SELECTED TOPICS ON THE TOPOLOGY OF IDEAL FLUID FLOWS

DANIEL PERALTA-SALAS

Abstract. This is a survey of certain geometric aspects of inviscid and incompressible fluid flows, which are described by the solutions to the Euler equations. We will review Arnold's theorem on the topological structure of stationary fluids in compact manifolds, and Moffatt's theorem on the topological interpretation of helicity in terms of knot invariants. The recent realization theorem by Enciso and Peralta-Salas of vortex lines of arbitrarily complicated topology for stationary solutions to the Euler equations will be also introduced. The aim of this paper is not to provide detailed proofs of all the stated results but to introduce the main ideas and methods behind certain selected topics of the subject known as Topological Fluid Mechanics. This is the set of lecture notes the author gave at the XXIV International Fall Workshop on Geometry and Physics held in Zaragoza (Spain) during September 2015.

1. The Euler equations on a Riemannian 3-manifold

The dynamics of an inviscid and incompressible fluid flow in a smooth domain of the Euclidean space $\Omega \subseteq \mathbb{R}^3$ is modeled by the *Euler equations*

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla P \,, \qquad \operatorname{div} u = 0 \,,$$

where $u(x,t)$ is a time-dependent vector field representing the velocity field of the fluid, and $P(x,t)$ is the pressure function. As a boundary condition it is customary to assume that the field u is tangent to the boundary of the domain, that is

$$u \cdot \nu = 0 \text{ on } \partial\Omega \,,$$

for all t, with ν a field normal to the boundary. The initial velocity field $u(x,0) = u_0(x)$ is prescribed, and the pressure $P(x,0) \equiv P_0(x)$ at $t=0$ is determined (up to a constant) by the Euler equations:

$$\Delta P_0 = -\operatorname{div}[(u_0 \cdot \nabla) u_0] \,.$$

It is well known that there exists a smooth solution to the Euler equations for short times provided that u_0 is smooth and compactly supported, but the global existence is still a challenging open problem.

Another time-dependent vector field that plays a crucial role in fluid mechanics is the *vorticity*, defined as

$$\omega := \operatorname{curl} u \,,$$

which is related to the rotation of the fluid and is a measure of the entanglement of the fluid particle trajectories. It is easy to check that an alternative form of the

Euler equations using the vorticity is

$$\frac{\partial u}{\partial t} - u \times \omega = -\nabla B\,, \qquad \operatorname{div} u = 0\,,$$

where $B := P + \frac{1}{2}|u|^2$ is the *Bernoulli function*. Here the symbol $\times$ denotes the standard vector product in $\mathbb{R}^3$.

The Euler equations can be written in any Riemannian 3-manifold, thus describing a fluid flow in a curved space. Indeed, let (M, g) be a smooth (possibly with boundary) Riemannian manifold of dimension 3, oriented but not necessarily compact, and denote the Riemannian volume form by μ. The Euler equations then read as

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla P\,, \qquad \operatorname{div} u = 0\,, \tag{1.1}$$

where ∇_u is the covariant derivative along u, and div and ∇ are the divergence and gradient operators, respectively, computed with the metric g. The unknowns are the velocity field u, which is a time-dependent vector field on M, and the pressure P, which is a time-dependent real-valued function on M.

The incompressibility condition $\operatorname{div} u = 0$ can be expressed using the volume form μ as

$$L_u \mu = 0\,,$$

where L_u is the Lie derivative along u. The vorticity $\omega := \operatorname{curl} u$ is defined as the only vector field satisfying

$$i_\omega \mu = d\alpha\,, \tag{1.2}$$

where i_X is the contraction operator of differential forms with vector fields, and

$$\alpha := i_u g$$

is the 1-form dual to the field u using the metric. It is easy to check that the dual 1-form of the vorticity is $\star d\alpha$, where the symbol $\star$ denotes the Hodge operator.

In terms of the vorticity, the Euler equations read as in the Euclidean case, that is

$$\frac{\partial u}{\partial t} - u \times \omega = -\nabla B\,, \qquad \operatorname{div} u = 0\,. \tag{1.3}$$

Here the Bernoulli function B is defined as above using the Riemannian norm $|u|^2$ of the velocity field, and the vector product $u \times \omega$ in (M, g) is defined as the only vector field that satisfies the equation

$$i_\omega i_u \mu = i_{u \times \omega} g\,. \tag{1.4}$$

In the following proposition we show how to write the Euler equations using differential forms:

Proposition 1.1. *The Euler equations are equivalent to*

$$\frac{\partial \alpha}{\partial t} + L_u \alpha = -dF\,, \qquad d(\star\alpha) = 0\,,$$

where $\alpha := i_u g$ *and* $F := P - \frac{1}{2}|u|^2$.

Proof. The equation $d(\star\alpha) = 0$ follows from the identity $\star\alpha = i_u\mu$ and the incompressibility condition $0 = L_u\mu = d(i_u\mu)$. Moreover, the standard identity

$$\nabla_u u = \frac{1}{2}\nabla|u|^2 - u \times \omega$$

implies that the dual 1-form of the vector field $\nabla_u u$ is

$$\frac{1}{2}d(|u|^2) - i_\omega i_u\mu = \frac{1}{2}d(|u|^2) + i_u d\alpha = L_u\alpha - \frac{1}{2}d(|u|^2)\,, \tag{1.5}$$

where we have used Eqs. (1.2) and (1.4), and Cartan's formula $L_u\alpha = i_u d\alpha + d(i_u\alpha)$. Writing the first of the Euler equations (1.1) in terms of differential forms, the proposition easily follows from Eq. (1.5). □

Remark 1.2. In view of Proposition 1.1, the Euler equations can also be equivalently written as

$$\frac{\partial\alpha}{\partial t} + i_u d\alpha = -dB\,, \qquad d(\star\alpha) = 0\,, \tag{1.6}$$

The goal of this article is to introduce certain selected topics on the geometry and topology of the solutions to the Euler equations. A general view of this subject can be found in Arnold and Khesin monograph [4]. In Section 2 we explain the notions of stream lines and vortex lines, and we prove Helmholtz's transport of vorticity, which gives rise to both local (helicity) and non-local (KAM-type invariants) conservation laws. The stationary solutions of the Euler equations are discussed in Section 3, where we construct some particularly relevant exact solutions, including Seifert foliations of the 3-sphere $\mathbb{S}^3$. In Section 4 we state and sketch the proof of Arnold's structure theorem for stationary solutions in compact manifolds, and in Section 5 we provide a detailed proof of Moffatt's theorem computing the helicity for divergence-free vector field modeled on a link. Finally, Beltrami flows are studied in detail in Section 6, where we also review the recent realization theorem for knotted vortex lines of Enciso and Peralta-Salas.

2. Stream lines, vortex lines and transport of vorticity

The motion of fluid particles is described by the integral curves of the velocity field u, that is, by the solutions of the non-autonomous ODE

$$\dot{x}(t) = u(x(t), t)$$

for some initial condition $x(0) = x_0$. These trajectories are usually called *particle paths.* The integral curves of $u(x,t)$ at fixed time t are called *stream lines* and thus the stream line pattern changes with time if the flow is non-stationary.

The integral curves of the vorticity $\omega(x,t)$ at fixed time t, that is to say, the solutions of the autonomous ODE

$$\dot{x}(\tau) = \omega(x(\tau), t)$$

for some initial condition $x(0) = x_0$, are the *vortex lines* of the fluid at time t. A remarkable property that was discovered by Helmholtz in the 19th century [4] is that the vortex lines pattern is the same for any t, up to an ambient diffeomorphism, because the vorticity is transported by the fluid flow. This mechanism places vorticity in a leading role in analyzing the Euler equations.

