EXISTENCE OF KNOTTED VORTEX STRUCTURES IN STATIONARY SOLUTIONS OF THE EULER EQUATIONS

ALBERTO ENCISO AND DANIEL PERALTA-SALAS

ABSTRACT. In this paper, we review recent research on certain geometric aspects of the vortex lines of stationary ideal fluids. We mainly focus on the study of knotted and linked vortex lines and vortex tubes, which is a topic that can be traced back to Lord Kelvin and was popularized by the works of Arnold and Moffatt on topological hydrodynamics in the 1960s. In this context, we provide a leisurely introduction to some recent results concerning the existence of stationary solutions to the Euler equations in Euclidean space with a prescribed set of vortex lines and vortex tubes of arbitrarily complicated topology. The content of this paper overlaps substantially with the one the authors published in the Newsletter of the European Mathematical Society in June 2015.

1. INTRODUCTION

The dynamics of an inviscid incompressible fluid flow in $\mathbb{R}^3$ is modeled by the Euler equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P, \qquad \operatorname{div} u = 0,$$

where $u(x, t)$ is the velocity field of the fluid, which is a time-dependent vector field, and $P(x, t)$ is the pressure function, which is defined by these equations up to a constant. This system of partial differential equations was first published by Leonhard Euler in 1757 and still stands as a major challenge for engineers, physicists and mathematicians.

The motion of the particles in the fluid is described by the integral curves of the velocity field, that is, by the solutions to the non-autonomous ODE

$$\dot{x}(t) = u(x(t), t)$$

for some initial condition $x(t_0) = x_0$, and are usually called *particle paths*. The integral curves of $u(x, t)$ at fixed time t are called *stream lines*, and thus the stream line pattern changes with time if the flow is unsteady.

Another time-dependent vector field that plays a crucial role in fluid mechanics is the *vorticity*, defined by

$$\omega := \operatorname{curl} u.$$

This quantity is related to the rotation of the fluid and is a measure of the entanglement of the stream lines. The integral curves of the vorticity $\omega(x, t)$ at fixed time t, that is to say, the solutions to the autonomous ODE

$$\dot{x}(\tau) = \omega(x(\tau), t)$$

for some initial condition $x(0) = x_0$, are the *vortex lines* of the fluid at time t. A domain in $\mathbb{R}^3$ that is the union of vortex lines and whose boundary is a smoothly embedded torus is called a (closed) *vortex tube*. Obviously, the boundary of a vortex tube is an invariant torus of the vorticity.

In this article we will be concerned with *stationary* solutions of the Euler equations, which describe an equilibrium configuration of the fluid. In this case, the velocity field u does not depend on time, and the Euler equations can then be written as

$$u \times \omega = \nabla B\,, \qquad \operatorname{div} u = 0\,,$$

where $B := P + \frac{1}{2}|u|^2$ is the Bernoulli function. This is a fully nonlinear system of partial differential equations, so a priori it is not easy to see for which choices of the function B there exist any solutions and which properties they can exhibit. It is obvious that for stationary flows, the particle paths coincide with the stream lines.

Our goal is to introduce some results in fluid mechanics whose common denominator is that the main objects of interest are the stream and vortex lines of ideal fluid flows. In particular, we shall review the recent construction of stationary solutions to the Euler equations in $\mathbb{R}^3$ describing topologically nontrivial fluid structures [5, 6]. Mathematically, these problems are extremely appealing because they give rise to remarkable connections between different areas of mathematics, such as partial differential equations, dynamical systems and differential geometry. From a physical point of view, these questions are often considered in the Lagrangian approach to turbulence and in the study of the hydrodynamical instability.

In this context, a major problem that has attracted considerable attention is the existence of knotted and linked vortex lines and tubes [1], see Fig. 1. The interest in this question dates back to Lord Kelvin [22], who developed an atomic theory in which atoms were understood as stable knotted thin vortex tubes in the ether, an ideal fluid modeled by the Euler equations. Kelvin's theory was inspired by the transport of vorticity discovered by Helmholtz [14], which in particular implies that the vortex tubes are frozen within the fluid flow and hence their topological structure does not change with time. Vortex tubes were called water twists by Maxwell, and were experimentally constructed by Tait by shooting smoke rings with a cannon of his own design. The stability required by Kelvin's atomic theory led him to conjecture in 1875 that thin vortex tubes of arbitrarily complicated topology can arise in stationary solutions to the Euler equations [23].

The mathematical elegance of Kelvin's theory, in which each knot type corresponds to a chemical element, captivated the scientific community for two decades. However, by the end of the XIX century, with the discovery of the electron and the experimental proof that the ether does not exist, it was clear that this theory was erroneous. Nevertheless, Kelvin's vortex tubes hypothesis was an important boon for the development of knot theory and fluid mechanics. In particular, Kelvin's conjecture has been a major open problem since then and has had a deep influence in mathematics.

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[1]We recall that a knot is a smooth closed curve in $\mathbb{R}^3$ without self intersections, and a link is a disjoint union of knots.
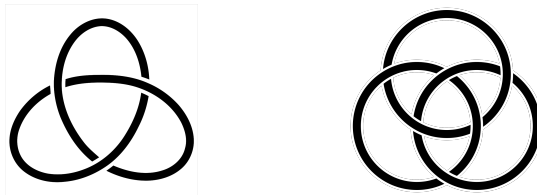
FIGURE 1. The problem involves showing that there are stationary solutions of the Euler equations realizing links, e.g. the trefoil knot and the Borromean rings depicted above, as vortex lines or vortex tubes.

In modern times, the study of knotted vortex tubes is a central topic in the so called topological hydrodynamics [3], a young area that was considerably developed after the foundational works of Arnold [1, 2] and Moffatt [16]. Arnold, in his celebrated structure theorem, classified the topological structure of the stationary solutions when the Bernoulli function is not identically constant, and he conjectured that a particular class of stationary solutions called Beltrami flows, should exhibit stream lines of arbitrarily complicated topology [2]. Moffatt introduced the concept of helicity to study the entangledness and knottedness of the fluid, and gave a heuristic argument supporting the existence of stationary states with stream lines of any knot type [17], leaving completely open the case of vortex lines and tubes [3]. A stronger conjecture was stated in the 1990s by R.F. Williams [24], who asked about the existence of a fluid flow having stream lines tracing out all knots. The main difficulty of these problems is that they lie somewhere between the partial differential equations and the dynamical systems, which explains why purely topological or analytical techniques have not been very successful in these kinds of problems.

