Perturbation theory, KAM theory and Celestial Mechanics
9. Breakdown and applications

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Sevilla, 25-27 January 2016

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Outline

1. Break–down of quasi–periodic tori and attractors

2. KAM break–down criterion

3. Partial justification of Greene's method

4. Complex perturbing parameter

5. Applications

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Break–down of quasi–periodic tori and attractors

- We can compute a rigorous lower bound of the break–down threshold of invariant tori by means of KAM theory.

- Which is the real break–down value?

- In physical problems one can compare KAM result with a measure of the parameter. For example in the 3-body problem, $\varepsilon = \frac{m_{Jupiter}}{m_{Sun}} \simeq 10^{-3}$.

- In model problems one needs to apply numerical techniques: KAM break–down criterion, Greene's technique, frequency analysis, etc.

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KAM break–down criterion [Calleja, Celletti 2010]

• Solve the invariance equation for (K, μ):

$$f_\mu \circ K(\theta) = K(\theta + \omega) \ .$$

• Numerically efficient criterion: close to breakdown, one has a blow up of the Sobolev norms of a trigonometric approximation of the embedding:

$$K^{(L)}(\theta) = \sum_{|\ell| \leq L} \widehat{K}_\ell \, e^{i\ell\theta} \ .$$

• A regular behavior of $\|K^{(L)}\|_m$ as ε increases (for λ fixed) provides evidence of the existence of the invariant attractor. Table: ε_{crit} for $\omega_r = 2\pi \frac{\sqrt{5}-1}{2}$.

Conservative case	Dissipative case	
ε_{crit}	λ	ε_{crit}
0.9716	0.9	0.9721
	0.5	0.9792

Greene's method, periodic orbits and Arnold's tongues

• Greene's method: breakdown of $\mathcal{C}(\omega)$ related to the stability of $\mathcal{P}(\frac{p_j}{q_j}) \to \mathcal{C}(\omega)$, but in the dissipative case: drift in an interval - *Arnold tongue* - admitting a periodic orbit.

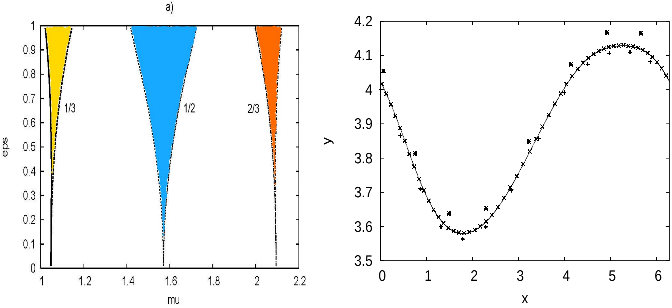

Figure: Left: Arnold's tongues providing μ vs. ε for 3 periodic orbits. Right: For $\lambda = 0.9$ and $\varepsilon = 0.5$ invariant attractor with frequency ω_r and approximating periodic orbits: 5/8 ($*$), 8/13 ($+$), 34/55 ($\times$).

• Greene's method: let $\varepsilon^{\omega_r}_{p_j,q_j}$ be the maximal ε for which the periodic orbit has a stability transition; the sequence converges to the breakdown threshold of $\omega_r = 2\pi \frac{\sqrt{5}-1}{2}$.

p_j/q_j	$\varepsilon^{\omega_r}_{p_j,q_j}(cons)$ $\varepsilon_{Sob} = [0.9716]$	$\varepsilon^{\omega_r}_{p_j,q_j}(\lambda = 0.9)$ $\varepsilon_{Sob} = [0.972]$	$\varepsilon^{\omega_r}_{p_j,q_j}(\lambda = 0.5)$ $\varepsilon_{Sob} = [0.979]$
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Partial justification of Greene's method [Calleja, Celletti, de la Llave, Falcolini 2014]

• Greene's criterion: originally developed for the standard map, gives the existence of an invariant curve with frequency ω if and only if the periodic orbits with frequencies given by the rational approximants p_j/q_j approaching ω are at the border of linear stability, measured by the *residue* $R(\frac{p_j}{q_j}) = \frac{1}{4}(2 - Tr(Df^q))$.

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• We use the linearization theorem and give 2 different proofs: deformation theory and NHIM theory.