Theorem 2.1 (Helmholtz's transport of vorticity)**.** *Let $u(x,t)$ be a time-dependent solution of the Euler equations, then the vorticity satisfies the transport equation*

$$\frac{\partial \omega}{\partial t} = -L_u \omega \,.$$

Proof. Taking the exterior derivative in the equation $\frac{\partial \alpha}{\partial t} + L_u \alpha = -dF$, cf. Proposition 1.1, we obtain

$$\frac{\partial (d\alpha)}{\partial t} = -L_u(d\alpha)\,,$$

where we have used that the exterior derivative and the Lie derivative commute. On the other hand, differentiating with respect to t in the definition of the vorticity $i_\omega \mu = d\alpha$, and using the previous equation, we get

$$i_{\frac{\partial \omega}{\partial t}} \mu = -L_u(d\alpha) = -L_u(i_\omega \mu)\,.$$

Finally, standard identities and the fact that u is divergence-free imply

$$L_u(i_\omega \mu) = i_\omega(L_u \mu) + i_{L_u \omega}\mu = i_{L_u\omega}\mu\,,$$

so that

$$i_{\frac{\partial \omega}{\partial t} + L_u \omega}\, \mu = 0\,,$$

and the theorem follows. □

This result implies that the vorticity at time t can be expressed in terms of the initial vorticity $\omega(x,0) = \omega_0(x)$ as

$$\omega(x,t) = (\phi_{t,0})_* \omega_0(x)\,,$$

where $(\phi_{t,0})_*$ denotes the push-forward of the non-autonomous local flow of the velocity field between the times 0 and t. Accordingly, the topology of the vortex lines is preserved as far as the solution does not blow-up. Since u is divergence-free, its local flow $\phi_{t,0}$ is volume-preserving.

An interesting consequence of the fact that the vorticity is transported by the velocity field is the existence of infinitely many non-local conservation laws for the Euler equations. Indeed, any topological property of the phase portrait of the initial vorticity ω_0 is preserved with time, as for example the number of stagnation points (i.e. zeros of the vorticity) or of periodic vortex lines. Inspired by this property and KAM theory, Khesin, Kuksin and Peralta-Salas introduced in [18] a functional acting on the space $\mathfrak{X}^1_{\text{ex}}(M)$ of exact divergence-free vector fields of class C^1 on M, which is a new conserved quantity for the Euler equations. Recall that a divergence-free vector field w on M is exact if the closed 1-form $i_w \mu$ is exact. More precisely, we have:

Definition 2.2. Let (M,g) be a compact Riemannian 3-manifold without boundary whose total volume is normalized by 1. The *partial integrability functional*

$$\varkappa : \mathfrak{X}^1_{\text{ex}}(M) \to [0,1]$$

assigns to a vectorfield $w \in \mathfrak{X}^1_{\text{ex}}(M)$ the (inner) measure of the set equal to the union of all ergodic, 2-dimensional and of class C^1, invariant tori of w.

Theorem 2.3. $\varkappa(\omega(t)) = \varkappa(\omega_0)$ *for all t for which the solution is defined.*

This theorem was proved in [18], where it was used to show that the Euler (local) flow in the space of divergence-free vector fields is not topologically mixing. We also gave a quantitative criterion estimating the distance (in the C^k norm) between a solution of the Euler equations and a given divergence-free vector field. We remark that the functional $\varkappa$ is not an integral operator (i.e. it is not given as the integral of a well-behaved local density), and in fact it is not even continuous in $\mathfrak{X}^1_{\rm ex}(M)$. Nevertheless, $\varkappa$ is lower semi-continuous at certain (partially) integrable divergence-free vector fields, which is enough to prove the aforementioned results.

The transport of vorticity also implies the conservation of the *helicity* of ω. In general, the helicity $\mathcal{H}(w)$ of an exact divergence-free vector field w is defined as

$$\mathcal{H}(w) := \int_M w \cdot \operatorname{curl}^{-1} w \,, \tag{2.1}$$

where $\operatorname{curl}^{-1} : \mathfrak{X}^1_{\rm ex}(M) \to \mathfrak{X}^1_{\rm ex}(M)$ is an operator assigning to w the only vector field $v \in \mathfrak{X}^1_{\rm ex}(M)$ such that $\operatorname{curl} v = w$. The inverse of curl is a generalization to manifolds of the Biot-Savart integral operator, and it is well known that can also be written in terms of a (matrix-valued) integral kernel $k(x,y)$ as

$$\operatorname{curl}^{-1} w(x) = \int_M k(x,y) w(y) \mu_y \,.$$

The relevance of the helicity in fluid mechanics, as well as its topological meaning, were unveiled by Moffatt in 1969 [21]. We will discuss about the relationship between the helicity and the knottedness of vortex lines in Section 5. Actually, the helicity is invariant under any volume-preserving diffeomorphisms of w, which in particular implies the conservation of the helicity of the vorticity, i.e. $\mathcal{H}(\omega(t)) = \mathcal{H}(\omega_0)$, as a consequence of Helmholtz's Theorem 2.1.

Theorem 2.4. *Let (M,g) be a compact Riemannian 3-manifold without boundary, and let $\Phi : M \to M$ be a volume-preserving diffeomorphism, that is $\Phi_*\mu = \mu$. Then, for any exact divergence-free vector field w, we have $\mathcal{H}(\Phi_* w) = \mathcal{H}(w)$.*

Proof. Let α be the dual 1-form with the metric of the vector field $v = \operatorname{curl}^{-1} w$. Then the dual 1-form of w is $\star d\alpha$, and the helicity can be written as

$$\mathcal{H}(w) = \int_M \alpha \wedge d\alpha \,.$$

Moreover, the equation $i_w \mu = d\alpha$ implies that $i_{\Phi_* w}\mu = d(\Phi_*\alpha)$ because Φ is volume-preserving and the exterior derivative and the push-forward commute. This implies, in particular, that $\Phi_* w \in \mathfrak{X}^1_{\rm ex}(M)$. Accordingly, we can compute the helicity of the field $\Phi_* w$ as

$$\mathcal{H}(\Phi_* w) = \int_M \Phi_*\alpha \wedge d(\Phi_*\alpha) = \int_M \Phi_*(\alpha \wedge d\alpha) = \int_M \alpha \wedge d\alpha = \mathcal{H}(w) \,,$$

as we wanted to prove. □

A key feature of the helicity, which distinguishes it from the KAM-type invariants introduced in [18], is that it is an integral operator with a well-behaved integral kernel. A natural question is whether there exist other quantities that are integrals of local densities and are invariant under any volume-preserving diffeomorphisms. This problem has been recently solved by Enciso, Peralta-Salas and Torres de Lizaur in [11], where they prove that given a functional $\mathcal{I}$ defined on $\mathfrak{X}^1_{\rm ex}(M)$ that is associated with a well-behaved integral kernel, then $\mathcal{I}$ is invariant under volume-preserving diffeomorphisms if and only if it is a function of the helicity.