It should be emphasised that the interest of Kelvin's conjecture is not merely academic; in fact spectacular recent experiments by Kleckner and Irvine at the University of Chicago [15] have physically supported the validity of Kelvin's conjecture through the experimental realisation of knotted and linked vortex tubes in actual fluids using cleverly designed hydrofoils, see Fig. 2. Furthermore, the existence of topologically complicated stream and vortex lines is crucial in the study of the Lagrangian theory to turbulence and in magnetohydrodynamics.

The article is organized as follows. In Section 2 we explain how Helmholtz's transport of vorticity gives rise to knotted structures in the time-dependent Euler equations (for short times), and review Moffatt's heuristic argument suggesting the existence of stream lines of any knot type in stationary Euler flows. In Section 3 we state Arnold's structure theorem and introduce Beltrami fields and Arnold's conjecture in this context; we also review the geometric approach of Etnyre and Ghrist to address the existence of knotted vortex lines and tubes in the stationary Euler equations. In Sections 4 and 5 we state the realisation theorems on vortex

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[2]In Arnold's words [1]: "Il est probable que les écoulements tels que curl $v = \lambda v$, $\lambda =$ cte, ont des lignes de courant à la topologie compliquée".

[3]In Moffatt's words [18]: "there may exist steady knotted vortex tubes configurations, but no technique has as yet been found to prove the existence of such configurations".
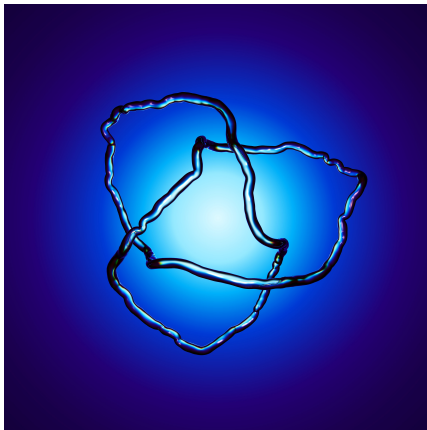
FIGURE 2. A knotted vortex tube of water obtained in the Irvine Lab at the University of Chicago. Figure courtesy of William Irvine.

lines [5] and vortex tubes [6] in $\mathbb{R}^3$, proved recently by the authors of this paper, which establish Kelvin's and related conjectures; we also include readable detailed sketches of the proofs of these results. Finally, in Section 6 we show the existence of high-frequency stationary solutions of the Euler equations on the 3-torus $\mathbb{T}^3$ and the 3-sphere $\mathbb{S}^3$ exhibiting vortex lines and tubes of arbitrarily complicated topology.

2. Helmholtz's transport of vorticity and Moffatt's magnetic relaxation argument

In 1858 Helmholtz [14] discovered that the vorticity is transported by ideal fluid flows, so that for different times t_0 and $t_1 > t_0$ the phase portraits of the autonomous vector fields $\omega(\cdot, t_0)$ and $\omega(\cdot, t_1)$ are topologically equivalent. This turned out to be a fundamental mechanism in fluid mechanics that placed the vorticity in a leading role in order to analyse the Euler equations.

Using the transport of vorticity, it is easy to construct time-dependent solutions of the Euler equations with vortex lines of complex topology. The basic idea is the following: Suppose that $u(x, t)$ is a time-dependent solution of the Euler equations. Then its vorticity satisfies the transport equation

$$\frac{\partial \omega}{\partial t} = [\omega, u],$$

with $[\cdot, \cdot]$ the commutator of vector fields. Therefore, the vorticity at time t can be expressed in terms of the vorticity $\omega_0(x)$ at time t_0 as

$$\omega(x, t) = (\phi_{t,t_0})_* \omega_0(x),$$

where $(\phi_{t,t_0})_*$ denotes the push-forward of the non-autonomous flow of the velocity field between the times t_0 and t.

From this expression for the vorticity it stems that the vortex lines at time t are diffeomorphic to those at time t_0. Accordingly, one can attempt to construct the initial vorticity ω_0 with a prescribed set of vortex lines and tubes. This is a problem in dynamical systems where the only constraint on the vector field ω_0 is

that $\operatorname{div} \omega_0 = 0$, which in $\mathbb{R}^3$ implies that ω_0 is exact, i.e. there exists a vector field u_0 such that $\operatorname{curl} u_0 = \omega_0$. The initial vorticity ω_0 can be constructed as follows. let L be the finite link in $\mathbb{R}^3$ that we want to realise as a set of vortex lines. As it has trivial normal bundle, a tubular neighbourhood N_k of each component L_k of L is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. We take each neighbourhood N_k so that the compact sets $\overline{N}_k$ are pairwise disjoint. Let us parameterize N_k with local coordinates $\alpha \in \mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ and $z = (z_1, z_2) \in \mathbb{R}^2$. In these coordinates, the Euclidean volume reads as

$$dx = f(\alpha, z)\, d\alpha\, dz_1\, dz_2$$

for some smooth positive function f. Using this parametrization, we can define a vector field v_k in each domain N_k as:

$$v_k := \frac{F(\rho^2)}{f}\Big(\partial_\alpha + G(\rho^2)\partial_\varphi\Big),$$

where we have used the polar coordinates (ρ, φ) defined as $z_1 = \rho \cos\varphi$ and $z_2 = \rho \sin\varphi$, and F and G are smooth functions such that $F(0) = 1$ and $F = 0$ for $\rho \geqslant 1$. By construction, v_k is a smooth vector field compactly supported in N_k, and it is straightforward to check that it is volume preserving for any choice of the functions F and G. Moreover, L_k is an integral curve of v_k, and for any $\rho_0 > 0$, the domain $\{\rho < \rho_0\}$, expressed in the coordinates (α, ρ, φ), is an invariant tube of v_k.

Using the fields v_k, we can prescribe the initial vorticity as the compactly supported divergence-free vector field

$$\omega_0(x) := \begin{cases} v_k(x) & \text{if } x \in N_k\,, \\ 0 & \text{if } x \in \mathbb{R}^3 \backslash \bigcup N_k\,. \end{cases}$$

Through the Biot–Savart operator, this initial vorticity corresponds to the initial velocity

$$u_0(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega_0(y)}{|x-y|^3}\, dy\,,$$

which falls off at infinity as $|u_0(x)| < C/|x|^2$.

By construction, the link L is a union of vortex lines of the initial vorticity ω_0. This field is integrable and nondegenerate in the sense that each tubular neighbourhood N_k is filled by vortex tubes, and the vortex lines are either periodic or quasi-periodic depending on whether the value of the function $G(\rho^2)$ on the invariant torus $\{\rho = \rho_0\}$ is rational or not. Therefore, the classical local (in time) existence theorem implies that there is a smooth solution to the Euler equations with initial datum u_0 which is defined for $t \in [0, T)$ (it is not known whether the maximal time of existence $T > 0$ is actually infinite). The solution u has a set of vortex lines diffeomorphic to the link L for all $t \in [0, T)$, and vortex tubes enclosing these vortex lines, as we wanted to show.

