Perturbation theory, KAM theory and Celestial Mechanics
8. Proof of the KAM theorem

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Outline

1. Sketch of the Proof for CS systems

2. The a-posteriori approach

3. KAM algorithm

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Sketch of the Proof for CS systems [Calleja, Celletti, de la Llave 2013]

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Step 2: determine the new approximation
Step 3: solve the cohomological equation
Step 4: convergence of the iterative step
Step 5: local uniqueness

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- Analytic tools:
  - exponential decay of Fourier coefficients of analytic functions;
  - estimates to bound the derivatives in smaller domains;
  - quantitative analysis of the cohomology equations;
  - abstract implicit function theorem.

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Step 1: approximate solution and linearization

- Let (K, μ) be an approximate solution: $f_\mu \circ K(\theta) - K(\theta+\omega) = E(\theta)$.

- Using the Lagrangian property in coordinates, $DK^T(\theta)\, J \circ K(\theta)\, DK(\theta) = 0$, the tangent space is

$$\text{Range}\Big(DK(\theta)\Big) \oplus \text{Range}\Big(V(\theta)\Big)$$

with $V(\theta) = J^{-1} \circ K(\theta)\, DK(\theta) N(\theta)$ and $N(\theta) = (DK(\theta)^T\, DK(\theta))^{-1}$.

- Define:

$$M(\theta) = [DK(\theta) \mid V(\theta)]\ .$$

Lemma

Up to a remainder R:

$$Df_\mu \circ K(\theta)\, M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} + R(\theta)\,. \qquad (R)$$

Proof: Recall $M(\theta) = [DK(\theta) \mid V(\theta)]$.

Part 1: taking the derivative of $f_\mu \circ K(\theta) = K(\theta+\omega) + E(\theta)$, one gets $Df_\mu \circ K(\theta)\, DK(\theta) = DK(\theta+\omega) + DE(\theta)$;

Part 2: due to the remark on the tangent space, one has:

$$Df_\mu \circ K(\theta)\, V(\theta) = DK(\theta+\omega)\, S(\theta) + V(\theta+\omega)\, \lambda \mathrm{Id} + h.o.t.$$

with

$$\begin{aligned} S(\theta) \;\equiv\;& N(\theta+\omega)^T DK(\theta+\omega)^T Df_\mu \circ K(\theta)\, J^{-1} \circ K(\theta)\, DK(\theta) N(\theta) \\ -\;& N(\theta+\omega)^T DK(\theta)^T J^{-1} \circ K(\theta) DK(\theta) N(\theta+\omega) \lambda \mathrm{Id}\,. \end{aligned}$$

Step 2: determine a new approximation $K' = K + MW$, $\mu' = \mu + \sigma$ satisfying

$$f_{\mu'} \circ K'(\theta) - K'(\theta + \omega) = E'(\theta) \, . \qquad (APPR - INV)'$$

• Expanding in Taylor series:

$$f_\mu \circ K(\theta) + Df_\mu \circ K(\theta)\, M(\theta) W(\theta) + D_\mu f_\mu \circ K(\theta) \sigma$$
$$- K(\theta + \omega) - M(\theta + \omega)\, W(\theta + \omega) + h.o.t. = E'(\theta) \, .$$

• Recalling that $f_\mu \circ K(\theta) - K(\theta + \omega) = E(\theta)$, the new error E' is quadratically smaller provided:

$$Df_\mu \circ K(\theta)\, M(\theta) W(\theta) - M(\theta + \omega)\, W(\theta + \omega) + D_\mu f_\mu \circ K(\theta) \sigma = -E(\theta) \, .$$

• Combine the previous formula

$$Df_\mu \circ K(\theta)\, M(\theta) W(\theta) - M(\theta+\omega)\, W(\theta+\omega) + D_\mu f_\mu \circ K(\theta)\sigma = -E(\theta)$$

and the Lemma:

$$Df_\mu \circ K(\theta)\, M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} + R(\theta)\,, \qquad (R)$$

to get equations for $W = (W_1, W_2)$ and σ:

$$\underline{M(\theta+\omega)} \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} W(\theta) - \underline{M(\theta+\omega)}\, W(\theta+\omega) = -E(\theta) - D_\mu f_\mu \circ K(\theta)\sigma\,.$$