We finish this section recalling that the Euler equations also admit conservation laws of Noether type, which are associated with continuous symmetries of the equations. The most important one is the kinetic energy

$$K(u) := \frac{1}{2}\int_M |u|^2\,,$$

which is well defined provided that M is compact or u decays fast enough at infinity. If the Riemannian manifold (M, g) has a Killing vector field ξ, there is another conserved quantity of Noether type, which is called the momentum of the fluid flow:

Proposition 2.5. *Let ξ be a Killing vector field of a compact manifold (M, g) without boundary. Then the momentum*

$$M(u) := \int_M u\cdot\xi$$

is conserved by the Euler equations.

Proof. The Euler equations (1.1) imply that

$$\frac{dM(u)}{dt} = -\int_M \xi\cdot\nabla P - \int_M \xi\cdot\nabla_u u = -\int_M u\cdot\nabla_u\xi + \int_M u\cdot\nabla(u\cdot\xi) = -\int_M u\cdot\nabla_u\xi\,,$$

where we have integrated by parts and used that $\operatorname{div} u = 0$. The proposition follows noticing that $u\cdot\nabla_u\xi = 0$ because ξ is a Killing field. □

Remark 2.6. A generic Riemannian metric does not admit Killing vector fields, so the momentum $M(u)$ does not provide a non-trivial conserved quantity. It is unknown if in such a case the only integral invariants of the Euler equations are the helicity and the kinetic energy; partial results in this direction in the Euclidean space were obtained by Serre in [23].

3. Stationary solutions of the Euler equations

In what follows, we will be concerned with stationary solutions of the Euler equations, which describe an equilibrium configuration of the fluid. In this case, the velocity field u does not depend on time, so it defines an autonomous dynamical system on the manifold M. The stationary Euler equations are

$$\nabla_u u = -\nabla P\,, \qquad \operatorname{div} u = 0\,, \tag{3.1}$$

or, equivalently in terms of the Bernoulli function and the vorticity,

$$u\times\omega = \nabla B\,, \qquad \operatorname{div} u = 0\,. \tag{3.2}$$

If the manifold M has a non-empty boundary, it is customary to assume that u is tangent to the boundary. A divergence-free vector field that satisfies the stationary Euler equations is sometimes called an *Euler vector field.*

Remark 3.1. Using the dual 1-form $\alpha = i_u g$, the stationary Euler equations can also be written as

$$i_u d\alpha = -dB\,, \qquad d(\star\alpha) = 0\,. \tag{3.3}$$

This is a fully nonlinear system of partial differential equations so, a priori, it is not easy to see for which choices of the function B there exist any solutions (even locally!) and which properties they can exhibit. In fact, only a few explicit solutions of the stationary Euler equations are known (even in $\mathbb{R}^3$). A particularly relevant class of solutions are the *Beltrami flows*, which satisfy the equations

$$\operatorname{curl} u = f u\,, \qquad \operatorname{div} u = 0\,, \tag{3.4}$$

for some function f. These vector fields are solutions of the stationary Euler equations with constant Bernoulli function, that is the pressure is given by $P = -\frac{1}{2}|u|^2$ up to the addition of a constant. Beltrami flows will be studied in detail in Section 6.

The analysis of the integral curves of the autonomous vector fields u and ω, i.e. the stream and vortex lines, is an important topic in the study of the Lagrangian turbulence and the hydrodynamical instability. In particular, the existence of Euler vector fields with periodic stream (or vortex lines) of arbitrary knot type has been a major open problem in the subject for decades. For example, the topologist and dynamicist R.F. Williams asked in [25] about the existence of a stationary fluid flow in $\mathbb{R}^3$ having stream lines tracing out all knots at the same time. We recall that a knot is a smoothly embedded circle in M (without self-intersections). It is obvious that, for stationary flows, the particle paths coincide with the stream lines.

In order to gain some intuition on the dynamics of the Euler vector fields, let us now present some interesting examples with geometric meaning. In the next sections we will review the state of the art of the aforementioned general questions.

Example 3.2. Any Killing vector field u on (M, g) is a stationary solution of the Euler equation with pressure $P = \frac{1}{2}|u|^2$ (so, in particular, $B = |u|^2$). Indeed, being a Killing field, $\operatorname{div} u = 0$. Moreover, an easy computation shows that $\nabla_u u = -\frac{1}{2}\nabla(|u|^2)$, thus proving the claim.

Example 3.3. If u is a divergence-free geodesic vector field, it is obviously a stationary solution of the Euler equations with constant pressure because $\nabla_u u = 0$.

These examples provide solutions of the stationary Euler equations for metrics admitting a Killing field or a geodesic field, which is a highly non-generic phenomenon. A simple class of solutions which exist for any compact (M, g) provided that the first De Rham cohomology group $H^1(M; \mathbb{R})$ is not trivial, are the harmonic fields, that is the solutions to the equations $\operatorname{div} u = 0$ and $\operatorname{curl} u = 0$. It is well known that the number of linearly independent harmonic fields on a compact Riemannian manifold is equal to the rank of $H^1(M; \mathbb{R})$ (the first Betti number of M). Observe that, although the existence of such fields is a purely topological issue, their dynamical properties strongly depend on the metric.

We finish this section introducing an explicit construction of solutions of the stationary Euler equations in the sphere $\mathbb{S}^3$ with stream and vortex lines of arbitrary torus knot type. This family was first presented in [18]:

Example 3.4. It is convenient to represent $\mathbb{S}^3$ as the unit sphere in $\mathbb{R}^4$,

$$\mathbb{S}^3 = \{(x, y, z, \xi) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + \xi^2 = 1\}\,.$$

It is well known that the Hopf fields $u_1 = (-y, x, \xi, -z)|_{\mathbb{S}^3}$ and $u_2 = (-y, x, -\xi, z)|_{\mathbb{S}^3}$ on $\mathbb{S}^3$ are Beltrami flows with eigenvalue 2 and -2 respectively, that is

$$\operatorname{curl} u_1 = 2u_1\,, \qquad \operatorname{curl} u_2 = -2u_2\,.$$

The name Hopf fields comes from the fact that they are tangent to two (isotopic) Hopf fibrations of $\mathbb{S}^3$, so that all the integral curves of u_1 and u_2 are periodic and linked once to each other. Observe that u_2 is somehow the mirror image of u_1.

Obviously, the function $F = (x^2 + y^2)|_{\mathbb{S}^3}$ is a common first integral of u_1 and u_2. The regular level sets of F are 2-dimensional tori, and the critical level sets consist of two circles C_0 and C_1, which correspond to the minimum and maximum values 0 and 1 of F. The integral curves of u_1 and u_2 lie on these level sets, and are trivial knots linked once with each core circle C_0 and C_1. The following properties are also easy to check:

- $|u_1|^2 = |u_2|^2 = 1$,
- $u_1 \cdot u_2 = 2F - 1$
- $u_1 \times u_2 = -\nabla F$

where both the dot and cross products $\cdot$ and $\times$ are computed using the (induced) round metric of $\mathbb{S}^3$.

We claim that the vector field

$$u = f_1(F)u_1 + f_2(F)u_2$$

is a smooth stationary solution of the Euler equations for any choice of the functions f_1 and f_2 in $C^\infty(\mathbb{R})$, with a Bernoulli function given by

$$B = \int_0^F H(s)ds\,, \qquad H(s) := f_1 f_1' + f_2 f_2' + 4 f_1 f_2 + (2s-1)(f_1 f_2' + f_2 f_1')\,.$$

Observe that the connected components of the level sets of B are given by the level sets of F. It is trivial to check that u is divergence-free using that u_1 and u_2 are divergence-free and F is a common first integral of both fields. The equation $u \times \operatorname{curl} u = \nabla B$ is more complicated to derive, the reader can find a detailed proof in [18].