The importance of this simple argument is that it suggests the existence of stationary solutions of the Euler equations with knotted and linked vortex lines and tubes. Heuristically, one can argue as follows. If there is a smooth global solution $u(x, t)$ that evolves, for large times, into an equilibrium state, characterized by a stationary solution to Euler $u_\infty(x)$, it is conceivable, although certainly not at all obvious, that this stationary solution should also have a set of closed vortex lines

diffeomorphic to L. Of course, these hypotheses prevent us from promoting this heuristic argument to a rigorous result.

In this direction, Moffatt [17] introduced a particularly influential scenario which was inspired by ideas of the physicists Zakharov and Zeldovitch. Moffatt's heuristic argument, based on the magnetic relaxation phenomenon, supports the existence of knotted stream lines, although making his approach precise seems to be way out of reach despite the recent rigorous results in this direction, see e.g. [12]. To explain this argument, let us consider the following magnetohydrodynamic system with viscosity μ:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P + \mu \, \Delta u + H \times \operatorname{curl} H \,,$$

$$\frac{\partial H}{\partial t} = [H, u] \,, \quad \operatorname{div} u = \operatorname{div} H = 0 \,.$$

In this equation, $u(x, t)$ represents the velocity field of a plasma, $H(x, t)$ is the associated magnetic field and $P(x, t)$ is the pressure of the plasma.

Just as in the case of the previous argument based on the vorticity transport, the idea is to take initial conditions (H_0, u_0) such that the vector field H_0 has a prescribed set of invariant closed lines, possibly knotted and linked. The construction of H_0, whose only constraint is being volume preserving, can be done exactly as in the case of vortex lines. Then one can argue that, if there is a global solution with this choice of initial conditions, it is reasonable that the viscous term $\mu \, \Delta u$ forces the velocity to become negligible as $t \to \infty$. If the magnetic field also has some definite limit $H_\infty(x)$ as $t \to \infty$, then this limit field satisfies

$$H_\infty \times \operatorname{curl} H_\infty = \nabla P_\infty \,, \qquad \operatorname{div} H_\infty = 0 \,.$$

Formally, these equations are the same as the stationary Euler equations, so H_∞ is then a stationary solution to the Euler equations. Since the magnetic field is transported by the flow of the velocity field, the same argument as above suggests that one can hope that H_∞ should have a set of integral curves (i.e., stream lines) diffeomorphic to any prescribed link. The problems that appear when one tries to make this argument rigorous are similar to those appearing in the case of vortex lines, e.g., it relies on the global existence of solutions to the aforementioned MHD system, which is currently not known.

3. Arnold's structure theorem, Beltrami fields and the contact geometry approach

In spite of the fact that it is very challenging to make rigorous the ideas introduced in Section 2, these arguments are the main theoretical basis for the firm belief in the validity of Kelvin's and related conjectures among the physics community.

A landmark in this direction is Arnold's structure theorem [1, 2], which asserts that, under mild technical assumptions, the stream and vortex lines of a stationary solution to Euler whose velocity field is not everywhere collinear with its vorticity, are nicely stacked in a rigid structure akin to those which appear in the study of integrable Hamiltonian systems with two degrees of freedom:

Theorem 3.1 (Arnold's structure theorem)**.** *Let u be a solution to the stationary Euler equations in a bounded domain $\Omega \subset \mathbb{R}^3$ with analytic boundary. Suppose*

*that u is tangent to the boundary and analytic in the closure of the domain. If u
and its vorticity ω are not everywhere collinear, then there is an analytic set C, of
codimension at least 1, so that $\Omega \backslash C$ consists of a finite number of subdomains in
which the dynamics of u and ω are of one of the following two types:*

- *The subdomain is trivially fibered by tori invariant under u and ω. On
each torus, the flows of u and ω are conjugate to linear flows (rational or
irrational).*
- *The subdomain is trivially fibered by cylinders invariant under u whose
boundaries sit on $\partial\Omega$. All the stream lines of u on each cylinder are peri-
odic.*

The proof of Arnold's structure theorem is based on two simple observations:
the Bernoulli function B is a nontrivial first integral of both u and ω, and on each
regular level set of B, the fields u and ω are linearly independent and commute.
For our purposes, the main consequence of Arnold's theorem is that when u and ω
are not collinear, there is not much freedom in choosing how the vortex lines and
vortex tubes can sit in space, so it should be difficult to construct topologically
complicated vortex structures. This rough idea was confirmed in [9] by showing
that, under appropriate hypotheses, the rigid structure predicted by Arnold indeed
leads to obstructions on admissible knot and link types for stream and vortex lines.

In fact, with suitable assumptions, it is not difficult to extend Arnold's theorem
to solutions defined on the whole $\mathbb{R}^3$, so the hypothesis that u is defined on a
bounded domain Ω is not essential. Actually, Arnold himself emphasised that the
key hypothesis is that the velocity and the vorticity should not be everywhere
collinear, and predicted that when this condition is not satisfied, i.e., when the
velocity and vorticity are everywhere parallel, then one should be able to construct
stationary solutions to the Euler equations with stream and vortex lines of arbitrary
topological complexity.

Therefore, if one tries to prove Kelvin's conjecture, or to construct stationary
solutions with stream and vortex lines of any link type, it is natural to consider
solutions of the form

$$(1) \qquad\qquad \operatorname{curl} u = f\, u\,, \qquad \operatorname{div} u = 0\,,$$

with f a smooth function on $\mathbb{R}^3$. Taking the divergence in this equation we infer
that $\nabla f \cdot u = 0$, i.e., that f is a first integral of the velocity field. As a consequence
of this, the trajectories of u must lie on the level sets of the function f. The
solutions to this equation are very difficult to handle. In fact, it can be shown [7]
that there are no nontrivial solutions for an open and dense set of factors f in the
C^k topology, $k \geqslant 7$. The reason is that the existence of a non-trivial solution to
Eq. (1) in a domain U implies that f must satisfy the constraint $P[f] = 0$ in U,
where P is a non-linear partial differential operator involving derivatives of order
at most 6.

