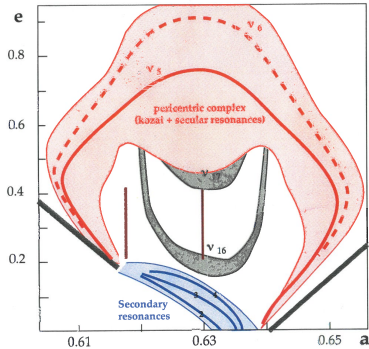


- Superposition of two dynamics : choice for the trajectory
- Very similar initial conditions : very different long term behavior
- Sensitivity to the initial conditions, to the computer accuracy
- Uncertainty, unpredictable behavior
- Lyapunov, Chirikov
- How to quantify this sensitivity ? How to measure the chaos ?

Planetary ephemerides

- Limited ephemerides : JPL + IMCCE INPOP
- Laskar: the motion of the planets in the Solar System is chaotic
- No accurate prediction of their trajectories beyond a few tens of millions of years : 60 Millions of years
- One possible motion, not THE motion
- The recent simulations show that planetary collisions or ejections are possible on a period of less than 5 billion years

Presence of chaos



Regularity and Chaoticity

Quantitative consideration: indicators of chaoticity .. or indicators of resonances

In chaotic (irregular) regions of phase space two initially nearby trajectories **separate roughly exponentially with time**; in quasi-periodic (regular) neighboring trajectories **separate roughly linearly with time** (Chirikov, 1979)

Consequences:

- Computation of **rate of separation (divergence)**
- Sensitive **dependence on initial conditions**

Definition

$$\gamma = \frac{1}{t - t_0} \ln \left(\frac{d(t)}{d(t_0)} \right)$$

d is the Euclidian distance between two initially nearby trajectories.

- Chaotic trajectories (irregular):
 d grows exponentially (on the average), γ approaches some positive constant
- Quasi-periodic trajectories (regular):
 d grows linearly, γ approaches zero as $\ln(t)/t$

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \alpha), \quad \mathbf{x} \in \mathbb{R}^{2n}$$

where α is a vector of parameters

$$\dot{\delta} = \frac{d}{dt}\delta(\phi(t)) = \mathbf{J}(\phi(t))\delta(\phi(t)), \quad \text{with} \quad \mathbf{J}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\phi(t)),$$

with $\delta = x(t) - x_0(t)$ and $\phi(t)$ is a solution of the flow.

The Lyapounov Characteristic Number λ (LCN)

$$\lambda = \lim_{t \rightarrow \infty} \lambda_1(t), \quad \text{with} \quad \lambda_1(t) = \frac{1}{t} \ln \frac{\|\delta(\phi(t))\|}{\|\delta(\phi(t_0))\|}$$

FLI = Fast Lyapunov Indicator

- Directly linked with Lyapunov number
- Introduced by Froeschlé, Lega, Gonczi (1997)
- Simplest and most complex (papers of Lega and Guzzo)
- Very popular
- $FLI(\vec{x}(0), \vec{x}(0), t) = \ln \|\vec{v}(t)\|$
- Reference : standard map
- Special model created for tests
- $H(\phi_1, \phi_2, \phi_3, I_1, I_2, I_3) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + I_3 + \epsilon \frac{1}{\cos \phi_1 + \cos \phi_2 + \cos \phi_3 + 4}$

Standard Map 2 dimensions

Pendulum : $H(x, y) = \frac{x^2}{2} + \epsilon \cos y$ associated with

$$\dot{x} = -\frac{\partial H}{\partial y} = \epsilon \sin y$$

$$\dot{y} = \frac{\partial H}{\partial x} = x$$

If we replace $\dot{x} \simeq x(t+1) - x(t)$ and $\dot{y} \simeq y(t+1) - y(t)$:

$$x(t+1) - x(t) = \epsilon \sin y(t)$$

$$y(t+1) - y(t) = x(t)$$

or for symplecticity :

$$y(t+1) = x(t) + y(t)$$

$$x(t+1) = x(t) + \epsilon \sin y(t+1) = x(t) + \epsilon \sin (x(t) + y(t))$$

Leap-Frog integrator of the pendulum

$$x(t+1) = x(t) + \epsilon \sin(x(t) + y(t)) \quad \text{Modulo } 2\pi$$

$$y(t+1) = x(t) + y(t) \quad \text{Modulo } 2\pi$$

with the associated linear variational system:

$$\begin{pmatrix} v_x(t+1) \\ v_y(t+1) \end{pmatrix} = \begin{pmatrix} 1 + \epsilon \cos(x + y) & \epsilon \cos(x + y) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}$$

FLI for the Standard Map

Particular case $\epsilon = 0$: $v_x(0) = \beta$ and $v_y(0) = 0$

$$v_x(t+1) = v_x(t) \text{ and } v_y(t+1) = v_x(t) + v_y(t)$$

or

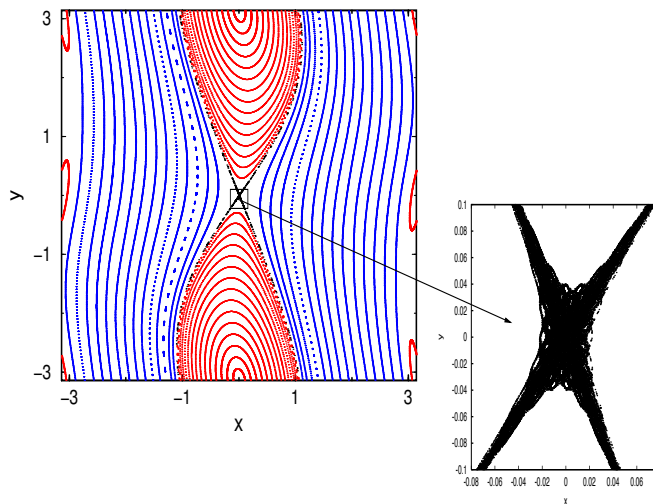
$$v_x(t) = v_x(0) = \beta$$

$$v_y(t) = t v_x(0) + v_y(0) = t v_x(0) = t \beta$$

Consequently : $\|\vec{v}\| \sim t \beta$ for large t and

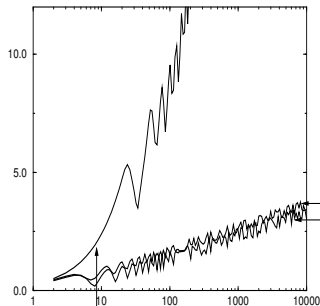
$$FLI(t) \sim \begin{cases} \ln \beta + \ln t & \text{for regular orbits} \\ \lambda t & \text{for chaotic orbits} \end{cases}$$

Examples (courtesy of Elena Lega)

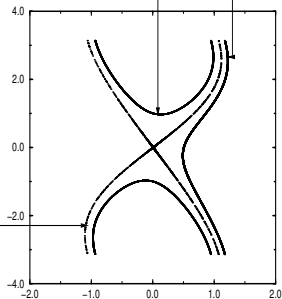


A set of orbits for $\epsilon = 0.3$

Examples (Elena Lega)