This construction gives rise to interesting particular solutions. First, notice that the fields $\{u_1, u_2\}$ form a 2-basis at each point of $F^{-1}(c)$ for $c \in (0, 1)$, so that $u(p) = 0$ at a point $p \in F^{-1}(c)$ if and only if $f_1(p) = f_2(p) = 0$. For example, we can obtain any Seifert foliation of the 3-sphere by choosing

$$f_1 = \frac{m-n}{2}\,, \qquad f_2 = \frac{m+n}{2}$$

with m, n coprime integers. With this choice, all the integral curves of u are (m, n)-torus knots except for the 2 exceptional fibres C_0 and C_1. We recall that a Seifert foliation of $\mathbb{S}^3$ is a foliation by circles. On the contrary, if

$$f_1 = c_1\,, \qquad f_2 = c_2$$

with $c_1/c_2 \notin \mathbb{Q}$, all the integral curves are quasi-periodic except for the circles C_1 and C_2. Analogous results hold for the vorticity $\omega = \operatorname{curl} u$, which in the case that f_1 and f_2 are constants, is given by the formula

$$\omega = 2f_1 u_1 - 2f_2 u_2\,.$$

This family also illustrates that the stationary Euler equations do not satisfy the unique continuation property. Indeed, we can choose f_1 and f_2 such that

$$f_1(s) = f_2(s) = 0 \qquad \text{if} \qquad s \in \Big[0, \frac{1}{4}\Big] \cup \Big[\frac{3}{4}, 1\Big]\,,$$

and non-zero otherwise. Therefore, u is zero only on the solid tori $F^{-1}([0,1/4])$ and $F^{-1}([3/4,1])$.

All the knotted stream lines and vortex lines that one can obtain within this family of explicit solutions are of torus knot type. The case of more general knots will be addressed in Section 6 using high-energy Beltrami fields on $\mathbb{S}^3$.

Remark 3.5. It is easy to check that any non-vanishing Euler vector field u in Example 3.4 has a periodic orbit. In fact, it is unknown if there exist smooth non-vanishing divergence-free vector fields in $\mathbb{S}^3$ without periodic orbits (C^1 examples were constructed in [20]).

Example 3.6. A similar, but simpler, example of a family of stationary solutions of the Euler equations can be constructed in the flat torus $\mathbb{T}^3$. We endow $\mathbb{T}^3$ with the 2π-periodic coordinates (x,y,z). Then, it is easy to check that for any 2π-periodic smooth functions $f_1:\mathbb{R}\to\mathbb{R}$ and $f_2:\mathbb{R}\to\mathbb{R}$, the vector field

$$u = f_1(z)\partial_x + f_2(z)\partial_y$$

is a stationary solution of the Euler equations with Bernoulli function

$$B = \frac{1}{2}(f_1^2 + f_2^2)\,,$$

so each connected component of the level sets of B is given by a surface $z=$ constant. Since z is a first integral of u, the trajectories of the velocity field lie on the 2-dimensional tori $z=$ constant. Taking $f_1 = c_1$ and $f_2 = c_2$, with c_i constants such that $c_1/c_2 \notin \mathbb{Q}$, it follows that u has no periodic orbits (all the integral curves of u are quasi-periodic).

4. Arnold's structure theorem of stationary solutions

In Examples 3.4 and 3.6 of Section 3 we saw that the Bernoulli function is a first integral of the velocity field u and the vorticity ω. In fact, this is a general property of any stationary solution of the Euler equations:

Proposition 4.1. *Let u be a stationary solution of the Euler equations with Bernoulli function B. Then B is a first integral of u and ω, that is $u\cdot\nabla B = \omega\cdot\nabla B = 0$. Moreover, $u(p)$ and $\omega(p)$ are linearly independent non-vanishing vectors at each point $p\in M$ such that $\nabla B(p)\neq 0$.*

Proof. It is immediate from the equation $u\times\omega = \nabla B$. □

Accordingly, the stream lines and vortex lines of a stationary solution lie on the level sets of B, which can be interpreted in physical terms as a *laminar behavior* for the fluid flow. The turbulent behavior understood as Lagrangian turbulence (which means that the dynamical system defined by u is chaotic) can only appear in the domains where B is constant, because in this case the first integral is trivial and the integral curves of u (and ω) are not constrained to lie on 2-dimensional surfaces. For a C^∞ stationary solution, domains where B is constant can coexist with domains where it is not a constant, see e.g. Example 3.4. However, in the analytic (C^ω) setting, Arnold observed the following dichotomy [1]. Assume that (M,g) is an analytic Riemannian manifold and u is a C^ω stationary solution of the Euler equations, then

- If B is not a constant, its critical set
$$\mathrm{Cr}(B) := \{p \in M : \nabla B(p) = 0\}$$
has codimension at least 1, and in particular it has en empty interior.
- If B is a constant, then the vorticity $\operatorname{curl} u$ is proportional to u everywhere, i.e. there exists a function f such that $\operatorname{curl} u = fu$ and $\operatorname{div} u = 0$, and therefore u is a Beltrami flow.

Arnold's observation is based on the fact that the critical set $\mathrm{Cr}(B)$ is defined as the zero set of the analytic function $(\nabla B)^2$, so it is an analytic set by definition. Since the structure of analytic sets is severely restricted [19] (for instance, they are stratified submanifolds of codimension greater or equal than one), the aforementioned dichotomy holds. In fact, Arnold proved in [1] a structure theorem for C^ω stationary solutions of the Euler equations on compact manifolds, which was one of the landmarks that marked the birth of the modern Topological Hydrodynamics [17]. It asserts that, in the analytic setting, the stream and vortex lines of a stationary solution of the Euler equations whose velocity field is not collinear with its vorticity are nicely stacked in a rigid structure analogous to those which appear in the study of integrable Hamiltonian systems with 2 degrees of freedom:

Theorem 4.2 (Arnold's structure theorem)**.** *Let u be a C^ω stationary solution of the Euler equations on an analytic compact manifold M with non-constant Bernoulli function. If M has a non-empty boundary ∂M, we assume that u is tangent to the boundary. Then, there exists an analytic set C of codimension greater or equal than one, such that $M\backslash C$ consists of finitely many domains M_i such that*

(i) *Either M_i is trivially fibred by invariant tori of u (so that, in particular, M_i is diffeomorphic to the product $\mathbb{T}^2 \times (-1, 1)$), and on each torus, the velocity field u is conjugate to a linear field.*
(ii) *Or M_i is trivially fibred by invariant cylinders of u whose boundaries lie on ∂M (so that, in particular, M_i is diffeomorphic to $\mathbb{S}^1 \times (-1, 1) \times (-1, 1)$), and on each cylinder, all the stream lines of u are periodic.*

Proof. We start defining the analytic set C. To this end, we consider the set Λ_1 consisting of all the critical level sets of B, that is
$$\Lambda_1 := \{B^{-1}(c) : c \text{ is a critical value of } B\} ,$$
and the set Λ_2 of all the level sets of B that are tangent at some point to the boundary ∂M. By construction, the set $C = \Lambda_1 \cup \Lambda_2$ is an analytic set of codimension greater or equal than one because it is the union of finitely many level sets of the analytic function B. The number of level sets in C is finite due to the facts that M is compact (and then the number of critical value of B is finite), and that M, and therefore its boundary ∂M, are analytic manifolds (and hence the number of values c for which $B^{-1}(c)$ is tangent to ∂M is finite as well). Observe that C does not need to be a topological submanifold (it may have self-intersections).

