Taylor series:

$$\begin{aligned}
F_\varepsilon(L, G, \ell, g) &= -(1 + \frac{a^2}{4} + \frac{9}{64}\, a^4 + \frac{3}{8}\, a^2 e^2) \\
&+ \Big(\frac{1}{2} + \frac{9}{16} a^2\Big)\, a^2 e\, \cos\ell - \Big(\frac{3}{8}\, a^3 + \frac{15}{64} a^5\Big) \cos(\ell + g) \\
&+ \Big(\frac{9}{4} + \frac{5}{4} a^2\Big)\, a^2 e\, \cos(\ell + 2g) - \Big(\frac{3}{4}\, a^2 + \frac{5}{16}\, a^4\Big) \cos(2\,\ell + 2\,g) \\
&- \frac{3}{4}\, a^2 e \cos(3\,\ell + 2\,g) - \Big(\frac{5}{8}\, a^3 + \frac{35}{128}\, a^5\Big) \cos(3\,\ell + 3\,g) \\
&- \frac{35}{64}\, a^4 \cos(4\,\ell + 4\,g) - \frac{63}{128} a^5 \cos(5\ell + 5g)\ .
\end{aligned}$$