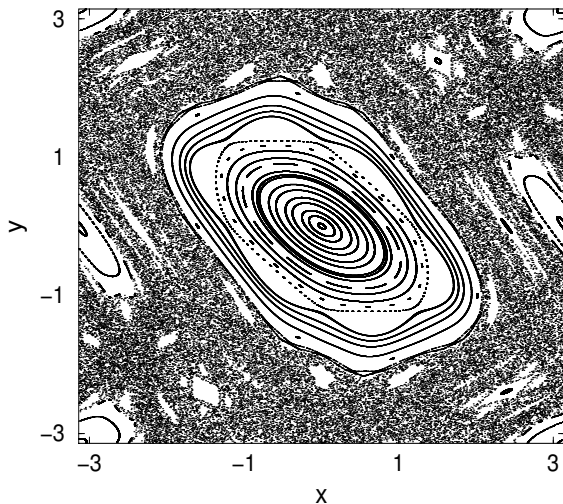


**Variation of the FLI with time
for 3 orbits of the standard map**

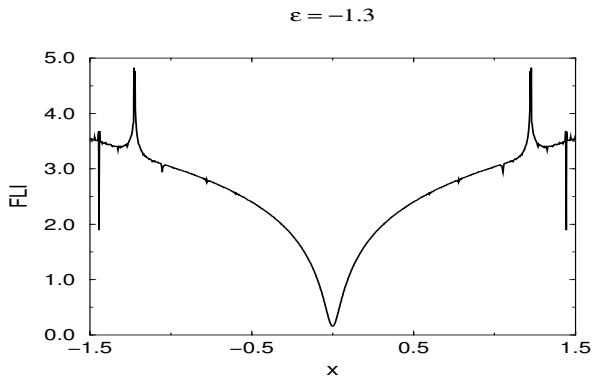


Examples (Elena Lega)

$$\varepsilon = -1.3$$

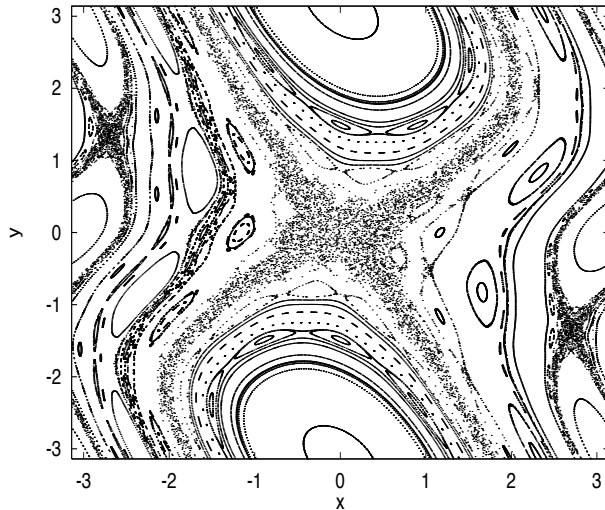


Examples (Elena Lega)

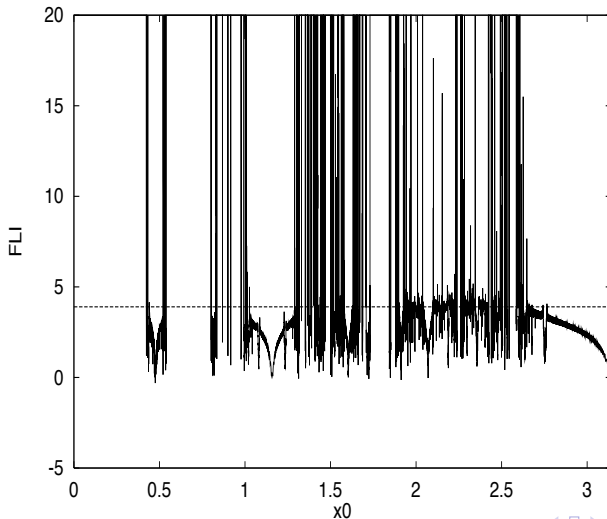


Examples (Elena Lega)

$$\epsilon = 0.9$$

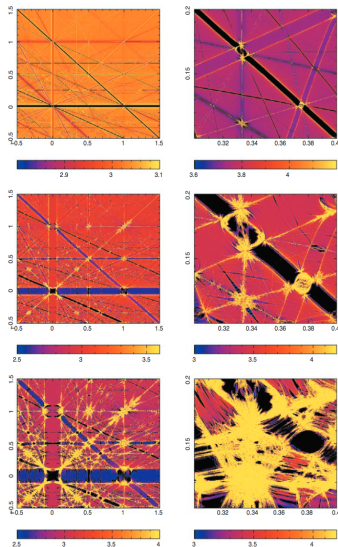


The FLI-map for $\epsilon = 0.9$

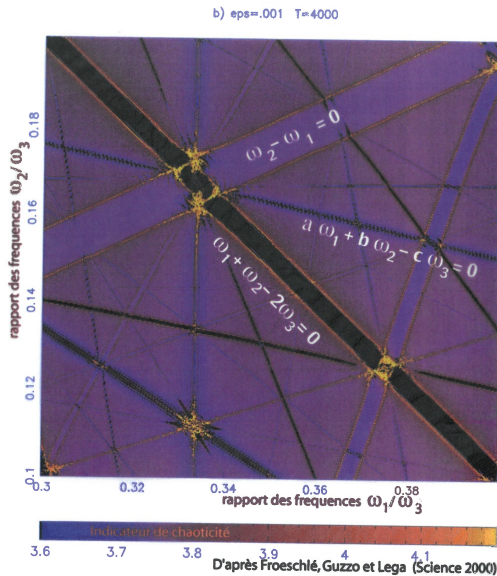


Examples : Science 2000 Froeschlé et al

$$H(\phi_1, \phi_2, \phi_3, l_1, l_2, l_3) = \frac{l_1^2}{2} + \frac{l_2^2}{2} + l_3 + \epsilon \frac{1}{\cos \phi_1 + \cos \phi_2 + \cos \phi_3 + 4}$$

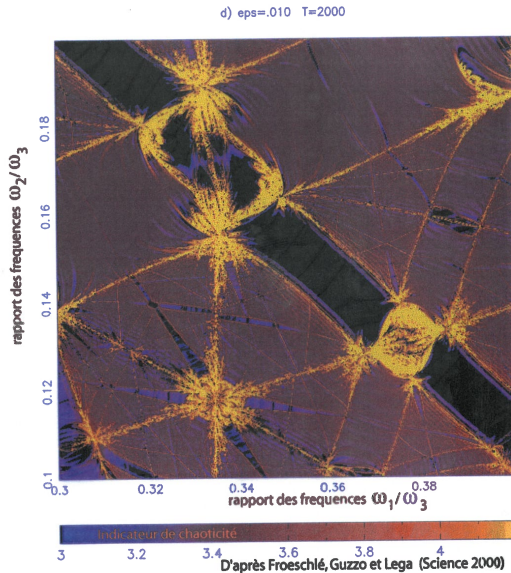


Exemples : Science 2000 Froeschlé et al

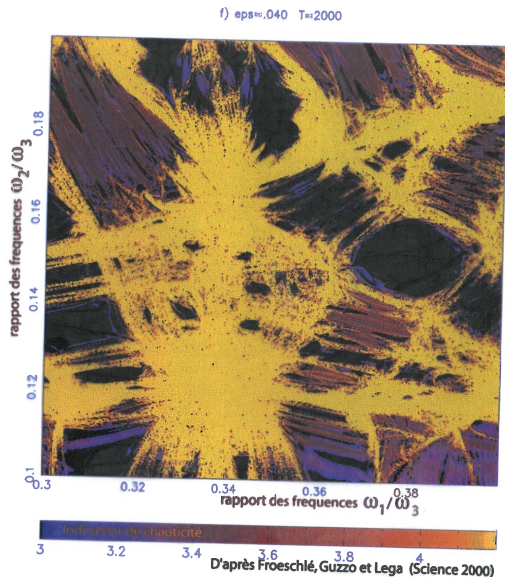


D'apr s Froeschl , Guzzo et Lega (Science 2000)