Accordingly, the set $M\backslash C$ consists of finitely many pre-compact domains M_i, on each M_i any level set of B that intersects ∂M do it transversely, and ∇B does not vanish at any point of M_i. It is easy to check that these properties imply that the union of the level sets of B in M_i defines a 2-dimensional foliation, which is a trivial bundle over the interval $(-1, 1)$ whose fiber is an orientable surface Σ_i that is a regular level set of B (possibly with boundary). Moreover, Proposition 4.1 implies that u and ω are linearly independent vectors at each point of M_i.

If Σ_i has an empty boundary, it must be diffeomorphic to a torus $\mathbb{T}^2$ because it supports the non-vanishing vector field u. Therefore, any level set of B in M_i is diffeomorphic to $\mathbb{T}^2$. Moreover, Helmholtz's transport of vorticity, cf. Theorem 2.1, implies for stationary fluids that

$$[u, \omega] = 0 \,, \tag{4.1}$$

where $[\cdot]$ is the commutator of vector fields. Since u and ω are linearly independent at any point of M_i, we conclude that their flows define an $\mathbb{R}^2$-action, and therefore both u and ω are conjugate to a linear vector field [3] on each level set of B in M_i.

If Σ_i has a boundary, it must lie on ∂M, and since both ∂M and Σ_i are invariant by the vector field u, which does not vanish on Σ_i, it follows that $\partial \Sigma_i$ consists of finitely many periodic orbits of u. Additionally, the vorticity ω is a transverse symmetry of u because both fields commute by Eq. (4.1), so all the integral curves of u in Σ_i are periodic. It is known that the only orientable surface with boundary that can be foliated by circles is the cylinder, and therefore Σ_i is diffeomorphic to $\mathbb{S}^1 \times (-1, 1)$. The same holds for any other level set of B in M_i, thus concluding the proof of the theorem. □

Remark 4.3. Arnold's theorem is an a posteriori consequence in the sense that it describes the topological structure of the stationary solutions of the Euler equations under certain assumptions, but it does not say anything on the existence of solutions of this type.

The C^ω assumption in Arnold's theorem is not key, actually it is only used to control the critical set of B and the intersections of the level sets of B and ∂M. It is not difficult to prove an analogous result assuming, for example, that $\partial M = \emptyset$ and B is a C^2 function that is Morse-Bott. The compactness of M is not key either, so that analogous results hold for non-compact manifolds assuming that u and ω satisfy certain growth conditions at infinity, e.g.

$$|u(x)| + |\omega(x)| < C$$

for all $x \in M$. In the non-compact case, apart from regular level sets of B diffeomorphic to $\mathbb{T}^2$ and $\mathbb{S}^1 \times \mathbb{R}$, we can have level sets diffeomorphic to $\mathbb{R}^2$, which correspond to the three possible leaves for $\mathbb{R}^2$-actions.

If M is a compact manifold without boundary, Theorem 4.2 implies that all the connected components of the regular level sets of B are tori. In particular, this implies that a necessary condition for the existence of C^ω stationary solutions of the Euler equations whose velocity field is not collinear with the vorticity, is that M must admit an analytic function all whose regular level sets are diffeomorphic to a torus. I do not know if such functions exist on any compact 3-manifold. However, an easy consequence of this observation is that for "most" functions B, the stationary Euler equations do not have a global solution:

Corollary 4.4. *Let M be a compact 3-manifold without boundary. Then, there exists an open and dense set of functions $\mathcal{S} \subset C^2(M)$ such that if $B \in \mathcal{S}$, then there are no C^1 global solutions of the stationary Euler equations in M with Bernoulli function B, for any choice of the metric g.*

Proof. The set $\mathcal{S}$ of C^2 Morse functions on M is open and dense in $C^2(M)$. Moreover, the compactness of M implies that any function $B \in \mathcal{S}$ attains its maximum value at some point $p \in M$, and therefore the regular level sets of B in a neighborhood of p are diffeomorphic to spheres. Then, the function B cannot be the Bernoulli function of a stationary solution of the Euler equations. □

This result contributes to understand the apparent contradiction between the fact that a "generic" stationary fluid is integrable (laminar) because the function B is generically non-constant, and the common wisdom that integrability is a rare phenomenon (turbulence should be typical). The point is that for a generic function B there are no stationary solutions.

Another consequence of Arnold's structure theorem is that the way the stream lines are arranged is severely restricted, which manifests in the fact that under appropriate assumptions the stream lines cannot be of certain knot types [13]:

Theorem 4.5. *Let u be a C^ω stationary solution of the Euler equations in $\mathbb{S}^3$ (for an arbitrary metric) with non-constant Bernoulli function. Assume that u does not vanish anywhere. Then the periodic stream lines of u can only be knots of the following types: torus knots, cablings of torus knots or connected sums of them.*

This theorem suggests that if we want to construct stationary solutions of the Euler equations with complicated dynamics (e.g. with stream or vortex lines of any knot type), we should consider Beltrami flows, cf. Eq. (3.4). In this case, the Bernoulli function B is a constant, so Arnold's structure theorem does not give any information (u and ω are collinear everywhere). We shall see in Section 6 that the Beltrami flows with constant proportionality factor

$$\operatorname{curl} u = \lambda u \,,$$

with $\lambda \in \mathbb{R}\backslash\{0\}$, are flexible enough in order to realize stream lines of arbitrarily complicated topology.

5. Helicity and Moffatt's theorem

H.K. Moffatt introduced in 1969 (inspired by an analogous quantity defined by Woltjer in the context of the magnetohydrodynamics [26]) a functional to measure the "degree of knottedness" of a divergence-free vector field: the helicity [21]. We have already introduced it in Section 2, see Eq. (2.1) and Theorem 2.4, where we showed that $\mathcal{H}$ is a conserved quantity for the time-dependent Euler equations and, more generally, it is invariant under the action of volume-preserving diffeomorphisms on exact divergence-free vector fields. The topological interpretation of the helicity was first unveiled by Moffatt for vector fields modeled on a link in the following sense:

Definition 5.1. We say that a divergence-free vector field w in $\mathbb{R}^3$ is modeled on the two-component link $\gamma_1 \cup \gamma_2$ (each γ_i is a knot) if w is supported in a tubular neighborhood $T_1 \cup T_2$ of the link, and $w|_{T_i}$ is tangent to a trivial fibration by circles of T_i where all fibres are unlinked to each other.

Before stating Moffatt's theorem, let us introduce some notation. For each tubular neighborhood T_i of the knot γ_i there is a diffeomorphism $\Psi_i : T_i \to \mathbb{S}^1 \times \mathbb{D}^2$, where $\mathbb{D}^2$ is the unit 2-dimensional disk. Then, denoting by $\Pi : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{D}^2$

the natural projection over the disk, the vector field $w|_{T_i}$ is tangent to the trivial fibration defined by $\Pi \circ \Psi_i : T_i \to \mathbb{D}^2$, and we choose the diffeomorphism Ψ_i such that all the fibres are unlinked to each other; such a trivialization of a tubular neighborhood of a knot is called *preferred*, and it always exists and is unique up to isotopy. We parameterize the set $\mathbb{S}^1 \times \mathbb{D}^2$ with coordinates $\theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $y \in \mathbb{D}^2$, and denote by Φ_i the flux of $w|_{T_i}$ across a transverse section of T_i; since w is divergence-free, it is straightforward to check that Φ_i does not depend on the section, which can be chosen to be $\Psi_i^{-1}(\{\theta = 0\})$.