• Let the periodic orbit have frequency $\nu = (a_1, ..., a_n)/L$ with $a_j \in \mathbb{Z}$, $L \in \mathbb{N}$. The spectrum has a pairing rule: $Spec(Df^L) = \{\gamma_i, \lambda^L \gamma_i^{-1}\}$.

Theorem (Calleja,A.C.,Falcolini,de la Llave, 2013)

Let f_μ be conformally symplectic, such that f_0 admits a Lagrangian invariant torus with frequency ω. Then, there exists a ngh. $\mathcal{U}$ of the torus, s.t. when the periodic orbit with $\nu = (a_1, ..., a_n)/L$ is in $\mathcal{U}$, there exists $C_N > 0$ s.t.

$$|\gamma_i - 1| \leq L\, C_N \, \|\mu\|^N \simeq C_N \, \|\omega - \nu\|^N \,, \qquad i = 1, ..., n \,.$$

• Thus we have bounds on the spectral numbers of the periodic orbits.

• We get also upper/lower bounds on the width of the Arnold tongues.

• Proof: deformation theory:

► Find a smooth change of variables (normal form) that reduces the system to $(\theta + S_\mu, \lambda I)$ up to an error (S_μ polynomial function)

► The spectrum is invariant under smooth changes of variables

► For the system in normal form neglecting the remainder, the spectral numbers are equal to 1 and the residue is zero

► Estimate the spectrum by bounding the error in the normal form (use the theory of deformations, [de la Llave, Banyaga, Wayne, Marco, Moriyón]).

• Proof: NHIM and averaging theory:

► NHIM theory (Fenichel, Hirsch, Pugh, Shub): $\mathcal{T}_\mu$ is a family of tori invariant under f_μ (the invariant torus for f_0 is a NHIM)

► We can write these manifolds as the image of the torus under a family of maps K_μ such that $f_\mu \circ K_\mu = K_\mu \circ R_\mu$, where R_μ denotes the dynamics of f_μ restricted to $\mathcal{T}_\mu$ (R_0 is the Diophantine rotation)

► Averaging theory tells us that for $N \leq N_0$ we can find a diffeomorphism $B_\mu^{(N)}$ and a rotation $T_{\omega_\mu^{(N)}}$ such that

$$(B_\mu^{(N)})^{-1} \circ R_\mu \circ B_\mu^{(N)} = T_{\omega_\mu^{(N)}} + O(\|\mu\|^{N+1})$$

► Periodic orbits (in the NHIM) have n Lyapunov exponents close to 1

► Pairing rule and Lagrangian character of the tori imply that the remaining exponents of the periodic orbit with $\rho = a/L$ are close to λ^L.

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Complex perturbing parameter [Calleja, Celletti 2010]

• We compute the solution of the functional equation assuming $\varepsilon \in \mathbb{C}$.
Applying Newton's method we follow the solution from $\varepsilon = 0$ increasing the real and imaginary parts of $\varepsilon = \varepsilon_r + i\varepsilon_i$ until blow-up.

• The expansion of the parametrization K in terms of the complex ε as the sum of a real and an imaginary part becomes ($K_j(\theta)$ are real)

$$\begin{aligned} K(\theta;\varepsilon) &= \sum_{j=1}^{\infty} K_j(\theta)(\varepsilon_r + i\varepsilon_i)^j \\ &= K_r(\theta;\varepsilon_r,\varepsilon_i) + iK_i(\theta;\varepsilon_r,\varepsilon_i) \end{aligned}$$

and the same for $g(\theta + K) = \sin(\theta + K)$:

$$\varepsilon g(\theta + K) = \varepsilon_r g_r - \varepsilon_i g_i + i(\varepsilon_r g_i + \varepsilon_i g_r)\ .$$

• Setting $\gamma = \omega(1-\lambda) - \mu = \gamma_r + i\gamma_i$, the functional equation corresponds to the following two equations:

$$\begin{aligned} D_1 D_\lambda K_r(\theta+\omega;\varepsilon_r,\varepsilon_i) - \varepsilon_r g_r(\theta) + \varepsilon_i g_i(\theta) - \gamma_r &= 0 \\ D_1 D_\lambda K_i(\theta+\omega;\varepsilon_r,\varepsilon_i) - \varepsilon_r g_i(\theta) - \varepsilon_i g_r(\theta) - \gamma_i &= 0\,. \end{aligned}$$

• Figure: domains of existence in the complex ε–plane for different mappings, for $\omega/(2\pi) = [3, 12, 1, 1, 1, 1, ...]$ and the golden ratio, for specific values of λ (cut of Figure top-right is possibly due to the fact that the frequency is close to a rational).