Exemples : Science 2000 Froeschlé et al

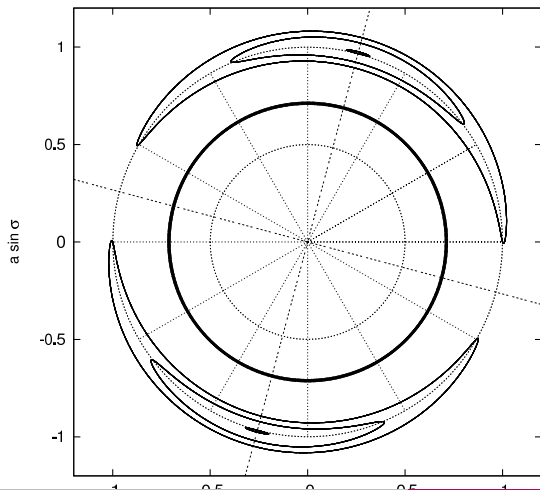


Exemples : Science 2000 Froeschlé et al



FLI chaos indicators and research

- FLI : Fast Lyapounov Indicator (Froeschlé, Lega, Guzzo, etc)
- intensively used by Celletti and collaborators



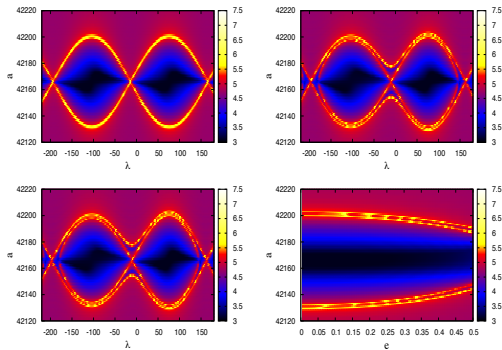


FIGURE 3. FLI (using Hamilton's equations) for the GEO 1:1 resonance for $e = 0.005$, $i = 0^\circ$, $\omega = 0^\circ$, $\Omega = 0^\circ$ under the effects of the J_2 and J_{22} terms (top left); all harmonics up to degree and order $n = m = 3$ (top right); all harmonics up to $n = m = 4$ (bottom left). The bottom right panel yields the FLI for $i = 0^\circ$, $\lambda = 75.07^\circ$ in the (e, a) plane under the effects of all harmonics up to $n = m = 4$.

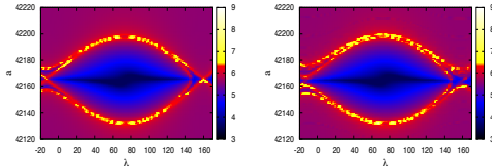
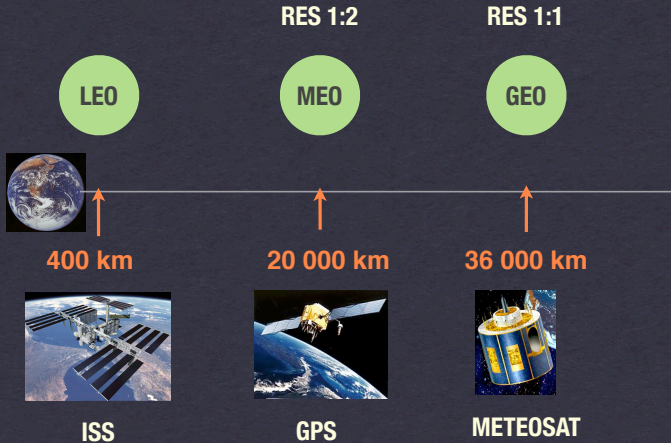


FIGURE 5. FLI (using Cartesian equations) for the GEO 1:1 resonance for $e = 0.005$, $i = 30^\circ$, $\omega = 0$, $\Omega = 0$, under all harmonics up to degree and order three (left panel), all harmonics up to degree and order three + Moon + Sun+ SRP with $A/m = 0.1$ (right panel).

EARTH ENVIRONMENT

resonance between the
orbital period of the satellite
and the rotation of the Earth
= gravitational resonance



Terms of the geopotential : 2:1

$$\begin{aligned}t_1 &= \frac{\mu_E R_E^2 J_{22}}{a^3} \left\{ \frac{3}{4} (1 + \cos i)^2 \left(-\frac{e}{2} \right) \cos(\sigma + \omega - 2\lambda_{22}) \right\} \\t_2 &= \frac{\mu_E R_E^2 J_{22}}{a^3} \left\{ \frac{3}{2} \sin^2 i \left(\frac{3}{2} e \right) \cos(\sigma - \omega - 2\lambda_{22}) \right\} \\t_3 &= \frac{\mu_E R_E^3 J_{32}}{a^4} \left\{ \frac{15}{8} \sin i (1 - 2 \cos i - 3 \cos^2 i) (1 + 2e^2) \sin(\sigma - 2\lambda_{32}) \right\}\end{aligned}$$

where $\sigma = 2\lambda$ with λ as in (3.14).

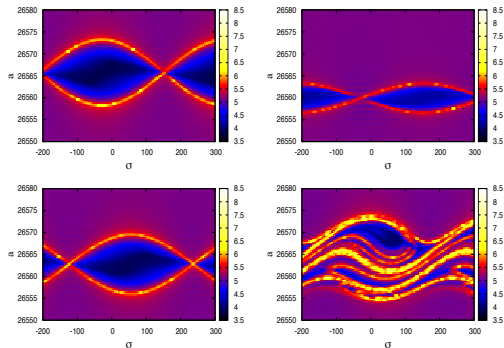


FIGURE 7. FLI for the toy-model (5.1), for $e = 0.1$, $i = 20^\circ$, $\omega = 0$, $\Omega = 0$, under various effects: $J_2 + t_1$ (top left); $J_2 + t_2$ (top right); $J_2 + t_3$ (bottom left); $J_2 + t_1 + t_2 + t_3$ (bottom right).

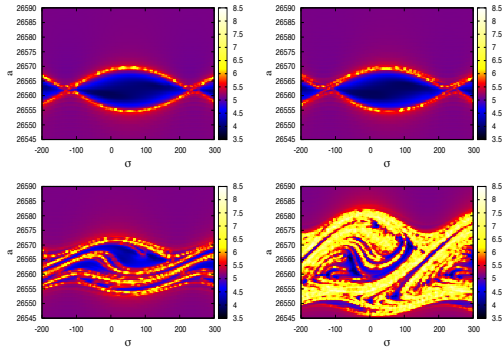


FIGURE 9. FLI (using Hamilton's equations) for the MEO 2:1 resonance, under the effects of all harmonics up to degree and order $n = m = 4$, for $i = 30^\circ$, $\omega = 0$, $\Omega = 0$: $e = 0.005$ (top left); $e = 0.01$ (top right); $e = 0.1$ (bottom left); $e = 0.5$ (bottom right).