Theorem 5.2 (Moffatt's theorem)**.** *Let w be a divergence-free vector field in $\mathbb{R}^3$ that is modeled on the link $\gamma_1 \cup \gamma_2$. Then*

$$\mathcal{H}(w) = 2\Phi_1\Phi_2 Lk(\gamma_1, \gamma_2)\,,$$

where $Lk(\gamma_1, \gamma_2)$ is the linking number of γ_1 and γ_2.

Proof. To simplify the computations, we shall assume that the metric in coordinates (θ, y) is flat, that is $ds^2 = d\theta^2 + dy^2$. By construction of the diffeomorphism Ψ_i, and since w is divergence-free, a straightforward computation shows that

$$(\Psi_i)_* w = F_i(y)\partial_\theta\,,$$

for some smooth function $F_i : \mathbb{D}^2 \to \mathbb{R}$ with $F_i(\partial\mathbb{D}^2) = 0$.

Now, consider the vector field $u := \operatorname{curl}^{-1} w$, which is smooth in $\mathbb{R}^3$ and can be computed using the Biot-Savart integral operator. In coordinates (θ, y) we can write this vector field as follows:

$$(\Psi_i)_* u = u_\theta^{(i)}\partial_\theta + u_y^{(i)}\partial_y\,,$$

where $u_\theta^{(i)}$ and $u_y^{(i)}$ are certain functions of θ and y whose explicit form is not relevant for our purposes. Using the expression of the metric in coordinates (θ, y), we compute the helicity of w as

$$\begin{aligned}\mathcal{H}(w) = \int_{T_1\cup T_2} u\cdot\omega &= \int_{\mathbb{S}^1\times\mathbb{D}^2} F_1(y)u_\theta^{(1)}d\theta\,dy + \int_{\mathbb{S}^1\times\mathbb{D}^2} F_2(y)u_\theta^{(2)}d\theta\,dy\\ &= \int_{\mathbb{D}^2} F_1(y)\Big(\int_{\mathbb{S}^1} u_\theta^{(1)}d\theta\Big)dy + \int_{\mathbb{D}^2} F_2(y)\Big(\int_{\mathbb{S}^1} u_\theta^{(2)}d\theta\Big)dy\,.\end{aligned}$$

For each value $y = c$, the integral of $u_\theta^{(i)}$ in the θ variable, is just the circulation of the vector field u along the circle $\gamma_i^c := \Psi_i^{-1}(\{y = c\})$, $|c| < 1$, which can be computed using Stokes theorem:

$$\int_{\mathbb{S}^1} u_\theta^{(i)}d\theta = \int_{\gamma_i^c} u\cdot ds = \int_{\Sigma_i^c} w\cdot\nu d\sigma\,. \tag{5.1}$$

In this formula Σ_i^c is any surface with $\partial\Sigma_i^c = \gamma_i^c$ (a *Seifert surface* of the knot γ_i^c, which is isotopic to γ_i), ν is a unit vector field normal to the surface and $d\sigma$ is the surface measure. Since the trivialization Ψ_i of the tubular neighborhood T_i is preferred, we can choose the surface Σ_i^c to be tangent to the fibres $\Psi_i^{-1}(\{y = c\})$, and hence to the vector field $w|_{T_i}$, and transverse to the fibres $\Psi_{i'}^{-1}(\{y = c\})$, and hence to the vector field $w|_{T_{i'}}$, with $i' = i + 1$ mod. 2. Therefore, since the flux of $w|_{T_{i'}}$ does not depend on the transverse section, and the linking number of γ_1 and

γ_2 is just the number of intersections with sign of γ_1 with any Seifert surface of γ_2 (and conversely), we easily deduce that

$$\int_{\Sigma_i^c} w \cdot \nu d\sigma = \Phi_{i'} \mathrm{Lk}(\gamma_1, \gamma_2)\,. \tag{5.2}$$

Using Eqs. (5.1) and (5.2) in the expression of the helicity, we finally obtain

$$\mathcal{H} = \Phi_2 \mathrm{Lk}(\gamma_1, \gamma_2) \int_{\mathbb{D}^2} F_1(y) dy + \Phi_1 \mathrm{Lk}(\gamma_1, \gamma_2) \int_{\mathbb{D}^2} F_2(y) dy = 2\Phi_1 \Phi_2 \mathrm{Lk}(\gamma_1, \gamma_2)\,,$$

as we wanted to prove. □

Remark 5.3. We observe that the assumption that the vector field $w|_{T_i}$ is tangent to a trivial fibration by circles of T_i where all fibres are unlinked to each other (or equivalently, that the trivialization Ψ_i is preferred) is crucially used for the existence of a Seifert surface Σ_i^c of γ_i^c which is tangent to the field $w|_{T_i}$. Indeed, if the trivialization is preferred, the fibre γ_i^c (provided with an orientation) defines the zero homology class in $H_1(\mathbb{R}^3 \backslash \gamma_i; \mathbb{Z})$, and hence the Seifert surface Σ_i^c can be taken to be disjoint from the neighborhood $\Psi_i^{-1}(\{|y| < c\})$ and tangent to a 1-parameter family of fibres $\gamma_i^{c'}$ with $|c'| \in [|c|, 1)$.

Moffatt's theorem holds for vector fields that are modeled on a link, which is a very restrictive class of flows. For a general divergence-free vector field, the relationship between helicity and the linkage/knottedness of the trajectories of the field is not so clear. In [2], Arnold proved for compact manifolds that the helicity can be interpreted as an asymptotic (in the sense of $t \to \infty$) and averaged (in the sense of integrating over the whole manifold) linking number of the trajectories. Nevertheless, the implications of the helicity on the dynamical properties of the field is still unclear. For example, Taubes proved [24] that there are no uniquely ergodic divergence-free vector fields with positive helicity, but non-vanishing helicity does not imply, however, the existence of periodic orbits, at least in class C^1:

Theorem 5.4. *For any constant $c \neq 0$, there exists a C^1 non-vanishing divergence-free vector field w in the 3-sphere $\mathbb{S}^3$ without periodic orbits and with helicity*

$$\mathcal{H}(w) = c\,.$$

Remark 5.5. The proof of this theorem is a variation of Kuperberg's construction [20] of C^1 non-vanishing divergence-free vector fields in $\mathbb{S}^3$ without periodic orbits. It is unknown if such vector fields exist in class C^2.

There is no obstruction on the values that the helicity can take for stationary solutions of the Euler equations. For instance, consider the family of stationary solutions in $\mathbb{S}^3$ introduced in Example 3.4 with constant coefficients $f_1 = c_1$ and $f_2 = c_2$, $c_i \in \mathbb{R}$. Then, the helicity of the vorticity $\mathcal{H}(\omega)$ is given by

$$\mathcal{H}(\omega) = 2(c_1^2 - c_2^2)\,,$$

which can be any real number. However, Beltrami flows with constant proportionality factor, which satisfy the equation $\omega = \lambda u$, always have non-zero helicity $\mathcal{H}(\omega) = \lambda^{-1} \|\omega\|^2_{L^2(M)}$.

We finish this section recalling Rechtman's theorem [22], which shows the existence of periodic stream lines and vortex lines for non-vanishing stationary solutions of the Euler equations in the analytic class (C^ω). In the proof of this result, the dichotomy observed by Arnold in the C^ω setting that was discussed in Section 4 turns out to be key: either u has a non-trivial analytic first integral, or it is a Beltrami flow.