• The shapes of the existence domains strongly depend on the choice of the function $g(\theta)$ (bottom panel).

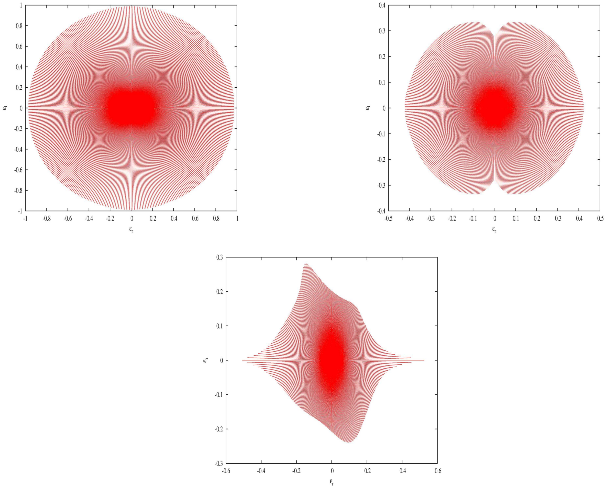

Figure: Axes: ε_r and ε_i. *a)* $g(x) = \sin x$, $\omega/(2\pi) = \frac{\sqrt{5}-1}{2}$, $\lambda = 0.9$; *b)* $g(x) = \sin x$, $\omega/(2\pi) = [3, 12, 1, 1, 1, 1, ...]$, $\lambda = 0.9$; *c)* $g(x) = \sin x + \frac{1}{20}\sin(4x) + \frac{1}{30}\sin(6x)$, $\omega/(2\pi) = \frac{\sqrt{5}-1}{2}$, $\lambda = 0.9$.

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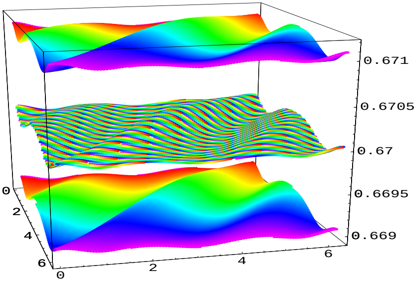
Applications

- Standard map
- Rotational dynamics: spin–orbit problem
- Orbital dynamics: three–body problem

KAM stability through confinement

• Confinement in 2–dimensional systems: dim(phase space)=4, dim(constant energy level)=3, dim(invariant tori)=2 $\rightarrow$ confinement in phase space for ∞ times between bounding invariant tori



• Confinement no more valid for $n > 2$: the motion can diffuse through invariant tori, reaching arbitrarily far regions (Arnold's diffusion).

Conservative standard map

Results of the '90s

• [A.C., L. Chierchia] Let $\omega = 2\pi\frac{\sqrt{5}-1}{2}$; $|\varepsilon| \leq 0.838$ (86% of Greene's value) there exists an invariant curve with frequency ω.
• [R. de la Llave, D. Rana] Using accurate strategies and efficient computer–assisted algorithms, the result was improved to 93% of Greene's value.
• Very recent results [J.-L. Figueras, A. Haro, A. Luque] in http://arxiv.org/abs/1601.00084 reaching 99.9%!!!

Dissipative standard map

- Using $K_2(\theta) = \theta + u(\theta)$, the invariance equation is

$$D_1 D_\lambda u(\theta) - \varepsilon \sin(\theta + u(\theta)) + \omega(1-\lambda) - \mu = 0 \qquad (1)$$

with $D_\lambda u(\theta) = u(\theta + \frac{\omega}{2}) - \lambda u(\theta - \frac{\omega}{2})$.

Proposition [dissipative standard map, R. Calleja, A.C., R. de la Llave (2016)]

Let $\omega = 2\pi \frac{\sqrt{5}-1}{2}$ and $\lambda = 0.9$; then, for $\varepsilon \leq \varepsilon_{KAM}$, there exists a unique solution $u = u(\theta)$ of (1), provided that $\mu = \omega(1-\lambda) + \langle u_\theta \, D_1 D_\lambda u \rangle$.