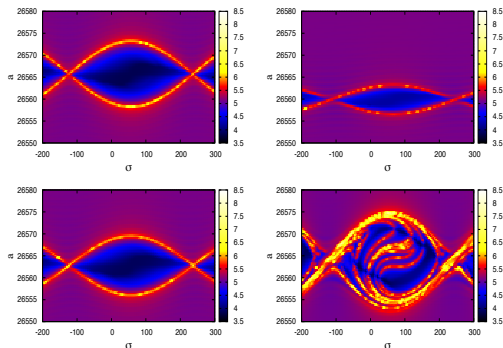


FIGURE 11. FLI for the toy-model (5.1), for $e = 0.1$, $i = 20^\circ$, $\omega = -85^\circ$, $\Omega = 0$, under various effects: $J_2 + t_1$ (top left); $J_2 + t_2$ (top right); $J_2 + t_3$ (bottom left); $J_2 + t_1 + t_2 + t_3$ (bottom right).

The MEGNO indicator - Integral formulation

$$\lambda = \frac{1}{t} \int_0^t \frac{\dot{\delta}(\phi(s))}{\delta(\phi(s))} ds$$

where $\delta = \|\delta\|$, $\dot{\delta} = \dot{\delta} \cdot \delta / \delta$.

MEGNO

The **M**ean **E**xponential **G**rowth factor of **N**earby **time-weighted version** of the integral form of the LCN. More precisely, Cincotta et al. introduced and defined the MEGNO indicator:

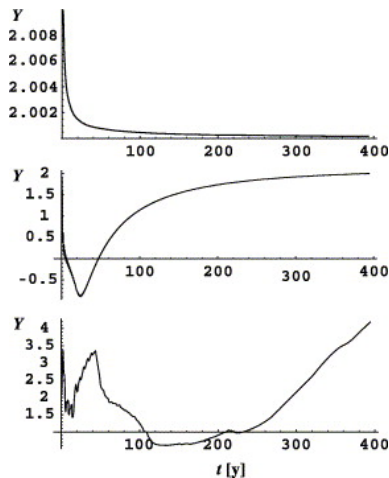
$$Y(\phi(t)) = \frac{2}{t} \int_0^t \frac{\dot{\delta}(\phi(s))}{\delta(\phi(s))} s ds, \quad \overline{Y}(\phi(t)) = \frac{1}{t} \int_0^t Y(\phi(s)) ds$$

Cincotta and Simó, A&A, 2000,

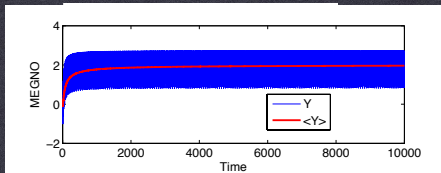
Cincotta, Giordano, and Simó, Physica D, 2003

The MEGNO indicator - properties

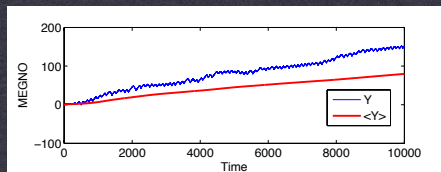
- Stable, isochronous periodic orbits $\overline{Y}(t) \rightarrow 0$
- Quasi-periodic (regular) $\overline{Y}(t) \rightarrow 2$
- Chaotic (irregular) $\overline{Y}(t) \simeq \lambda/2 t$



Breiter et al 2001

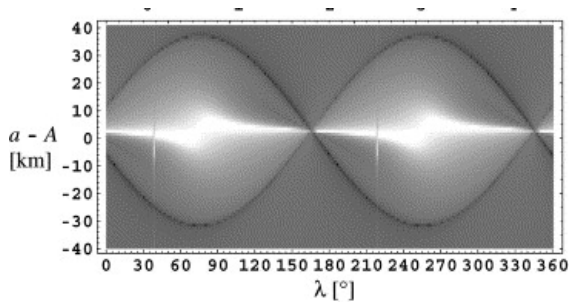


Stable
quasi-periodic
orbit



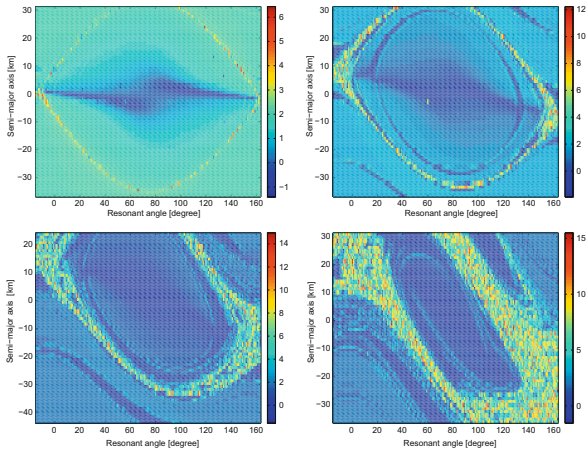
Unstable orbit

time in days

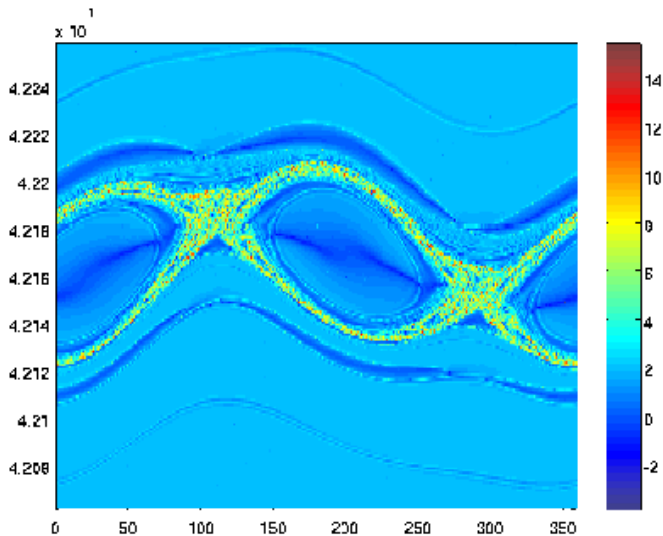


Chaos maps

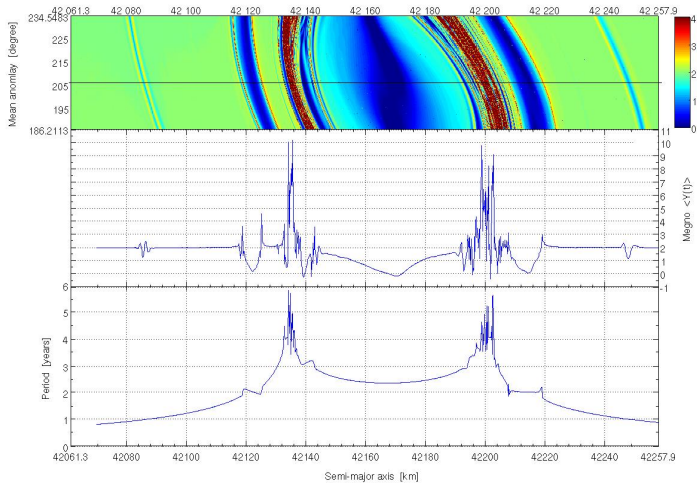
MEGNO : GEO - 30 years - $A/M = 1, 5, 10, 20 \text{ m}^2/\text{kg}$



Geo resonance



Geo resonance



- Earth + $J_2 + J_{22}$
- Very stable case :
 $e_0 = 0.002$
- 30 years and 300 years of integration

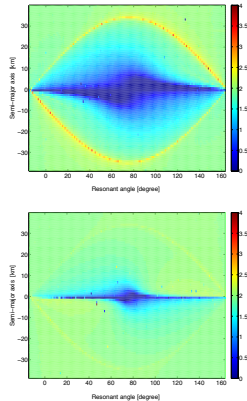


Figure 5.2: The MEGNO computed as a function of initial mean longitudes λ_0 and osculating semi-major axis a_0 . The equations of motion include the central body attraction as well as the second degree and order harmonics J_2, C_{22} and S_{22} . The mean longitude grid is 1° and the semi-major axis grid is 1 km, spanning the $42\,164 \pm 35$ km range. The initial conditions are $e_0 = 0.002$, $i_0 = 0.004$, $\Omega_0 = \omega_0 = 0$. Time at epoch is 25 January 1991. The patterns have been obtained using two different integration times, $t_I = 30$ years [top] and $t_I = 300$ years [bottom].