Observe that Arnold's structure theorem does not apply to stationary solutions
satisfying Eq. (1) because the vorticity is parallel to the velocity field, so the com-
pact regular level sets of f do not need to be diffeomorphic to a torus. Nevertheless,
it is not difficult to show [7] that f cannot have a connected component of a regular
level set diffeomorphic to the sphere $\mathbb{S}^2$:

Proposition 3.2. *Let u be a non-trivial solution of Eq. (1) with proportionality factor f in a neighborhood U of a regular level set $\Lambda_c := f^{-1}(c)$. Then no connected component of Λ_c can be diffeomorphic to $\mathbb{S}^2$.*

Proof. Assume that a connected component Σ of Λ_c is diffeomorphic to $\mathbb{S}^2$. Since u is divergence-free and f is a first integral, it is easy to check that the induced vector field j^*u on Σ preserves the area 2-form

$$\mu_2 := j^*\left(i_{\frac{\nabla f}{|\nabla f|^2}} dx\right).$$

Here $j : \Sigma \to U$ is the inclusion of the surface Σ in U and dx is the Euclidean volume form. Then, j^*u being divergence-free on a surface Σ diffeomorphic to $\mathbb{S}^2$, it is standard that it has a periodic trajectory $\gamma \subset \Sigma$ (because j^*u has a non-trivial first integral on Σ). An easy application of Stokes theorem allows us to write

$$0 < \int_\gamma u = \int_D \operatorname{curl} u \cdot \nu \, d\sigma = c \int_D u \cdot \nu \, d\sigma = 0 \,,$$

where $D \subset \Sigma$ is a disk with boundary $\partial D = \gamma$, ν is a normal field to Σ and $d\sigma$ is the induced surface measure on Σ. To pass to the second equality we have used Eq. (1) and $f = c$ on Σ, and in the last equality we have noticed that u is tangent to Σ. This contradiction shows that no connected component of a regular level set of f can be diffeomorphic to $\mathbb{S}^2$. $\qquad\square$

In light of the previous comments, in order to keep things simple, we are naturally led to consider a constant proportionality factor f to construct stationary solutions with complex vortex patterns. Then, we will focus our attention on *Beltrami fields*, which satisfy the equation

$$\operatorname{curl} u = \lambda u$$

for some nonzero constant λ. This equation immediately implies that $\operatorname{div} u = 0$. Notice that Beltrami fields satisfy the equation $\Delta u = -\lambda^2 u$, and hence by standard elliptic regularity they are real analytic. However, they cannot be in $L^2(\mathbb{R}^3)$ so they do not have finite energy. Actually, it is an open question whether the Euler equations in $\mathbb{R}^3$ admit any (nonzero) stationary solutions with finite energy. Obviously the stream lines of a Beltrami field are the same as its vortex lines, so henceforth we will only refer to the latter.

After establishing his structure theorem, Arnold conjectured that, contrary to what happens in the non-collinear case, Beltrami fields could present vortex lines of arbitrary topological complexity, which is fully consistent with Kelvin's conjecture. Indeed, there is abundant numerical evidence and some analytical results that suggest that the dynamics of a Beltrami field can be extremely complex. The most thoroughly studied examples are the ABC fields, introduced by Arnold in [1]:

$$u(x) = \big(A\sin x_3 + C\cos x_2, \, B\sin x_1 + A\cos x_3,$$
$$C\sin x_2 + B\cos x_1\big) \,.$$

Here A, B, C are real parameters. It is remarkable that all our intuition about Beltrami fields comes from the analysis of a few exact solutions, which basically consist of fields with Euclidean symmetries and the ABC family.

From the experimental viewpoint, it was observed in actual fluid flows [20] that in turbulent regions of low dissipation, and hence governed by the Euler equations,

the velocity and vorticity vectors have a tendency to align, which is precisely the Beltrami condition. This is an additional support in order to consider Beltrami fields as the right solutions if one wants to construct topologically complicated vortex structures. As a matter of fact, these fields also play an important role in magnetohydrodynamics, where they are known as force-free magnetic fields. These force-free solutions model the dynamics of plasmas in stellar atmospheres, where complicated magnetic tubes, which are the analogues of vortex tubes, have been observed.

An interesting approach to the problem on the existence of knotted and linked vortex lines in stationary Euler flows is due to Etnyre and Ghrist. It hinges on the connection of Beltrami fields with contact geometry [10]. The main observation is the following. Let u be a Beltrami field and α its dual 1-form, so that the Beltrami equation can be written using the Hodge $*$-operator as

$$*d\alpha = \lambda\alpha\,.$$

Therefore, if the Beltrami field does not vanish anywhere, we have that

$$\alpha \wedge d\alpha = \lambda|u|^2\, dx_1 \wedge dx_2 \wedge dx_3$$

does not vanish either, so that by definition α defines a contact 1-form. Conversely, if α is a contact 1-form in $\mathbb{R}^3$, there is a smooth Riemannian metric g adapted to the form α so that this 1-form satisfies the Beltrami equation above with the Hodge $*$-operator corresponding to the metric g. The vector field dual to the 1-form α is a Beltrami field with respect to the adapted metric g, and is called Reeb field in contact geometry.

The reason why this observation is useful is that the machinery of contact geometry is very well suited for the construction of contact forms whose associated Reeb fields have a prescribed invariant set, e.g. a set of closed integral curves or invariant tori. Therefore, one finds that there is a metric in $\mathbb{R}^3$, which in general is neither flat nor complete, such that the Euler equations in this metric admit a stationary solution of Beltrami type having a set of vortex lines and vortex tubes of any knot and link type. The geometric properties of a metric adapted to a contact 1-form are very rigid [11], so this strategy cannot work when we consider the Euler equations for a fixed (e.g. Euclidean) metric.

4. A realisation theorem for knotted vortex lines

In this section we shall discuss a realisation theorem showing the existence of Beltrami fields with a set of closed vortex lines diffeomorphic to any given link [5]:

Theorem 4.1. *Let $L \subset \mathbb{R}^3$ be a finite link and let λ be any nonzero real number. Then one can deform the link L by a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in the C^m norm, such that $\Phi(L)$ is a set of vortex lines of a Beltrami field u, which satisfies the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$. Moreover, u falls off at infinity as $|D^j u(x)| < C_j/|x|$.*

We have only considered the case of finite links, but the case of locally finite links can be tackled similarly at the expense of losing the decay condition of the velocity field. In particular, taking into account the fact that the knot types modulo diffeomorphism are countable, it follows that there exists a stationary solution to

the Euler equations whose stream lines realise all knots at the same time, thus yielding a positive answer to a question of Williams [24].

The closed vortex lines in the set $\Phi(L)$ are hyperbolic, i.e. their associated monodromy matrices do not have any nontrivial eigenvalues of modulus 1. Since $\operatorname{div} u = 0$, this immediately implies that these vortex lines are unstable. Notice, however, that the theorem does not guarantee that $\Phi(L)$ contains all closed vortex lines of the Beltrami field.