• Multiplying by $M(\theta+\omega)^{-1}$ and writing $W=(W_1,W_2)$, one gets

$$\begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} \left(\begin{array}{c} W_1(\theta) \\ W_2(\theta) \end{array} \right) - \left(\begin{array}{c} W_1(\theta+\omega) \\ W_2(\theta+\omega) \end{array} \right) = \left(\begin{array}{c} -\tilde{E}_1(\theta) - \tilde{A}_1(\theta)\sigma \\ -\tilde{E}_2(\theta) - \tilde{A}_2(\theta)\sigma \end{array} \right) .$$

with $\tilde{E}_j(\theta) = -(M(\theta+\omega)^{-1}E)_j$, $\tilde{A}_j(\theta) = (M(\theta+\omega)^{-1}D_\mu f_\mu \circ K)_j$.

• In components:

$$\begin{aligned} W_1(\theta) - W_1(\theta+\omega) &= -\widetilde{E}_1(\theta) - S(\theta)W_2(\theta) - \widetilde{A}_1(\theta)\,\sigma && (A) \\ \lambda W_2(\theta) - W_2(\theta+\omega) &= -\widetilde{E}_2(\theta) - \widetilde{A}_2(\theta)\,\sigma && (B) \end{aligned}$$

• Cohomological eq.s with constant coefficients for (W_1, W_2), σ for known S, $\widetilde{E} \equiv (\widetilde{E}_1, \widetilde{E}_2)$, $\widetilde{A} \equiv [\widetilde{A}_1 | \widetilde{A}_2]$:

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• (A) involves small (zero) divisors, since for $k = 0$ one has $1 - e^{ik\cdot\omega} = 0$ in

$$W_1(\theta) - W_1(\theta+\omega) = \sum_k \widehat{W}_{1,k}\, e^{ik\cdot\theta} (1 - e^{ik\cdot\omega})\ .$$

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• Non–degeneracy condition: computing the averages of eqs. (A), (B), determine $\langle W_2 \rangle$, σ by solving ($W_2 = \langle W_2 \rangle + B^0 + \sigma \tilde{B}^0$)

$$\begin{pmatrix} \langle S \rangle & \langle SB^0 \rangle + \langle \widetilde{A}_1 \rangle \\ (\lambda - 1)\mathrm{Id} & \langle \widetilde{A}_2 \rangle \end{pmatrix} \begin{pmatrix} \langle W_2 \rangle \\ \sigma \end{pmatrix} = \begin{pmatrix} -\langle S\tilde{B}^0 \rangle - \langle \widetilde{E}_1 \rangle \\ -\langle \widetilde{E_2} \rangle \end{pmatrix}\ .$$

Step 3: solve the cohomological equations

- Non–average parts of W_1, W_2: solve cohomological equations of the form

$$\lambda w(\theta) - w(\theta+\omega) = \eta(\theta)$$

with $\eta : \mathbb{T}^n \to \mathbb{C}$ known and with zero average.

Lemma

Let $|\lambda| \in [A, A^{-1}]$ for $0 < A < 1$, $\omega \in \mathcal{D}(C,\tau)$, $\eta \in \mathcal{A}_\rho$, $\rho > 0$ or $\eta \in H^m$, $m \geq \tau$, and

$$\int_{\mathbb{T}^n} \eta(\theta)\, d\theta = 0 \ .$$

Then, there is one and only one solution w with zero average and

$$\begin{aligned} \|w\|_{\mathcal{A}_{\rho-\delta}} &\leq C_6\, C\, \delta^{-\tau} \|\eta\|_{\mathcal{A}_\rho} \ , \\ \|w\|_{H^{m-\tau}} &\leq C_7\, C\, \|\eta\|_{H^m} \ . \end{aligned}$$