- Earth + J_2 + J_{22} + Moon + Sun + SRP
- $A/m =$
1, 5, 10, $20m^2/kg$
- $e_0 = 0.002$
- 30 years

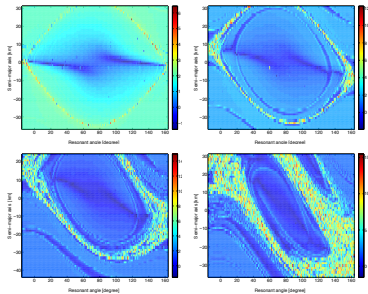


Figure 5.6: The MEGNO computed as a function of initial mean longitudes λ_0 and initial (osculating) semi-major axis a_0 . The equations of motion include the central body attraction, the second degree and order harmonics J_2 , C_{22} and S_{22} , the luni-solar interaction as well as the perturbing effects of the solar radiation pressure. The mean longitude grid is 1° and the semi-major axis grid is 1 km, spanning the $42\,164 \pm 35$ km range. The initial conditions are $(i_0 = 0.002, i_0 = 0.004, \Omega_0 = \omega_0 = 0)$. The integration time is 30 years from epoch fixed at 25 January 1991. The patterns have been obtained using four different area-to-mass ratios, $A/m = 1, 5, 10, 20\,m^2/kg$, respectively top left, top right, bottom left and bottom right panel.

- Earth + J_2 + J_{22} + Moon + Sun + SRP
- $A/m = 10 \text{ m}^2/\text{kg}$
- $e_0 = 2$ and $e = 0.4$

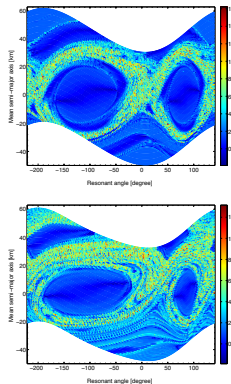


Figure 5.11: The MEGNO computed as a function of initial mean longitudes λ_0 and semi-major axis a_0 . The equations of motion include the central body attraction, the second degree and order harmonics J_2 , C_{22} and S_{22} , the luni-solar interaction as well as the perturbing effects of the solar radiation pressure. The mean longitude grid is 1° and the semi-major axis grid is 1 km spanning the $42\,164 \pm 35$ km range. The initial conditions are $i_0 = 0.004$, $\Omega_0 = \omega_0 = 0$ with an area-to-mass ratio $A/m = 10 \text{ m}^2/\text{kg}$. Time at epoch is 21 December 2000. The patterns have been obtained using two initial eccentricities, $e_0 = 0.2$ [top] and $e_0 = 0.4$ [bottom].

- Earth + J_2 + J_{22} + Moon + Sun + SRP
- $A/m = 10 \text{ m}^2/\text{kg}$
- $e_0 = 0.002$

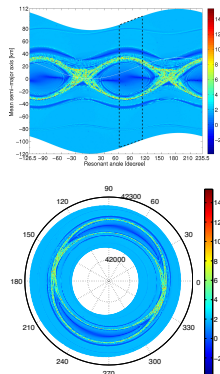


Figure 5.12: The MEGNO computed as a function of initial mean longitudes λ_0 and semi-major axis a_0 . [Top] Enlargement of the complete phase space. [Bottom] Enlargement of the complete phase space shown in polar coordinates. The equations of motion include the central body attraction, the second degree and order harmonics J_2 , C_{22} and S_{22} as well as the luni-solar perturbations. The mean longitude grid is 1° and the semi-major axis grid is 1 km, spanning the 42164 ± 100 km range. The initial conditions are $e_0 = 0.002$, $i_0 = 0.0004$, $\Omega_0 = \omega_0 = 0$. The area-to-mass ratio is $A/m = 10 \text{ m}^2/\text{kg}$. Time at epoch is 25 January 1991.

Variation of the frequencies - second derivatives

Principle

The main purpose of the FAM is to determine the approximation $f'(t)$ of a signal $f(t)$, where both are developed in Fourier series:

$$f'(t) = \sum_{k=1}^N p'_k e^{i v'_k t} \text{ approximation of the initial signal } f(t) = \sum_{k=1}^{\infty} p_k e^{i v_k t}.$$

The frequencies v'_k for $k = 1, \dots, N$ and their associated decreasing amplitudes p'_k for $k = 1, \dots, N$ are determined through an iterative scheme.

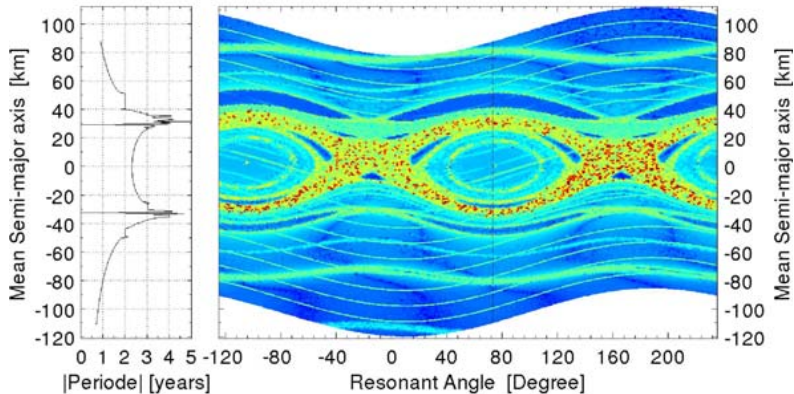
Laskar, 1993

Frequency map

$$A/m = 10m^2/kg$$

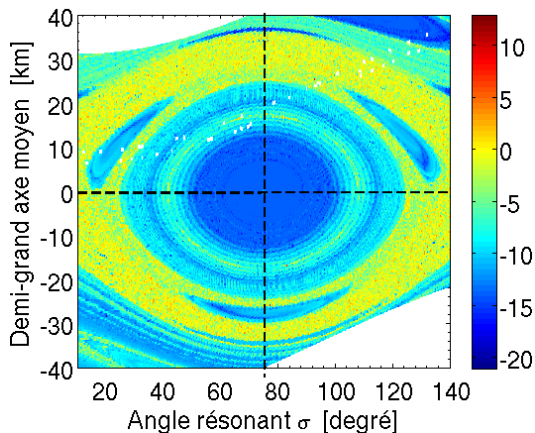
~Stable Point

$M=73.2144^\circ$



Further analysis stability : zoom

Primary and secondary resonance analysis : stability zones



Secondary resonances

Hamiltonian

$$\mathcal{H} = -\frac{\mu^2}{2L^2} - \dot{\theta}L + \frac{\mu}{a^3} R_e^2 (F_{200}(i) G_{200}(e) S_{2200} + F_{221}(i) G_{212}(e) S_{2212})$$