The $1/|x|$ decay we have is optimal within the class of Beltrami solutions, not necessarily with constant proportionality factor [19], so our solutions belong to the space $L^p(\mathbb{R}^3)$ for all $p > 3$. Notice that the $1/|x|$ decay was not proved in [5] (indeed, in this paper the Beltrami field was not shown to satisfy any conditions at infinity), but follows from the more refined global approximation theorem that we proved in [6].

We shall next sketch the proof of Theorem 4.1. The heart of the problem is that one needs to extract topological information from a PDE. Our basic philosophy is to use the methods of differential topology and dynamical systems to control auxiliary constructions and those of PDEs to realise these auxiliary constructions in the framework of solutions to the Euler equations. For concreteness, to explain the gist of the proof we will concentrate on constructing a solution for which we are prescribing just one vortex line L, which is a (possibly knotted) curve in $\mathbb{R}^3$.

Step 1: a geometric construction. It is well know that, perturbing the knot a little through a small diffeomorphism, we can assume that L is analytic. Since the normal bundle of a knot is trivial, we can take an analytic ribbon Σ around L. More precisely, there is an analytic embedding h of the cylinder $\mathbb{S}^1 \times (-\delta, \delta)$ into $\mathbb{R}^3$ whose image is Σ and such that $h(\mathbb{S}^1 \times \{0\}) = L$.

In a small tubular neighbourhood N of the knot L we can take an analytic coordinate system
$$(\theta, z, \rho) : N \to \mathbb{S}^1 \times (-\delta, \delta) \times (-\delta, \delta)$$
adapted to the ribbon Σ. Basically, θ and z are suitable extensions of the angular variable on the knot and of the signed distance to L as measured along the ribbon Σ, while ρ is the signed distance to Σ.

The reason why this coordinate system is useful is that it allows us to define a vector field w in the neighbourhood N that is key in the proof: simply, w is the field dual to the closed 1-form
$$d\theta - z\, dz\,.$$
From this expression and the definition of the coordinates it stems that w is an analytic vector field tangent to the ribbon Σ and that L is a stable hyperbolic closed integral curve of the pullback of w to Σ.

Step 2: a robust local Beltrami field. The field w we constructed in Step 1 will now be used to define a local Beltrami field v. To this end we will consider the Cauchy problem
$$(2) \qquad\qquad \operatorname{curl} v = \lambda v\,, \qquad v|_\Sigma = w\,.$$
One cannot apply the Cauchy–Kowalewski theorem directly because the curl operator does not have any non-characteristic surfaces as its symbol is an skew-symmetric matrix. In fact, a direct computation shows that there are some analytic Cauchy

data w, tangent to the surface Σ, for which this Cauchy problem does not have any solutions: a necessary condition for the existence of a solution, when the field w is tangent to Σ, is that the pullback to the ribbon of the 1-form dual to the Cauchy datum must be a closed 1-form.

Through a more elaborate argument that involves a Dirac-type operator, one can prove that this condition is not only necessary but also sufficient. Therefore, the properties of the field w constructed in Step 1 allow us to ensure that there is a unique analytic field v in a neighbourhood of the knot L which solves the Cauchy problem (2).

It is obvious that the knot L is a closed vortex line of the local Beltrami field v. As a matter of fact, it is easy to check that this line is hyperbolic (and therefore robust under small perturbations). The idea is that, by construction, the ribbon Σ is an invariant manifold under the flow of v that contracts into L exponentially. As the flow of v preserves volume because $\operatorname{div} v = 0$, there must exist an invariant manifold that is exponentially expanding and intersects Σ transversally on L, which guarantees its hyperbolicity.

Accordingly, L is a robust closed vortex line. More concretely, by the hyperbolic permanence theorem any field u that is close enough to v in the $C^m(N)$ norm, $m \geqslant 1$, has a closed integral curve diffeomorphic to L, and this diffeomorphism can be chosen C^m-close to the identity (and different from the identity only in N).

Step 3: a Runge-type global approximation theorem. The global Beltrami field u is obtained through a Runge-type theorem for the operator $\operatorname{curl} -\lambda$. This result allows us to approximate the local Beltrami field v by a global Beltrami field u in the $C^m(N)$ norm. More precisely, for any positive δ and any positive integer m there is a global Beltrami field u such that

$$\|u - v\|_{C^m(N)} < \delta\,.$$

Besides, the field u falls off at infinity as

$$|D^j u(x)| < \frac{C_j}{|x|}\,.$$

Basically, the proof of our Runge-type theorem [6] consists of two steps. In the first step we use functional-analytic methods and Green's functions estimates to approximate the field v by an auxiliary vector field $\tilde{v}$ that satisfies the elliptic equation $\Delta\tilde{v} = -\lambda^2\tilde{v}$ in a large ball of $\mathbb{R}^3$ that contains the set N. To prove this result it is crucially used that the complement $\mathbb{R}^3\backslash N$ of the set N has no compact components. In the second step, we define the approximating global Beltrami field u in terms of a truncation of a Fourier-Bessel series representation of the field $\tilde{v}$ and a simple algebraic trick.

To conclude the proof of the theorem it is enough to take δ small enough so that the hyperbolic permanence theorem ensures that if $\|u - v\|_{C^m(N)} < \delta$ then there is a diffeomorphism Φ of $\mathbb{R}^3$ such that $\Phi(L)$ is a closed vortex line of u and $\Phi - \mathrm{id}$ is supported in N with $\|\Phi - \mathrm{id}\|_{C^m(\mathbb{R}^3)}$ as small as wanted.

5. A REALISATION THEOREM FOR KNOTTED VORTEX TUBES

In Theorem 4.1 we have used Beltrami fields to prove the existence of stationary solutions to the Euler equations with vortex lines of any link type. Let us now show that one can construct stationary solutions with knotted vortex tubes, as predicted by Kelvin, using Beltrami fields as well. To state this result, let us denote by $\mathcal{T}_\epsilon(L)$ the ϵ-thickening of a given link L in $\mathbb{R}^3$, that is, the set of points that are at distance at most ϵ from L. The realisation theorem for vortex tubes can then be stated as follows [6]:

Theorem 5.1. *Let L be a finite link in $\mathbb{R}^3$. For any small enough ϵ, one can transform the collection of pairwise disjoint thin tubes $\mathcal{T}_\epsilon(L)$ by a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in the C^m norm, so that $\Phi[\mathcal{T}_\epsilon(L)]$ is a set of vortex tubes of a Beltrami field u, which satisfies the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ for some nonzero constant λ. Moreover, the field u decays at infinity as $|D^j u(x)| < C_j/|x|$.*

The parameter λ in the theorem cannot be chosen freely: it must be of order $\mathcal{O}(\epsilon^3)$. In fact, if we allow a diffeomorphism Φ that is not close to the identity, we can get any nonzero constant λ' just by considering the rescaled field

$$u'(x) := u\Big(\frac{\lambda' x}{\lambda}\Big),$$

which satisfies the Beltrami equation $\operatorname{curl} u' = \lambda' u'$. However, the fact that the vortex tubes are thin in the sense that their width is much smaller than their length, is a crucial ingredient in the proof of the theorem.