- The drift μ must be suitably tuned and cannot be chosen independently from ω.

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• The drift μ must be suitably tuned and cannot be chosen independently from ω.

• Preliminary result: conf. symplectic version, careful estimates, continuation method using the Fourier expansion of the initial approximate solution $\Rightarrow$

$$\varepsilon_{KAM} = \boxed{99\% \text{ of the critical breakdown threshold .}}$$

Rotational dynamics

The **Moon** and all evolved satellites, always point the same face to the host planet: 1:1 resonance, i.e. 1 rotation = 1 revolution (Phobos, Deimos - Mars, Io, Europa, Ganimede, Callisto - Jupiter, Titan, Rhea, Enceladus, etc.).
Only exception: **Mercury** in a 3:2 spin–orbit resonance (3 rotations = 2 revolutions).

• Important dissipative effect: **tidal torque,** due to the non–rigidity of planets and satellites.

Conservative spin–orbit problem

• Spin–orbit problem: triaxial satellite $\mathcal{S}$ (with $A < B < C$) moving on a Keplerian orbit around a central planet $\mathcal{P}$, assuming that the spin–axis is perpendicular to the orbit plane and coincides with the shortest physical axis.

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• Equation of motion:

$$\ddot{x} + \varepsilon(\frac{a}{r})^3 \sin(2x - 2f) = 0 , \qquad \varepsilon = \frac{3}{2}\frac{B-A}{C} .$$

• The (Diophantine) frequencies of the bounding tori are for example:

$$\omega_- \equiv 1 - \frac{1}{2 + \frac{\sqrt{5}-1}{2}} , \qquad \omega_+ \equiv 1 + \frac{1}{2 + \frac{\sqrt{5}-1}{2}} .$$

Proposition [spin–orbit model, A.C. (1990)]

Consider the spin–orbit Hamiltonian defined in $U \times \mathbb{T}^2$ with $U \subset \mathbb{R}$ open set. Then, for the true eccentricity of the Moon $e = 0.0549$, there exist invariant tori, bounding the motion of the Moon, for any $\varepsilon \leq \varepsilon_{Moon} = 3.45 \cdot 10^{-4}$.

Dissipative spin–orbit problem

• Possible forthcoming estimates: spin–orbit equation with tidal torque given by

$$\ddot{x} + \varepsilon \left(\frac{a}{r}\right)^3 \sin(2x - 2f) = -\lambda(\dot{x} - \mu) , \qquad (2)$$

where λ, μ depend on the orbital (e) and physical properties of the satellite.

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Proposition [A.C., L. Chierchia (2009)]

Let $\lambda_0 \in \mathbb{R}_+$, ω Diophantine. There exists $0 < \varepsilon_0 < 1$, such that for any $\varepsilon \in [0, \varepsilon_0]$ and any $\lambda \in [-\lambda_0, \lambda_0]$ there exists a unique function $u = u(\theta, t)$ with $\langle u \rangle = 0$, such that

$$x(t) = \omega\, t + u(\omega t, t)$$

solves the equation of motion with $\mu = \omega\, (1 + \langle u_\theta^2 \rangle)$.

Conservative three–body problem

• Consider the motion of a small body (with negligible mass) under the gravitational influence of two primaries, moving on Keplerian orbits about their common barycenter (*restricted* problem).

• Assume that the orbits of the primaries are circular and that all bodies move on the same plane: *planar, circular, restricted three–body problem* (PCR3BP).

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• Adopting suitable normalized units and action–angle Delaunay variables $(L, G) \in \mathbb{R}^2$, $(\ell, g) \in \mathbb{T}^2$, we obtain a 2 d.o.f. Hamiltonian function:

$$\mathcal{H}(L, G, \ell, g) = -\frac{1}{2L^2} - G + \varepsilon R(L, G, \ell, g) \; .$$

• ε primaries' mass ratio ($\varepsilon = 0$ Keplerian motion). Actions: $L = \sqrt{a}$, $G = L\sqrt{1 - e^2}$.