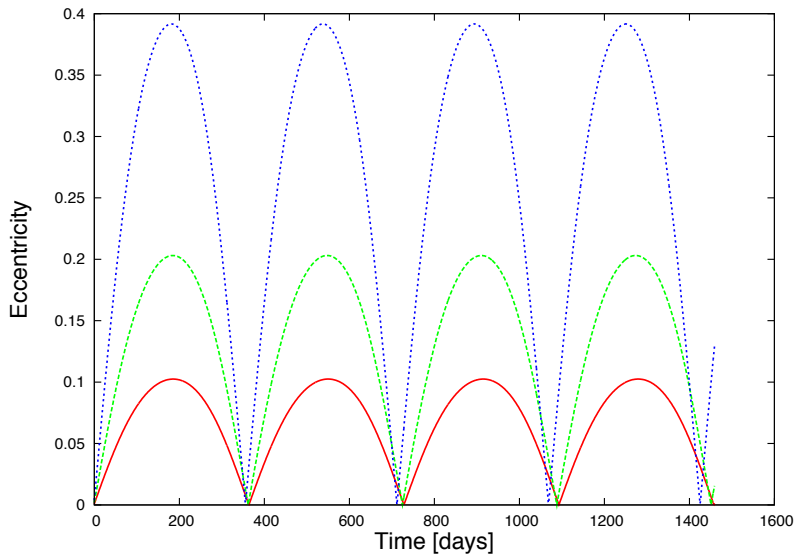
Truncation

$$\mathcal{H} = -\frac{\mu^2}{2L^2} - \dot{\theta}L + \frac{3\mu^4}{L^6} R_e^2 J_{22} \cos 2(\sigma - \sigma_0) - \frac{15\mu^4}{2L^6} R_e^2 e^2 J_{22} \cos 2(\sigma - \sigma_0).$$

$$\mathcal{Z} = \kappa = \frac{3}{2} C_r P_r \frac{A}{m} \frac{a}{\sqrt{L}}$$

$$\begin{aligned} x_1 &= -\kappa \sin \lambda_{\odot} + C_x &= -\kappa (\sin \lambda_{\odot} - D_x) \\ y_1 &= \kappa \cos \lambda_{\odot} \cos \epsilon + C_y &= \kappa (\cos \lambda_{\odot} \cos \epsilon + D_y). \end{aligned}$$

The eccentricity - pericenter motion : 1 year



Secondary resonances

Resonance

$$e^2 = \frac{\mathcal{Z}^2}{L^2 n_S^2} + \gamma^2 + \frac{2\mathcal{Z}}{Ln_S} \gamma \cos(\lambda_S + \delta)$$

and the final (with all these successive approximations) Hamiltonian K is:

$$K(L, \sigma) = -\frac{\mu^2}{2L^2} - \dot{\theta}L + \cos(2\sigma - 2\sigma_0) \left[\frac{F}{L^6} - \frac{2G}{L^6} \cos(\lambda_S + \delta) \right],$$

with

$$F = 3\mu^4 R_e^2 J_{22} - \frac{15\mu^4}{2} R_e^2 J_{22} \left(\frac{\mathcal{Z}^2}{L^2 n_S^2} + \gamma^2 \right)$$

$$G = \frac{15\mu^4}{2} R_e^2 J_{22} \frac{\mathcal{Z}}{Ln_S} \gamma$$

Secondary resonances

Expansion in the GEO region : $L \simeq L^*$

Introduction of normalized variables (r, R) on the pendulum and on the perturbation

Resonance

$$\begin{aligned} K &= -\beta \left(\frac{R^2}{2} - b \cos r \right) - 2 \frac{G}{L_0^6} \cos r \cos (\lambda_S + \delta) \\ &= -\beta \left(\frac{R^2}{2} - b \cos r + 2 G' \cos r \cos \eta \right) \quad \text{with } G' = -\frac{G}{\beta L_0^6} \\ h &= \frac{K}{-\beta} = \frac{R^2}{2} - b \cos r + G' \cos (r + \eta) + G' \cos (r - \eta), \end{aligned}$$

where $\eta = \lambda_S + \delta$.

Perturbed pendulum

Action-angle (ψ, ρ) on the pendulum and on the perturbation q a small parameter (0.5 on the separatrix, smaller above)

$$\begin{aligned} h &\simeq \frac{\rho^2}{2} - b + n_S \Lambda \\ &+ G' J_0(2q) \cos(\psi + \eta) + G' J_0(2q) \cos(\psi - \eta) \\ &+ G' J_2(2q) \cos(3\psi + \eta) + G' J_2(2q) \cos(3\psi - \eta) + G' J_2(2q) \cos(\psi + \eta) \\ &+ G' J_2(2q) \cos(\psi - \eta) - G' J_1(2q) \cos(\eta) - G' J_1(2q) \cos(-\eta) \\ &- G' J_1(2q) \cos(2\psi + \eta) - G' J_1(2q) \cos(2\psi - \eta) \end{aligned}$$

$\psi \equiv \sigma$ (GEO resonance- 2.5 years) and $\eta \equiv \lambda_S$ (Sun - 1 year)

$$\dot{\psi} = \frac{\partial h}{\partial J}, \quad \dot{\eta} = \frac{\partial h}{\partial \Lambda}, \quad \dot{J} = -\frac{\partial h}{\partial \psi} \quad \text{and} \quad \dot{\Lambda} = \frac{\partial h}{\partial \eta}.$$

We isolate a resonance $\phi = j\psi \pm \eta$

we build a canonical transformation between (ψ, η, J, Λ) to (ϕ, η, J, Γ) defined by :

$$\psi = \phi \mp \frac{\eta}{j} \quad \Lambda = \Gamma \pm \frac{J}{j}.$$

The Hamiltonian becomes :

$$\begin{aligned} h &\simeq \frac{\rho^2}{2} - b + n_S \left(\Gamma \mp \frac{J}{j} \right) \\ &+ G' J_0(2q) \cos\left(\phi \mp \frac{\eta}{j} + \eta\right) + G' J_0(2q) \cos\left(\phi \mp \frac{\eta}{j} - \eta\right) \\ &+ G' J_2(2q) \cos\left(3\phi \mp 3\frac{\eta}{j} + \eta\right) + G' J_2(2q) \cos\left(3\phi \mp 3\frac{\eta}{j} - \eta\right) \\ &+ G' J_2(2q) \cos\left(\phi \mp \frac{\eta}{j} + \eta\right) + G' J_2(2q) \cos\left(\phi \mp \frac{\eta}{j} - \eta\right) \\ &- G' J_1(2q) \cos\left(2\phi \mp 2\frac{\eta}{j} + \eta\right) - G' J_1(2q) \cos\left(2\phi \mp 2\frac{\eta}{j} - \eta\right) \\ &- 2 G' J_1(2q) \cos(\eta) \end{aligned}$$

Choice of the resonance $j = \pm 1$, or $j = \pm 2$ or $j = \pm 3$, we keep one resonant combination and we average over η so to get :

$$\bar{h}_1 = \frac{\rho^2}{2} - b + n_S (\Gamma \mp J) + G' (J_0(2q) + J_2(2q)) \cos(\phi)$$

$$\bar{h}_2 = \frac{\rho^2}{2} - b + n_S \left(\Gamma \mp \frac{J}{2} \right) - G' J_1(2q) \cos 2\phi$$

$$\bar{h}_3 = \frac{\rho^2}{2} - b + n_S \left(\Gamma \mp \frac{J}{3} \right) + G' J_2(2q) \cos 3\phi$$