The proof of Theorem 5.1 also yields information on the structure of the vortex lines inside each vortex tube:

(i) There are infinitely many nested invariant tori (which bound vortex tubes). On each of these tori, the vortex lines are ergodic.

(ii) In the region bounded by any pair of these invariant tori there are infinitely many closed vortex lines, not necessarily of the same knot type as the curves in the link L.

(iii) There is a set of elliptic [4] closed vortex lines diffeomorphic to the link L near the core of the vortex tubes. Being elliptic, they are linearly stable.

(iv) The vortex tubes are both Lyapunov stable and structurally stable.

The proof of Theorem 5.1 also relies on the combination of a robust local construction and a global approximation result, as in the case of Theorem 4.1. In fact, this global approximation result was used in the statement of Theorem 4.1 to ensure that our Beltrami fields fall off at infinity. However, the construction of the robust local solution is much more sophisticated than in the case of vortex lines and requires entirely different ideas.

Basically, the robustness of the tubes follows from a KAM-theoretic argument with two small parameters: the thinness ϵ of the tubes and the constant λ. The local solution must now be defined in the whole tubes, not just on a neighbourhood of the boundary. This makes it impossible to construct the local solution using a theorem

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[4]We recall that a closed integral curve of a vector field is elliptic if its associated monodromy matrix has all its eigenvalues of modulus 1.

of Cauchy-Kowalewski type, as we did in Step 2 of Theorem 4.1. Instead, we need to consider a boundary value problem for the curl operator in which the tangential part of the field cannot be prescribed. As a consequence of this, one cannot directly take local Beltrami fields which satisfy the non-degeneracy conditions of the KAM-type theorem: these conditions must be extracted from the equation using fine PDE estimates. This is in great contrast with the prescription of the Cauchy datum that we made in Step 1 of Theorem 4.1, which readily ensures the hyperbolicity of the closed vortex lines, and leads to very subtle problems with a deep interplay of PDE and dynamical systems techniques.

As we did in the sketch of proof of Theorem 4.1, we will concentrate on constructing a solution for which we are prescribing just one vortex tube $\mathcal{T}_\epsilon \equiv \mathcal{T}_\epsilon(L)$, where L is a (possibly knotted) curve in $\mathbb{R}^3$.

Step 1: a local Beltrami field in a tube. We will obtain a local Beltrami field v in $\mathcal{T}_\epsilon$ as the unique solution to certain boundary value problem for the Beltrami equation. To specify this problem, let us fix a (nonzero) harmonic field h in $\mathcal{T}_\epsilon$, which satisfies

$$\operatorname{div} h = 0 \quad \text{and} \quad \operatorname{curl} h = 0$$

in the tube and is tangent to the boundary. By Hodge theory, it is standard that there is a unique harmonic field in $\mathcal{T}_\epsilon$ up to a multiplicative constant. For concreteness, let us assume that $\|h\|_{L^2(\mathcal{T}_\epsilon)} = 1$.

The boundary problem we will then consider is

$$\operatorname{curl} v = \lambda v$$

in $\mathcal{T}_\epsilon$, supplemented with the boundary condition $\nu \cdot v = 0$ and a condition on the harmonic part of v such as

$$\int_{\mathcal{T}_\epsilon} v \cdot h \, dx = 1 \,.$$

Notice that in this boundary problem we are specifying the normal component of v on the boundary (which we set to zero, to ensure that $\partial \mathcal{T}_\epsilon$ is an invariant torus) but not the tangential component. This will be important later on.

Through a duality argument, it is not hard to prove that for any λ outside some discrete set, and in particular whenever $|\lambda|$ is smaller than some ϵ-independent constant, there is a unique solution to this problem. An easy consequence of the proof is that the field v becomes close to h for small λ, in the sense that

$$(3) \qquad \qquad \|v - h\|_{H^k(\mathcal{T}_\epsilon)} \leqslant C_{k,\epsilon}|\lambda| \,.$$

The problem now is that, when one tries to verify the conditions for the preservation of the invariant torus $\partial \mathcal{T}_\epsilon$ under small perturbations of v, one realizes that the above existence result is far from enough: the robustness of the invariant torus depends on KAM arguments, which require very fine information on the behavior of v in a neighbourhood of $\partial \mathcal{T}_\epsilon$.

An important simplification is suggested by the estimate (3): if we take small nonzero values of λ, it should be enough to understand the behavior of the harmonic field h, since the local solution v is going to look basically like this field (more refined estimates are needed to fully exploit this fact, but this is the basic idea.)

Therefore, our next goal is to estimate various analytic properties of the harmonic field h. To simplify this task, we will introduce coordinates adapted to the tube $\mathcal{T}_\epsilon$, which essentially correspond to an arc-length parametrization of the knot L and to rectangular coordinates in a transverse section of the tube defined using a Frenet frame. Thus we consider an angular coordinate α, taking values in $\mathbb{S}^1_\ell := \mathbb{R}/\ell\mathbb{Z}$ (with ℓ the length of the knot L), and rectangular coordinates $y = (y_1, y_2)$ taking values in the unit 2-disk $\mathbb{D}$.

To extract information about h, we start with a good guess of what h should look like: one can check that there is some function of the form $1 + \mathcal{O}(\epsilon)$ such that the vector field

$$h_0 := [1 + \mathcal{O}(\epsilon)] \left(\partial_\alpha + \tau \, \partial_\theta \right)$$

is "almost harmonic", in the sense that it is curl-free, tangent to the boundary and satisfies

$$\rho := -\operatorname{div} h_0 = \mathcal{O}(\epsilon) \,.$$

Here τ is the torsion of the curve L and θ is the angular polar coordinate in the 2-disk. The actual form of h_0 and ρ is important, but we will not write these details to keep the exposition simple.

From the above considerations we infer that the harmonic field is given by

$$h = h_0 + \nabla \psi \,,$$

where ψ solves the Neumann boundary value problem

$$(4) \qquad \Delta\psi = \rho \quad \text{in } \mathcal{T}_\epsilon \,, \qquad \partial_n\psi|_{\partial\mathcal{T}_\epsilon} = 0 \,, \qquad \int_{\mathcal{T}_\epsilon} \psi \, dx = 0 \,.$$

When written in the natural coordinates (α, y), we obtain a boundary value problem in the domain $\mathbb{S}^1_\ell \times \mathbb{D}$, the coefficients of the Laplacian in these coordinates depending on the geometry of the tube strongly through its thickness ϵ and the curvature and torsion of L.