• Degenerate Hamiltonian, but Arnold's isoenergetic non–degenerate (persistence of invariant tori on a fixed energy surface), i.e. setting $h(L, G) = -\frac{1}{2L^2} - G$:

$$\det \begin{pmatrix} h''(L,G) & h'(L,G) \\ h'(L,G)^T & 0 \end{pmatrix} = \det \begin{pmatrix} -\frac{3}{L^4} & 0 & \frac{1}{L^3} \\ 0 & 0 & -1 \\ \frac{1}{L^3} & -1 & 0 \end{pmatrix} = \frac{3}{L^4} \neq 0 \quad \text{for all } \; L \neq 0 \, .$$

• Dimension phase space = 4 , fix the energy: dim = 3; dimension invariant tori = 2.

Result: The stability of the small body can be obtained by proving the existence of invariant surfaces which confine the motion of the asteroid on a preassigned energy level.

Sample: Sun, Jupiter, asteroid 12 Victoria with

$$a_{\rm V} \simeq 0.449 \ , \quad e_{\rm V} \simeq 0.220 \ , \quad \imath_{\rm V} \simeq \frac{8.363-1.305}{360} = 1.961 \cdot 10^{-2} \ .$$

• Size of the perturbing parameter: $\varepsilon_J = 0.954 \cdot 10^{-3}$.
• Approximations: disregard $e_J = 4.82 \cdot 10^{-2}$ (worst physical approximation), inclinations, gravitational effects of other bodies (Mars and Saturn), dissipative phenomena (tides, solar winds, Yarkovsky effect,...)

• Empirical criterion: *expand the perturbation in e and a, neglecting contributions smaller than e_J. Neglect terms of order $O(\varepsilon)$ in F_ε (i.e. replace F_ε by F_0).*
• One-parameter family of Hamiltonians ($0 < G < L$):

$$H_{\rm SJV}(L, G, \ell, g; \varepsilon) = -\frac{1}{2L^2} - G + \varepsilon H_1(L, G, \ell, g) ,$$

with ($a = L^2$, $e = \sqrt{1 - \frac{G^2}{L^2}}$)

$$\begin{aligned} H_1(L, G, \ell, g) &= -(1 + \frac{a^2}{4} + \frac{9}{64}\, a^4 + \frac{3}{8}\, a^2 e^2) \\ + &\quad \left(\frac{1}{2} + \frac{9}{16} a^2\right) a^2 e \ \cos \ell - \left(\frac{3}{8}\, a^3 + \frac{15}{64} a^5\right) \cos(\ell + g) \\ + &\quad \left(\frac{9}{4} + \frac{5}{4} a^2\right) a^2 e \ \cos(\ell + 2g) - \left(\frac{3}{4}\, a^2 + \frac{5}{16}\, a^4\right) \cos(2\,\ell + 2\,g) \\ - &\quad \frac{3}{4}\, a^2 e \cos(3\,\ell + 2\,g) - \left(\frac{5}{8}\, a^3 + \frac{35}{128}\, a^5\right) \cos(3\,\ell + 3\,g) \\ - &\quad \frac{35}{64}\, a^4 \cos(4\,\ell + 4\,g) - \frac{63}{128} a^5 \cos(5\ell + 5g) . \end{aligned}$$

• Fixing the perturbation parameter at the value $\varepsilon = \varepsilon_J$, we obtain the *Sun–Jupiter–Victoria Hamiltonian*:

$$\begin{aligned} \overline{H}_{\rm SJV}(L,G,\ell,g) &= -\frac{1}{2L^2} - G + \varepsilon_J H_1(L,G,\ell,g)\ , \\ &= H_0(L,G) + \varepsilon_J H_1(L,G,\ell,g)\ . \end{aligned}$$

• Observed values: $L_{\rm V} = \sqrt{a_{\rm V}} \simeq 0.670$, $G_{\rm V} = L_{\rm V}\ \sqrt{1-e_{\rm V}^2} \simeq 0.654$.
• Define the "osculating energy value" in terms of the Keplerian approximation and in terms of the "secular" effects; define $E_{\rm V}^{(0)}$ and $E_{\rm V}^{(1)}$ as

$$\begin{aligned} H_0(L_{\rm V},G_{\rm V}) &= -\frac{1}{2L_{\rm V}^2} - G_{\rm V} \simeq -1.768 = E_{\rm V}^{(0)}\ , \\ \big\langle H_1(L_{\rm V},G_{\rm V},\cdot,\cdot)\big\rangle &\simeq -1.060 = E_{\rm V}^{(1)}\ , \\ E_{\rm V}(\varepsilon) &= E_{\rm V}^{(0)} + \varepsilon E_{\rm V}^{(1)}\ . \end{aligned}$$