Theorem 5.6. *Let u be a C^ω stationary solution of the Euler equations on a compact manifold M without boundary, and assume that u does not vanish anywhere. Then, if M is not a $\mathbb{T}^2$-bundle over a circle, the velocity field u has a periodic stream line.*

Remark 5.7. It is not known if any C^∞ non-vanishing stationary solution of the Euler equations, on a 3-manifold that is not a torus bundle over the circle, has a periodic stream line. If the stationary solution is a Beltrami flow (with possibly non-constant proportionality factor), then it always has a period stream line if the manifold is not a torus bundle over the circle, see Theorem 6.3 in the following section.

Compare Theorem 5.6 with Examples 3.4 and 3.6. In the first example, we constructed a family of stationary solutions in $\mathbb{S}^3$ such that any non-vanishing velocity field in this family has a periodic stream line (either the circle C_0 or the circle C_1). In the second example, we constructed a family of stationary solutions in $\mathbb{T}^3$, and showed that there is a non-vanishing velocity field in this family without periodic stream lines (all the integral curves are quasi-periodic).

6. Beltrami flows and knotted vortex structures

In Section 3 we introduced a special class of stationary solutions of the Euler equations, the Beltrami flows, which satisfy Eq. (3.4). Obviously the stream lines of a Beltrami field are the same as its vortex lines, so henceforth we will only refer to the former. A straightforward consequence of the Beltrami equation is that the proportionality factor f is a first integral of u (and ω), that is $u \cdot \nabla f = \omega \cdot \nabla f = 0$. Accordingly, the stream lines of the stationary solution lie on the level sets of the function f, which implies a laminar-type behavior of the fluid on the regions where f is not a constant. In fact, Theorem 4.5 also holds for C^ω non-vanishing Beltrami fields in $\mathbb{S}^3$ with non-constant proportionality factor, thus showing that not any knot type is admissible for the stream lines of these kinds of fields.

Despite its apparent simplicity, the solutions of the Beltrami equation are very difficult to handle. In particular, it can be shown [10] that there are no nontrivial (local) solutions for an open and dense set of factors f in the C^k topology, $k \geqslant 7$. The reason is that the existence of a non-trivial solution of the Beltrami equation in a domain U implies that f must satisfy the constraint $P[f] = 0$ in U, where P is a non-linear partial differential operator involving derivatives of order at most 6. Observe that Arnold's structure theorem does not apply in this setting because the vorticity is parallel to the velocity field, so the compact regular level sets of f do not need to be diffeomorphic to a torus. Nevertheless, it is not difficult to show that f cannot have a (connected component of a) regular level set diffeomorphic to the sphere $\mathbb{S}^2$:

Proposition 6.1. *Let u be a non-trivial solution of the Beltrami equation with proportionality factor f in a neighborhood U of a regular level set $\Lambda_c := f^{-1}(c)$. Then no connected component of Λ_c can be diffeomorphic to $\mathbb{S}^2$.*

Proof. Assume that a connected component Σ of Λ_c is diffeomorphic to $\mathbb{S}^2$. Since u is divergence-free and f is a first integral, it is easy to check that the induced vector field j^*u on Σ preserves the area 2-form

$$\mu_2 := j^*(i_{\frac{\nabla f}{|\nabla f|^2}}\mu) .$$

Here $j : \Sigma \to U$ is the inclusion of the surface Σ in U. Then, j^*u being divergence-free on a surface Σ diffeomorphic to $\mathbb{S}^2$, it is standard that it has a periodic trajectory $\gamma \subset \Sigma$ (because j^*u has a non-trivial first integral on Σ). An easy application of Stokes theorem allows us to write

$$0 < \int_\gamma u = \int_D \operatorname{curl} u \cdot \nu \, d\sigma = c \int_D u \cdot \nu \, d\sigma = 0 ,$$

where $D \subset \Sigma$ is a disk with boundary $\partial D = \gamma$, ν is a normal field to Σ and $d\sigma$ is the induced surface measure on Σ. To pass to the second equality we have used that u is a Beltrami flow and $f = c$ on Σ, and in the last equality we have noticed that u is tangent to Σ. This contradiction shows that no connected component of a regular level set of f can be diffeomorphic to $\mathbb{S}^2$. □

In light of the previous comments, we are naturally led to consider a constant proportionality factor f to construct stationary solutions with complex vortex patterns. Then, we will focus our attention on Beltrami fields which satisfy the equation

$$\operatorname{curl} u = \lambda u$$

for some nonzero constant λ. This equation immediately implies that $\operatorname{div} u = 0$.

Beltrami flows do not need to be integrable, in fact Arnold conjectured in [1] that they could present stream lines of arbitrarily complicated topology, which is fully consistent with Williams' problem stated in Section 2. From the analytic viewpoint, Beltrami flows with constant factor have some nice properties:

- (i) The Beltrami PDE is linear, although not elliptic (the principal symbol of the curl operator is an antisymmetric matrix).
- (ii) They satisfy the Helmholtz equation $\Delta u + \lambda^2 u = 0$ and hence, by standard elliptic regularity, they are C^ω if the manifold (M, g) is analytic. However, in the Euclidean space $\mathbb{R}^3$ they cannot have finite energy (there are no $L^2(\mathbb{R}^3)$ eigenfunctions of the Laplacian).
- (iii) There are infinitely many (linearly independent) Beltrami fields on any manifold. Indeed, if (M, g) is compact without boundary, the curl operator has a (non-empty) discrete spectrum; if the manifold is open, there is a non-trivial solution for any constant λ, and typically the multiplicity is infinite (e.g. in $\mathbb{R}^3$). Analogous results hold for manifolds with boundary provided that u is tangent to the boundary.

Remark 6.2. In the context of magnetohydrodynamics, Beltrami fields are called force-free fields. They describe magnetic fields H created by a plasma current J through Maxwell's equation $\operatorname{curl} H = J$, and they do not exert any force on the plasma because the Lorentz force $J \times H$ vanishes by the Beltrami condition.

In Example 3.4 we introduced the Hopf fields, which are Beltrami flows on $\mathbb{S}^3$ with the lowest possible eigenvalues (in absolute value). Other paradigmatic examples are the *ABC flows*, which are the eigenfields of the curl operator in the flat torus $\mathbb{T}^3$ with the lowest (positive) eigenvalue:

$$u(x,y,z) = [A\sin z + C\cos y]\partial_x + [B\sin x + A\cos z]\partial_y + [C\sin y + B\cos x]\partial_z\,,$$

where $(x,y,z)\in(\mathbb{R}/2\pi\mathbb{Z})^3$, and A,B,C are real parameters. If one of the parameters is zero, then the field has a non-trivial first integral. For other values, the ABC fields are typically chaotic and non-integrable, exhibiting transverse homoclinic intersections, see e.g. [5]. An interesting open problem in this context is whether all knot types of stream lines in $\mathbb{T}^3$ can be realized within the ABC family.