Sketch of the proof. Expand η as

$$\eta(\theta) = \sum_{j\in\mathbb{Z}^n} \widehat{\eta}_j e^{2\pi i j\cdot\theta}$$

and using

$$\lambda w(\theta) - w(\theta+\omega) = \eta(\theta)$$

find

$$\widehat{w}_j = (\lambda - e^{2\pi i j\cdot\omega})^{-1}\, \widehat{\eta}_j \;;$$

when $\lambda = 1, j = 0$, it must be $\widehat{\eta}_0 = 0$.

Estimate the multipliers using Cauchy bounds and use the Diophantine condition ([Rüssmann]).

Step 4: convergence of the iterative step

• The invariance equation is satisfied with an error quadratically smaller, i.e.

$$\|E'\|_{\mathcal{A}_{\rho-\delta}} \leq C_8 \delta^{-2\tau} \|E\|^2_{\mathcal{A}_\rho} \ , \qquad \|E'\|_{H^{m-\tau}} \leq C_9 \|E\|^2_{H^m} \ .$$

• The procedure can be iterated to get a sequence of approximate solutions, say $\{K_j, \mu_j\}$. Convergence: through an *abstract implicit function theorem,* alternating the iteration with carefully chosen smoothings operators defined in a scale of Banach spaces (analytic functions or Sobolev spaces).

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Step 5: local uniqueness

• Under smallness conditions, if there exist two solutions (K_a, μ_a), (K_b, μ_b), then there exists $\psi \in \mathbb{R}^n$ such that

$$K_b(\theta) = K_a(\theta + \psi) \qquad \text{and} \qquad \mu_a = \mu_b\ .$$

- Step 4 requires the following (technical) estimates.
- Given an approximate solution (K, E) with $\|E\|$ sufficiently small.

Lemma

The torus $K(\mathbb{T}^n)$ is approximately Lagrangian:

$$\begin{aligned} \|K^*\Omega\|_{\rho-\frac{\delta}{2}} &\leq C_0\, \delta^{-1}\, \|E\|_\rho \\ \|K^*\Omega\|_{m-1} &\leq C_0\, \|E\|_m\,. \end{aligned}$$

Lemma

Bound R in $Df_\mu \circ K(\theta)\, M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} + R(\theta)$ *by*

$$\begin{aligned} \|R\|_{\rho-\frac{\delta}{2}} &\leq C_0\, \delta^{-1}\, \|E\|_\rho \\ \|R\|_{m-1} &\leq C_0\, \|E\|_m\,. \end{aligned}$$

Lemma

Estimates for the corrections (W, σ)*:*

$$\begin{aligned} \|W\|_{\rho-\delta} &\leq C_0 \nu^{-1}\, \delta^{-\tau}\, \|E\|_\rho\,, \qquad & |\sigma| \leq C_0 \|E\|_\rho \\ \|W\|_{m-\tau} &\leq C_0 \nu^{-1}\, \|E\|_m\,, \qquad & |\sigma| \leq C_0 \|E\|_m\,. \end{aligned}$$

• In Nash–Moser theory the convergence of the iterative step is obtained implementing a quadratic iterative scheme. The quadratic estimates on the step amount to prove that:

Lemma

If $\|E\|$ *is sufficiently small:*

$$\begin{aligned} \|E(K + MW, \mu + \sigma)\|_{\rho-\delta} &\leq C_0 \nu^{-2} \delta^{-2\tau}\, \|E\|_\rho^2 \\ \|E(K + MW, \mu + \sigma)\|_{m-\tau} &\leq C_0 \nu^{-2}\, \|E\|_m^2\,. \end{aligned}$$

• **Abstract implicit function theorem.** Define a scale of Banach spaces with smoothing operators:

$$\mathcal{X}^0 \supseteq \mathcal{X}^{r'} \supseteq \mathcal{X}^r \supseteq \mathcal{X}^\infty , \qquad 0 \le r' \le r \le \infty ,$$

with norms satisfying $\|g\|_{\mathcal{X}^{r'}} \le \|g\|_{\mathcal{X}^r}$, $0 \le r' \le r$.