In the derivation of the result on preservation of the invariant torus we will need to solve approximately the boundary value problem (4), thus showing that ψ is of the following form:

- $\psi = \mathcal{O}(\epsilon^2)$,
- $D_y\psi = (\text{certain explicit function}) + \mathcal{O}(\epsilon^4)$,
- $\partial_\theta\psi = (\text{certain explicit function}) + \mathcal{O}(\epsilon^5)$.

The explicit expressions above are important, but we will omit them so as not to obscure the main points of the proof.

To obtain these expression, we need estimates for the L^2 norm of ψ and its derivatives that are optimal with respect to the parameter ϵ. The reason for this is that standard energy estimates of the form

$$\|\psi\|_{H^{k+2}(\mathcal{T}_\epsilon)} \leqslant C_{\epsilon,k} \|\rho\|_{H^k(\mathcal{T}_\epsilon)}$$

are of little use to us because for the preservation of the torus we will need to be very careful in dealing with powers of the small parameter ϵ. In particular, it is crucial to distinguish between estimates for derivatives of ψ with respect to the "slow" variable α and the "fast" variable y, and even to trade some of the gain of derivatives associated with the elliptic equation (4) (in some cases) for an improvement of the dependence on ϵ of the constants. Estimates optimal with

respect to ϵ are also derived for the equation $\operatorname{curl} v = \lambda v$ in $\mathcal{T}_\epsilon$ to help us exploit the connection between Beltrami fields with small λ and harmonic fields.

Step 2: A KAM theorem for Beltrami fields. To analyse the robustness of the invariant torus $\partial \mathcal{T}_\epsilon$ of the local solution v, the natural tool is KAM theory. At first, it may not be immediate to see why we can apply KAM-type arguments, as v is a divergence-free vector field in a three-dimensional domain and KAM theory is usually discussed in the context of integrable Hamiltonian systems in even-dimensional spaces.

The key here is to consider the Poincaré (or first return) map of v. To define this map, we take a normal section of the tube $\mathcal{T}_\epsilon$, say $\{\alpha = 0\}$. Given a point x_0 in this section, the Poincaré map Π associates to x_0 the point where the vortex line $x(\tau)$ with initial condition $x(0) = x_0$ cuts the section $\{\alpha = 0\}$ for the first positive time. The analysis in Step 1 gives that the harmonic field h is of the form

$$(5) \qquad h = \partial_\alpha + \tau(\alpha)\,(y_1\,\partial_2 - y_2\,\partial_1) + \mathcal{O}(\epsilon)\,,$$

so with a little work one can prove that the Poincaré map is well defined for small enough ϵ and λ. Identifying this section with the disk $\mathbb{D}$ via the coordinates y, this defines the Poincaré map as a diffeomorphism

$$\Pi : \mathbb{D} \to \mathbb{D}\,.$$

Since the vector field v is divergence-free, one can prove that the Poincaré map preserves some measure on the disk.

Notice that the invariant torus $\partial \mathcal{T}_\epsilon$ manifests itself as an invariant circle (namely, $\partial \mathbb{D}$) of the Poincaré map. To establish the robustness of the invariant torus $\partial \mathcal{T}_\epsilon$, we will resort to a KAM theorem [13] to prove that the invariant circle of Π is preserved under small area-preserving perturbations. After taking care of several technicalities that will be disregarded here, thanks to this theorem we can conclude that the invariant torus $\partial \mathcal{T}_\epsilon$ is robust provided two conditions are met: that the rotation number of Π on the invariant circle is Diophantine and that Π satisfies a nondegeneracy twist condition.

We would like to emphasize that computing the rotation number ω_Π and the twist $\mathcal{N}_\Pi$ of the Poincaré map amounts to obtaining quantitative information about the vortex lines of v. This is a hard, messy, lengthy calculation that we carry out by combining an iterative approach to control the integral curves of the associated dynamical system (i.e., the vortex lines) with small parameter ϵ and the PDE estimates, optimal with respect to ϵ, that we obtained for v in Step 1. The final formulas are

$$\omega_\Pi = \int_0^\ell \tau(\alpha)\,d\alpha + \mathcal{O}(\epsilon^2)\,,$$

$$(6) \qquad \mathcal{N}_\Pi = -\frac{5\pi\epsilon^2}{8} \int_0^\ell \kappa(\alpha)^2\,\tau(\alpha)\,d\alpha + \mathcal{O}(\epsilon^3)\,,$$

where κ and τ respectively denote the curvature and torsion of the knot L. The leading term of ω_Π is the total torsion of the curve L, while the leading term of the twist $\mathcal{N}_\Pi$ is proportional to the helicity of the velocity field associated with the

vortex filament motion under LIA [21]. These quantities are the first and the third constants of the motion for the LIA equation [5].

These expressions allow us to prove that for a "generic" curve L the rotation number is Diophantine and the twist is nonzero, so the hypotheses of the KAM theorem are satisfied. Hence the invariant torus $\partial\mathcal{T}_\epsilon$ of the local Beltrami field v is robust: if u is a divergence-free vector field in a neighbourhood of the tubes that is close enough to v in a suitable sense (e.g., in the C^m norm with $m \geqslant 4$), then u also has an invariant tube diffeomorphic to $\mathcal{T}_\epsilon$, and moreover the corresponding diffeomorphism can be taken close to the identity.

It is worth mentioning that the formula (6) provides some intuition about the question of why one needs to be so careful with the dependence on ϵ of the various estimates: the twist, which must be nonzero, is of order $\mathcal{O}(\epsilon^2)$. Another way of understanding this is by looking at the expression (5) for the harmonic field, which implies that our local solution v is an ϵ-small perturbation of the most degenerate kind of vector field from the point of view of KAM theory: a field with constant rotation number.

Step 3: a Runge-type global approximation theorem. To complete the proof of the theorem, we use the same Runge-type theorem as in Step 3 of the outline of the proof of Theorem 4.1, to show that there is a Beltrami field u in $\mathbb{R}^3$ close to the local solution:
$$\|u - v\|_{C^m(\mathcal{T}_\epsilon)} < \delta\,,$$
falling off at infinity as
$$|D^j u(x)| < \frac{C_j}{|x|}\,.$$
Putting all three steps together, this gives the outline of the proof of Theorem 5.1.

6. Knotted vortex structures on the torus and the sphere

The stationary solutions in $\mathbb{R}^3$ we constructed in Sections 4 and 5 fall off at infinity as $|x|^{-1}$, this decay not being fast enough for the velocity field to be in the energy space $L^2(\mathbb{R}^3)$. In fact, there are no Beltrami fields in $\mathbb{R}^3$ with finite energy even if the proportionality factor f (see Eq. (1)) is allowed to be nonconstant, as has been recently shown in [19]. On the contrary, Beltrami fields in a compact Riemannian 3-manifold are stationary solutions to the Euler equations that do have finite energy.