In terms of the dual 1-form $\alpha = i_u g$, the Beltrami equation is

$$\star d\alpha = \lambda\alpha\,.$$

In particular, $\alpha\wedge d\alpha = \lambda|u|^2\mu$, so if the velocity field u does not vanish, α is a contact 1-form on M and the rescaled field $|u|^{-2}u$ is the Reeb field of α. Conversely, Etnyre and Ghrist observed [14] that if $\mathcal{R}$ is the Reeb field of a contact 1-form α, there exists a metric g such that $\operatorname{curl}\mathcal{R} = \mathcal{R}$ and $|\mathcal{R}| = 1$ computed with this metric. So, $\mathcal{R}$ is a (non-vanishing) Beltrami field on (M,g) (in fact it is also geodesic). The metric g is not unique, and does not need to be complete if M is open. Since any Reeb flow has a closed integral curve by Taubes' theorem [15], the Beltrami-contact correspondence implies a remarkable consequence:

Theorem 6.3. *Any non-vanishing Beltrami field with constant proportionality factor on a compact manifold (without boundary) has a periodic stream line. The same result holds for Beltrami fields with non-constant proportionality factor, provided that the manifold is not a torus bundle over the circle.*

Remark 6.4. The second claim in this theorem does not follow from Taubes' theorem, but from a more general result proved by Hutchings and Taubes [16]. Beltrami fields with non-constant proportionality factor are related to the so called *stable Hamiltonian structures*, or equivalently to volume-preserving geodesible fields, see [16, 22] for details.

The existence of Beltrami fields on any open Riemannian manifold (in $\mathbb{R}^3$, in particular) with a set of periodic stream lines diffeomorphic to any given link was proved in [7]. This result settles Arnold's conjecture for non-compact manifolds and yields a positive answer to Williams' question.

Theorem 6.5 (Realization theorem for stream/vortex lines)**.** *Let (M,g) be a C^ω open Riemannian 3-manifold, and L a locally finite link in M. Then, for any constant $\lambda\neq 0$, any non-negative integer r and any $\epsilon>0$, there exist a vector field u on M satisfying the equation* $\operatorname{curl} u = \lambda u$*, and a diffeomorphism $\Phi: M\to M$ close to the identity as $\|\Phi - id\|_{C^r(M)} < \epsilon$ in the C^r norm, such that $\Phi(L)$ is a set of hyperbolic periodic stream/vortex lines of u.*

We recall that a link is a disjoint union of knots, and that locally finite means that the number of connected components of the link intersecting any compact set is finite. We also assume that each knot is a smoothly embedded circle. A periodic orbit of a vector field is *hyperbolic* if all the eigenvalues of the corresponding monodromy matrix (i.e. the solution to the normal variational equation evaluated on the periodic trajectory) have absolute values different from 1. It follows from

$\operatorname{div} u = 0$ that the hyperbolic periodic stream lines are of saddle type with both a stable manifold and an unstable manifold of dimension 2.

We remark that the Beltrami field u in this theorem may vanish at some points (so it does not need to define a Reeb flow), and it may have (and typically will have) other periodic stream lines. The assumption that the manifold is C^ω is used to apply the Cauchy-Kowalewski theorem in the proof.

The growth of u at infinity is not controlled in general, although if we assume that the link L is finite and the manifold is the Euclidean space $\mathbb{R}^3$, we can bound the Beltrami field as

$$|D^j u(x)| < \frac{C_j}{|x|}\,,$$

thus implying that u defines a complete flow. Under these same assumptions, Enciso and Peralta-Salas proved in [8] a realization theorem analogous to Theorem 6.5 where the periodic stream lines $\Phi(L)$ are *elliptic* (and hence linearly stable). We recall that a periodic orbit is elliptic if all the eigenvalues of the corresponding monodromy matrix are purely imaginary and have absolute values equal to 1. Actually, the theorem proved in [8] gives much more information because it shows the existence of a set of invariant solid tori of u (and hence of ω) that is a tubular neighborhood of the link L, and u is non-degenerate in a KAM sense and ergodic on the boundary of each solid torus. In the context of fluid mechanics these solid tori are called *vortex tubes*, so we prove the existence of stationary solutions of the Euler equations with vortex tubes that are knotted in an arbitrarily complicated way. The existence of these structures was conjectured by Lord Kelvin in the 19th century, see e.g. [9] for a recent account of this problem.

A key tool in order to prove Theorem 6.5 (and the aforementioned realization theorem for vortex tubes) is the following *Runge-type global approximation theorem* [7, 8]. It establishes the flexibility of Beltrami fields in open manifolds in the sense that any phenomenon that can be realized by a local solution can be realized by a global solution as well. As customary, we will say that a PDE is satisfied in a closed set K if it holds in some open set containing K.

Theorem 6.6. *Let v be a vector field that solves the Beltrami equation* $\operatorname{curl} v = \lambda v$, $\lambda \in \mathbb{R}\backslash\{0\}$, *in a compact set $K \subset M$ such that all the connected components of $M\backslash K$ are unbounded. Then, for any non-negative integer r and any positive constant ϵ, there exists a vector field u satisfying the Beltrami equation* $\operatorname{curl} u = \lambda u$ *in M such that $\|u - v\|_{C^r(K)} < \epsilon$.*

The topological assumption on the compact set K implies that this approximation theorem cannot be applied if M is compact (with or without boundary). In fact, an analogous theorem cannot hold in compact manifolds without boundary because in such a case the spectrum of the curl operator is discrete, so not any value of λ is allowed for global solutions, while locally the equation $\operatorname{curl} u = \lambda u$ admits a non-trivial solution for any λ.

Nevertheless, for the flat torus $\mathbb{T}^3$ and the round sphere $\mathbb{S}^3$ (and quotients of $\mathbb{S}^3$ with a finite subgroup of isometries), a realization theorem for periodic stream lines (and invariant solid tori) that is analogous to Theorem 6.5 can be proved [12] using Beltrami fields with high energy λ. A key point is that, in these manifolds, the multiplicity of λ tends to infinity as $\lambda \to \infty$ (the spectrum of curl is very degenerate), which provides a large set of solutions for each large enough λ. In the realization theorem proved in [12], the link L is finite and is assumed to be contained in a contractible subset (this is always the case in $\mathbb{S}^3$, but not in $\mathbb{T}^3$), and the diffeomorphism Φ transforming L into a union of periodic stream lines of a Beltrami field has the effect of contracting L into a ball of radius λ^{-1}. More precisely, the theorem we proved is the following. In the statement, we write $\mathbb{M}^3$ to denote either $\mathbb{T}^3$ or $\mathbb{S}^3$. Notice that the spectrum of the curl operator in $\mathbb{M}^3$ contains all the integers.

Theorem 6.7. *Let L be a finite link in $\mathbb{M}^3$. In the case of the torus, we also assume that L is contained in a contractible subset of $\mathbb{T}^3$. Then for any large enough odd integer λ there exists a Beltrami field u satisfying the equation* $\operatorname{curl} u = \lambda u$ *and a diffeomorphism $\Phi : \mathbb{M}^3 \to \mathbb{M}^3$ connected with the identity such that $\Phi(L)$ is a union of stream/vortex lines of u.*

For each compact and without boundary 3-manifold M there is a generic set of metrics (generic in the sense of a residual set in the C^k norm) such that the spectrum of the curl operator is simple [6], i.e. for each λ in the spectrum of curl the equation $\operatorname{curl} u = \lambda u$ has a unique solution up to a multiplicative constant factor. Therefore, the idea used to prove Theorem 6.7 does not hold, and hence the following important question remains open:

Open problem: Let (M, g) be a compact Riemannian 3-manifold without boundary. For each knot $L \subset M$, does there exist a Beltrami field u satisfying $\operatorname{curl} u = \lambda u$ that realizes L as a periodic stream line, up to a diffeomorphism of M?

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Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, 28049 Madrid, Spain

E-mail address: dperalta@icmat.es