♢ Banach spaces: analytic or Sobolev spaces with smoothing operators.

Analytic: smoothing obtained by rescaling the size of the strip on which analytic functions are defined.

Sobolev: $\{S_t\}_{t\ge 0}$ is a family of smoothing operators such that it satisfies interpolation inequalities and $\widehat{(S_t u)_k} = \hat{u}_k\, e^{-|k|/t}$:

i) $\lim_{t\to\infty} \|(S_t - \mathrm{Id})u\|_{\mathcal{X}^0} = 0$,

ii) $\|S_t u\|_{\mathcal{X}^m} \le C t^{m-\ell} \|u\|_{\mathcal{X}^\ell}$, $\qquad 0 \le \ell \le m$, $\quad u \in \mathcal{X}^\ell$,

iii) $\|(\mathrm{Id} - S_t)u\|_{\mathcal{X}^\ell} \le C t^{-(m-\ell)} \|u\|_{\mathcal{X}^m}$, $\qquad 0 \le \ell \le m$, $\quad u \in \mathcal{X}^m$.

• For generic Banach spaces:

Theorem

(Abstract IFT) Let $\alpha > 0$, $p > \alpha$, $p - \alpha \leq q \leq p + 13\alpha$ and let $\mathcal{X}^q$ be a scale of Banach spaces with smoothing operators; let (K, μ) be an approximate solution with error E and let (W, σ) be the improvement. Assume:

a) $\|(W, \sigma)\|_{\mathcal{X}^{q-\alpha}} \leq C_0 \|E\|_{\mathcal{X}^q}$,

b) $\|DE(K, \mu)\, MW + D_\mu E(K, \mu)\sigma + E\|_{\mathcal{X}^{p-\alpha}} \leq C_0 \|E\|^2_{\mathcal{X}^p}$,

c) $\|E\|_{\mathcal{X}^{p+13\alpha}} \leq C_0(1 + \|(K, \mu)\|_{\mathcal{X}^{p+13\alpha}})$.

Then, if (K_0, μ_0) is an approximate solution with $\|E_0\|$ sufficiently small, there exists (K_e, μ_e) such that $E(K_e, \mu_e) = 0$ and

$$\|(K_e - K_0, \mu_e - \mu_0)\|_{\mathcal{X}^p} \leq C_0 \|E_0\|_{\mathcal{X}^{p-\alpha}} \ .$$

• **Analytic spaces:** at each step the domains shrink by δ_h:

$$\rho_0 = \rho \ , \qquad \delta_h = \frac{\rho_0}{2^{h+2}} \ , \qquad \rho_{h+1} = \rho_h - \delta_h \ , \qquad h \geq 0 \ .$$

Then, the error is quadratic and for $a, b > 0$:

$$\|E(K_{h+1}, \mu_{h+1})\|_{\rho_{h+1}} \leq C_0 \ \nu^a \delta_h^b \ \|E(K_h, \mu_h)\|_{\rho_h}^2$$

and, if $\varepsilon_0 \equiv \|E(K_0, \mu_0)\|_{\rho_0}$ is small enough, then:

$$\|K_h - K_0\|_{\rho_h} \leq C_K \varepsilon_0 \ , \qquad |\mu_h - \mu_0| \leq C_\mu \varepsilon_0 \ .$$

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The a-posteriori approach

• Following [LGJV2005], for conformally symplectic systems, by adjusting the parameters under a suitable non-degeneracy condition *near an approximately invariant torus, there is a true invariant torus*, [CCL].