Unfortunately, the strategy we used to prove the realisation theorems presented in Sections 4 and 5 does not work for compact manifolds. The reason is that the proof of the aforementioned theorems is based on the construction of a local Beltrami field in a certain domain U (that is, the neighborhood N of the knot L in the case of Theorem 4.1 and the tube $\mathcal{T}_\epsilon$ in the case of Theorem 5.1), which is then approximated by a global Beltrami field in $\mathbb{R}^3$ using a Runge-type global approximation theorem. For compact manifolds the complement of U is compact, so we cannot apply the global approximation theorem. This is not just a technical issue, but a fundamental obstruction in any approximation theorem of this sort. Indeed,

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[5]This connection between the quantities measuring the nondegeneracy of the KAM argument for the vortex tubes and the LIA equation is quite surprising, and we do not see any obvious explanation for it.

for compact manifolds (with appropriate boundary conditions) the spectrum of the curl operator is discrete, so not any value of λ is allowed for global solutions, while locally the equation $\operatorname{curl} u = \lambda u$ admits a non-trivial solution for any λ.

Nevertheless, for the flat torus $\mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z})^3$ and the round sphere $\mathbb{S}^3$ (and quotients of $\mathbb{S}^3$ with a finite subgroup of isometries), a realisation theorem for knotted vortex lines and knotted vortex tubes that is analogous to Theorems 4.1 and 5.1 can be proved [8] using Beltrami fields with high frequency λ. A key point is that, in these manifolds, the multiplicity of λ tends to infinity as $\lambda \to \infty$ (the spectrum of curl is very degenerate), which provides a large set of solutions for each large enough λ. In the realisation theorem we proved in [8], the set $\mathcal{S}$ of closed curves and tubes is assumed to be contained in a contractible subset (this is always the case in $\mathbb{S}^3$, but not in $\mathbb{T}^3$), and the diffeomorphism Φ transforming $\mathcal{S}$ into a union of vortex lines and tubes of a Beltrami field contracts $\mathcal{S}$ into a ball of radius λ^{-1}. More precisely, the theorem we proved is the following. In the statement, we write $\mathbb{M}^3$ to denote either $\mathbb{T}^3$ or $\mathbb{S}^3$. Notice that the spectrum of the curl operator in $\mathbb{M}^3$ contains all the integers.

Theorem 6.1. *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{M}^3$. In the case of the torus, we also assume that $\mathcal{S}$ is contained in a contractible subset of $\mathbb{T}^3$. Then for any large enough odd integer λ there exists a Beltrami field u satisfying the equation $\operatorname{curl} u = \lambda u$ and a diffeomorphism $\Phi : \mathbb{M}^3 \to \mathbb{M}^3$ connected with the identity such that $\Phi(L)$ is a union of vortex lines and tubes of u.*

As mentioned above, the effect of the diffeomorphism Φ is to uniformly rescale a contractible subset of the manifold that contains $\mathcal{S}$ to have a diameter of order λ^{-1}. Furthermore, the set $\Phi(\mathcal{S})$ of vortex structures of u is structurally stable in the sense that any divergence-free vector field on $\mathbb{M}^3$ which is sufficiently close to u in the $C^{4,\alpha}$ norm will also have this collection of vortex lines and tubes, up to a diffeomorphism.

The proof of Theorem 6.1 involves an interplay between rigid and flexible properties of high-frequency Beltrami fields. Indeed, rigidity appears because high-frequency Beltrami fields in any 3-manifold behave, locally in sets of diameter λ^{-1}, as Beltrami fields in $\mathbb{R}^3$ with parameter $\lambda = 1$ do in balls of diameter 1. The catch here is that, in general, one cannot check whether a given Beltrami field in $\mathbb{R}^3$ actually corresponds to a high-frequency Beltrami field on the compact manifold. To prove a partial converse implication in this direction, it is key to exploit some flexibility that arises in the problem as a consequence of the fact that large eigenvalues of the curl operator in the torus or in the sphere have increasingly high multiplicities. More precisely, the key to prove Theorem 6.1 is the following lemma. In the statement, $\Psi : \mathbb{B} \to B$ is a patch of normal geodesic coordinates centered at a fixed point $p_0 \in \mathbb{M}^3$, with $\mathbb{B} \subset \mathbb{M}^3$ the geodesic ball of radius 1 centered at p_0 and B the unit ball in $\mathbb{R}^3$.

Lemma 6.2. *Let v be a Beltrami field in $\mathbb{R}^3$, satisfying $\operatorname{curl} v = v$. Let us fix any positive numbers ϵ and m. Then for any large enough odd integer λ there is a Beltrami field u, satisfying $\operatorname{curl} u = \lambda u$ in $\mathbb{M}^3$, such that*

$$\left\| \Psi_* u\left(\frac{\cdot}{\lambda}\right) - v \right\|_{C^m(B)} < \epsilon.$$

Assuming this key lemma, the proof of Theorem 6.1 is essentially as follows. First, we shrink the set $\mathcal{S}$ into the ball $\mathbb{B}$. The realisation theorems 4.1 and 5.1 imply that there is a Beltrami field v in $\mathbb{R}^3$ with a set of vortex lines and tubes diffeomorphic to $\Psi(\mathcal{S})$. Then, Lemma 6.2 implies the existence of a Beltrami field u in $\mathbb{M}^3$ whose "localization" $\Psi_* u\left(\frac{\cdot}{\lambda}\right)$ is C^m-close to v. Since the set of vortex structures of v is structurally stable, the theorem follows.

Lemma 6.2 does not hold for generic Riemannian 3-manifolds. Indeed, for each compact and without boundary 3-manifold there is a residual set of metrics for which the spectrum of the curl operator is simple [4], i.e. for each λ in the spectrum of curl the equation $\operatorname{curl} u = \lambda u$ has a unique solution up to a multiplicative constant factor. Therefore, the idea used to prove Theorem 6.1 cannot work for general manifolds, and hence the following important question remains open:

Open problem: Let (M, g) be a compact Riemannian 3-manifold without boundary. For each set $\mathcal{S} \subset M$ of closed lines and tubes, does there exist a Beltrami field u satisfying $\operatorname{curl} u = \lambda u$ that realises $\mathcal{S}$ as a set of vortex lines and tubes, up to a diffeomorphism?

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Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, 28049 Madrid, Spain

E-mail address: `aenciso@icmat.es, dperalta@icmat.es`