Initial map

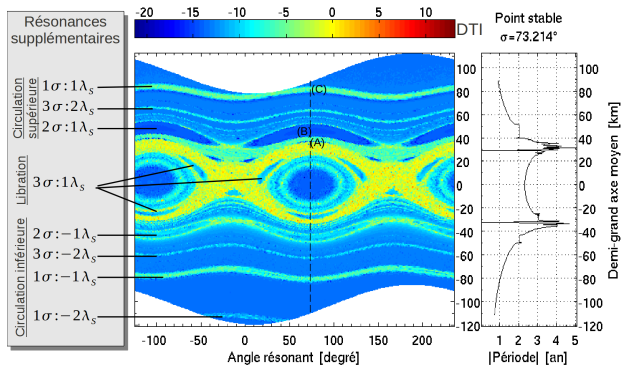
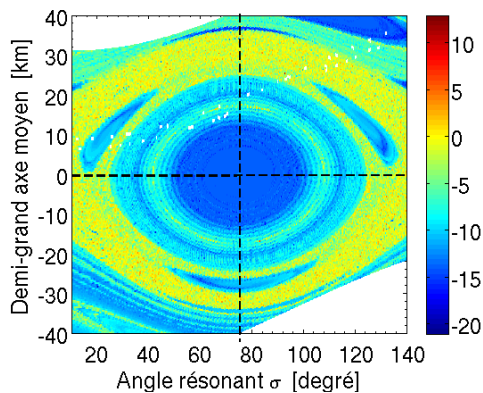


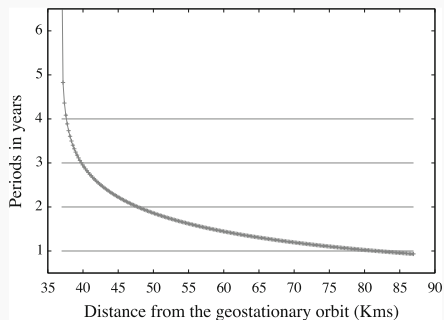
FIGURE 3.14 – Graphique central : carte de l'indicateur du taux de diffusion (DTI) de la fréquence principale de l'angle $\sigma = \lambda - \theta$ en fonction de la condition initiale en demi-grand axe moyen et en angle résonant. Le modèle de force contient les harmoniques sphériques liées aux coefficients C_{20} , C_{22} et S_{22} ainsi que la pression de radiation ($A/m = 10 \text{ m}^2/\text{kg}$). Graphique de droite : évolution de la période fondamentale de l'angle σ en fonction du demi-grand axe moyen initial pour un ensemble d'orbites dont l'angle résonant initial $\sigma_0 = 73.214^\circ$. Panneau de gauche : identification des résonances de type $k\sigma : j\lambda_s$ pour $k \in \{1, 2, 3\}$ et $j \in \{1, 2\}$. N.B. : Un demi-grand axe égal à 0 correspond à 42 164.140 km.

Initial map



Secondary resonances

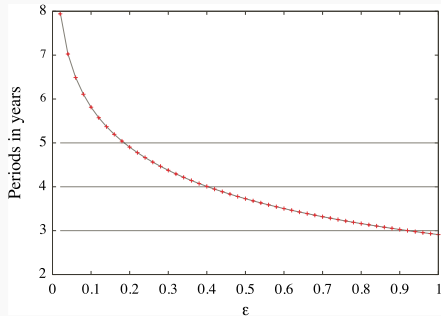
Circulation cases



Secondary resonances

Libration case

Fig. 4 Libration case: The period T (in years) calculated by a numerical integration of the pendulum differential equations (*dots*) and through the analytical expressions (*lines*), as functions of ϵ in the libration zone; ϵ is an angle measured from the separatrix



3:1 secondary resonance

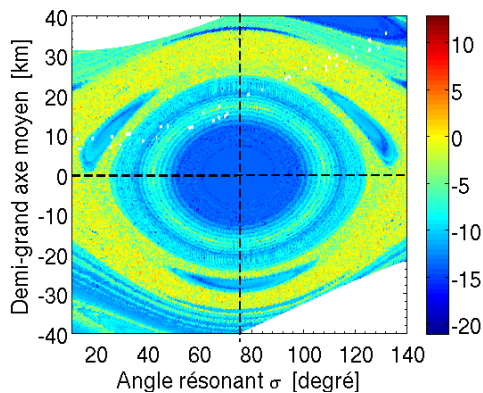
- $h_3 = b \cos \epsilon(J) - 2G' k^2 q \cos(3\psi - \eta) + n_S \Lambda$
- Λ is the momentum conjugated to η .
- $3\Phi = 3\psi - \eta$, Φ being always conjugated to J
- a new momentum $\Gamma = \Lambda + \frac{J}{3}$ conjugated to η :
- $h_3 = b \cos \epsilon(J) - \frac{n_S}{3} J - 2G' k^2 q \cos 3\Phi$
- $n_S \Gamma$ has been dropped.
- $\frac{\partial h_3}{\partial J} = 0 : \epsilon(J)$ corresponding to the resonance 3 : 1

Equation defining $\epsilon(J)$

$$-b \sin \epsilon \frac{\partial \epsilon}{\partial J} = \frac{n_S}{3} \quad \text{or} \quad \frac{\pi \sqrt{b}}{2 \mathbf{K}(k)} = \frac{n_S}{3} \quad \text{with} \quad k = \cos \frac{\epsilon}{2}.$$

- The three equilibrium islands : $\Phi = \frac{2\pi}{3}$, $\Phi = \pi$ and $\Phi = \frac{5\pi}{3}$
- $\psi = \Phi - \frac{\delta + \lambda_S}{3} = \Phi - \frac{\eta}{3}$.
- The three values of ϕ are then 60.26° 180.26° and 300.26°
- Measured from the vertical positive axis on which $\psi = r = 0^\circ$.

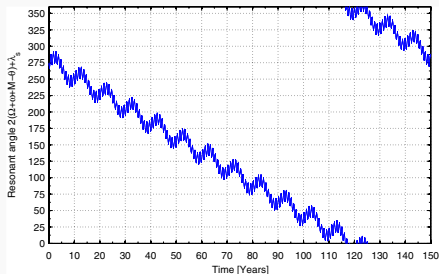
Initial map



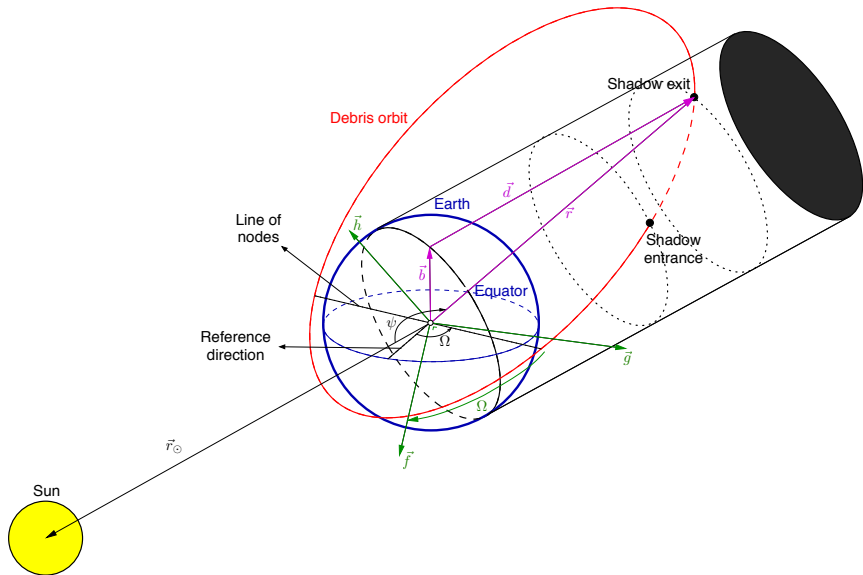
Long term evolution

Libration cases

Fig. 8 Evolution of the resonant angle $2(l - \theta) + \lambda_S$. The used model is the numerical model of Fig. 2, involving a long supplementary period of 150 years. The initial conditions are the same of the Fig. 2 with $M = 199.8^\circ$