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Advantages of the a-posteriori approach:

► it can be developed in any coordinate frame, not necessarily in action-angle variables;

► the system is not assumed to be nearly integrable;

► instead of constructing a sequence of coordinate transformations on shrinking domains as in the perturbation approach, we shall compute suitable corrections to the embedding and the drift.

Consequences of the a-posteriori approach for conformally symplectic systems (with R. Calleja, R. de la Llave):

- the method provides an efficient algorithm to determine the breakdown threshold, very suitable for computer implementations;
  - very refined quantitative estimates;
- local behavior near quasi–periodic solutions;
- partial justification of Greene's criterion (also with C. Falcolini);
- a bootstrap of regularity, which allows to state that all smooth enough tori are analytic, whenever the map is analytic;
- analyticity domains of the quasi–periodic attractors in the symplectic limit;
- whiskered tori for conformally symplectic systems.

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KAM algorithm

Given $K_0 : \mathbb{T}^n \to \mathcal{M}$, $\mu_0 \in \mathbb{R}^n$, let $\lambda \in \mathbb{R}$ be the conformal factor for f_{μ_0}.

1) $E_0 \leftarrow f_{\mu_0} \circ K_0 - K_0 \circ T_\omega$
2) $\alpha \leftarrow DK_0$
3) $N_0 \leftarrow [\alpha^\top \alpha]^{-1}$
4) $M_0 \leftarrow [\alpha | \, J^{-1} \circ K_0 \alpha N_0]$
5) $\beta \leftarrow M_0^{-1} \circ T_\omega$
6) $\widetilde{E}_0 \leftarrow \beta E_0$
7) $P_0 \leftarrow \alpha N_0$

   $S_0 \leftarrow (P_0 \circ T_\omega)^\top Df_{\mu_0} \circ K_0 \, J^{-1} \circ K_0 P_0$

   $\widetilde{A}_0 \leftarrow M_0^{-1} \circ T_\omega \, D_\mu f_{\mu_0} \circ K_0$

8) $(B_{a0})^0$ solves $\lambda(B_{a0})^0 - (B_{a0})^0 \circ T_\omega = -(\widetilde{E}_0^{(2)})^0$

$(B_{b0})^0$ solves $\lambda(B_{b0})^0 - (B_{b0})^0 \circ T_\omega = -(\widetilde{A}_0^{(2)})^0$

9) Find $\overline{W}_0^{(2)}$, σ_0 solving

$$0 = -\overline{S}_0\, \overline{W}_0^{(2)} - \overline{S_0(B_{a0})^0} - \overline{S_0(B_{b0})^0}\sigma_0 - \overline{\widetilde{E}_0^{(1)}} - \overline{\widetilde{A}_0^{(1)}}\sigma_0$$
$$(\lambda - 1)\overline{W}_0^{(2)} = -\overline{\widetilde{E}_0^{(2)}} - \overline{\widetilde{A}_0^{(2)}}\sigma_0 \,.$$

10) $(W_0^{(2)})^0 = (B_{a0})^0 + \sigma_0 (B_{b0})^0$

11) $W_0^{(2)} = (W_0^{(2)})^0 + \overline{W}_0^{(2)}$

12) $(W_0^{(1)})^0$ solves
$(W_0^{(1)})^0 - (W_0^{(1)})^0 \circ T_\omega = -(S_0 W_0^{(2)})^0 - (\widetilde{E}_0^{(1)})^0 - (\widetilde{A}_0^{(1)})^0 \sigma_0$

13) $K_1 \leftarrow K_0 + M_0 W_0$

$\mu_1 \leftarrow \mu_0 + \sigma_0 \,.$

Remark

• *Steps 2), 8), 10), 11), 12) involve diagonal operations in the Fourier space.*
• *The other steps are diagonal in the real space (while steps 10), 11) are diagonal in both spaces).*
• *If we represent a function in discrete points or in Fourier space, then we can compute the other functions by applying the Fast Fourier Transform (FFT). This implies that if we use N Fourier modes to discretize the function, then we need $O(N)$ storage and $O(N \log N)$ operations.*