• Osculating energy level of the Sun-Jupiter-Victoria model:

$$\overline{E}_{\rm V} = E_{\rm V}(\varepsilon_J) = E_{\rm V}^{(0)} + \varepsilon_J E_{\rm V}^{(1)} \simeq -1.769\ .$$

• From now on we will be concerned with such one–parameter family of energy surfaces:

$$\mathcal{S}_{\varepsilon,\mathrm{V}} = H_{\mathrm{SJV}}^{-1}\big(E_{\mathrm{V}}(\varepsilon)\big) \ .$$

• We consider two invariant tori on $\mathcal{S}_{0,\mathrm{V}}$, which bound from above and below the observed value L_{V}: we define

$$\tilde{L}_{\pm} = L_V \pm 0.001 \ .$$

• The corresponding frequencies are:

$$\underline{\tilde{\omega}}_{\pm} = \frac{\partial H_0}{\partial(L,G)} = \left(\frac{1}{\tilde{L}_{\pm}^3}, -1\right) = (\tilde{\alpha}_{\pm}, -1) \ .$$

• Since we need Diophantine frequencies, we compute the continued fraction representation up to the order 5 of $\tilde{\alpha}_{\pm}$ and then we modify the frequencies by adding a tail of all one's.

• Result: two quadratic “noble" numbers $\alpha_\pm$ given by:

$$\begin{aligned} \alpha_- &= [3;3,4,2,1^\infty] = 3.30976937631389... \\ \alpha_+ &= [3;2,1,17,5,1^\infty] = 3.33955990647860... \end{aligned}$$

We can now define the Diophantine frequencies

$$\underline{\omega}_\pm = (\alpha_\pm, -1) \ ,$$

with corresponding Diophantine constants

$$\tau_\pm = \tau = 1 \ , \quad \gamma_- = 7.224496 \cdot 10^{-3} \ , \quad \gamma_+ = 3.324329 \cdot 10^{-2} \ .$$

• We are interested in the KAM continuation of the following unperturbed tori, which lie on the energy level $H_0^{-1}(E_V^{(0)})$:

$$\mathcal{T}_0^\pm = \{(L_\pm, G_\pm)\} \times \mathbb{T}^2 \ ,$$

with

$$L_\pm = \frac{1}{\alpha_\pm^{1/3}} \ , \qquad G_\pm = -\frac{1}{2L_\pm^2} - E_V^{(0)} \ .$$

• Concrete example: Sun, Jupiter, asteroid 12 Victoria with $a = 0.449$ (in Jupiter–Sun unit distance) and $e = 0.22$, so that $L_{\rm V} \simeq 0.670$, $G_{\rm V} \simeq 0.654$.

• Select the energy level as $E^*_{\rm V} = -\frac{1}{2L^2_{\rm V}} - G_{\rm V} + \varepsilon_J \langle R(L_{\rm V}, G_{\rm V}, \ell, g) \rangle \simeq -1.769$, where $\varepsilon_J \simeq 10^{-3}$ is the observed Jupiter–Sun mass–ratio. On such (3–dim) energy level prove the existence of two (2–dim) trapping tori with frequencies $\omega_\pm$.

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Proposition [three–body problem, A.C., L. Chierchia (2007)]

Let $E = E^*_{\rm V}$. Then, for $|\varepsilon| \leq 10^{-3}$ the unperturbed tori with trapping frequencies $\omega_\pm$ can be analytically continued into KAM tori for the perturbed system on the energy level $\mathcal{H}^{-1}\big(E^*_{\rm V}\big)$ keeping fixed the ratio of the frequencies.

• Due to the link between a, e and L, G, this result guarantees that a, e remain close to the unperturbed values within an interval of size of order ε.

Corollary: The values of the perturbed integrals $L(t)$ and $G(t)$ stay close forever to their initial values $L_{\rm V}$ and $G_{\rm V}$ and the actual motion (in the mathematical model) is nearly elliptical with osculating orbital values close to the observed ones.