The Earth umbra



The shadow equation

Simple geometrical problem : cylinder $\cap$ ellipse

cylinder : axis in the Sun direction

ellipse : debris orbit

$$s_c(\mathbf{r}) = \frac{\mathbf{r} \cdot \mathbf{r}_\odot}{r_\odot} + \sqrt{r^2 - R_\oplus^2} \quad \begin{array}{ll} < 0 & \text{inside Earth's shadows} \\ > 0 & \text{outside Earth's shadows} \\ = 0 & \text{entry and exit} \end{array}$$

4th degree polynomial in $\tan \frac{E}{2}$ solved by Cardan formula

E_1 entry eccentric anomaly = $E_1(a, e, i, \omega, \Omega, \bar{r}_\odot)$

E_2 exit eccentric anomaly = $E_2(a, e, i, \omega, \Omega, \bar{r}_\odot)$

The toy model with shadow

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \begin{cases} C_r P_r \frac{A}{m} r \bar{r}_{\odot} \cos(\phi) & \text{outside Earth's shadows} \\ 0 & \text{inside Earth's shadows} \end{cases}$$

ϕ the angle between $\mathbf{r}$ and $\mathbf{r}_{\odot}$, $L = \sqrt{\mu a}$, $\bar{r}_{\odot} = \frac{r_{\odot}}{a_{\odot}}$.

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \begin{cases} C_r P_r \frac{A}{m} a(u\xi + v\eta) & \text{outside Earth's shadows} \\ 0 & \text{inside Earth's shadows} \end{cases}$$

Debris orbital motion : $u = \cos E - e$ and $v = \sin E \sqrt{1 - e^2}$.

Debris orbit orientation and Sun orbital motion :

$$\xi = \xi_1 \bar{r}_{\odot,1} + \xi_2 \bar{r}_{\odot,2} + \xi_3 \bar{r}_{\odot,3}$$

$$\eta = \eta_1 \bar{r}_{\odot,1} + \eta_2 \bar{r}_{\odot,2} + \eta_3 \bar{r}_{\odot,3}$$

$$\begin{aligned} \xi_1 &= \cos \Omega \cos \omega - \sin \Omega \cos i \sin \omega \\ \xi_2 &= \sin \Omega \cos \omega + \cos \Omega \cos i \sin \omega \\ \xi_3 &= \sin i \sin \omega \end{aligned}$$

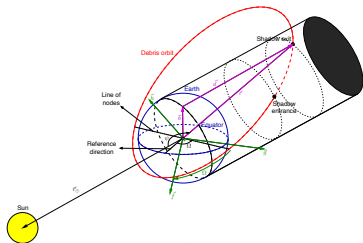
$$\begin{aligned} \eta_1 &= -\cos \Omega \sin \omega - \sin \Omega \cos i \cos \omega \\ \eta_2 &= -\sin \Omega \sin \omega + \cos \Omega \cos i \cos \omega \\ \eta_3 &= \sin i \cos \omega \end{aligned}$$

Averaging over the short periods : 1 day with shadow

Periods : **1 day** (Orbital motion E) and **1 year** (Sun $\bar{r}_{\odot,i}$)

Averaging over the fast variable (M the mean anomaly) :

$$\begin{aligned}\overline{\mathcal{H}} &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H} dM \\ &= -\frac{\mu^2}{2L^2} \\ &+ \frac{1}{2\pi} C_r P_r \frac{A}{m} \bar{a} \left[\int_0^{M_1} (u \xi + v \eta) dM + \int_{M_2}^{2\pi} (u \xi + v \eta) dM \right]\end{aligned}$$



$$dM = (1 - e \cos E) dE, \quad M_1 = E_1 - e \sin E_1, \quad M_2 = E_2 - e \sin E_2.$$

Aksnes 1976

The averaged Hamiltonian with shadow

$$\begin{aligned}\overline{\mathcal{H}} &= -\frac{\mu^2}{2\overline{L}^2} - \frac{3}{2} C_r P_r \frac{A}{m} \frac{\overline{L}^2}{\mu} \overline{e} \xi + \frac{1}{2\pi} C_r P_r \frac{A}{m} \frac{\overline{L}^2}{\mu} [\xi \mathcal{A} + \eta \mathcal{B}] \\ &= \overline{\mathcal{H}}_0(\overline{L}, \overline{G}, \overline{H}, -, \overline{\omega}, \overline{\Omega}, \overline{r}_\odot) + \overline{\mathcal{H}}_1(\overline{L}, \overline{G}, \overline{H}, -, \overline{\omega}, \overline{\Omega}, \overline{r}_\odot) \\ &= \overline{\mathcal{H}}(D=0) + \overline{\mathcal{H}}_1(D)\end{aligned}$$

$$\mathcal{A} = -2(1 + \overline{e}^2) \cos \frac{S}{2} \sin \frac{D}{2} + \frac{3}{2} \overline{e} D + \frac{\overline{e}}{2} \cos S \sin D$$

$$\mathcal{B} = \sqrt{1 - \overline{e}^2} \left(-2 \sin \frac{S}{2} \sin \frac{D}{2} + \frac{\overline{e}}{2} \sin S \sin D \right)$$

$$S = E_1 + E_2$$

$$D = E_2 - E_1 = D(\overline{L}, \overline{G}, \overline{H}, -, \overline{\omega}, \overline{\Omega}, \overline{r}_\odot)$$

The perturbed problem : with the shadows

$$\mathcal{H} = \mathcal{H}(L, P, Q, \lambda, p, q) \quad \Rightarrow \quad \overline{\mathcal{H}} = \overline{\mathcal{H}}(\overline{L}, \overline{P}, \overline{Q}, -, \overline{p}, \overline{q})$$

At first order :

$$\begin{aligned} \langle \dot{P} \rangle &= \dot{\overline{P}} = -\frac{\partial \overline{\mathcal{H}}}{\partial \overline{p}} = -\left\langle \frac{\partial \mathcal{H}}{\partial p} \right\rangle \\ \langle \dot{p} \rangle &= \dot{\overline{p}} = \frac{\partial \overline{\mathcal{H}}}{\partial \overline{P}} = \left\langle \frac{\partial \mathcal{H}}{\partial P} \right\rangle \\ \langle \dot{Q} \rangle &= \dot{\overline{Q}} = -\frac{\partial \overline{\mathcal{H}}}{\partial \overline{q}} = -\left\langle \frac{\partial \mathcal{H}}{\partial q} \right\rangle \\ \langle \dot{q} \rangle &= \dot{\overline{q}} = \frac{\partial \overline{\mathcal{H}}}{\partial \overline{Q}} = \left\langle \frac{\partial \mathcal{H}}{\partial Q} \right\rangle \end{aligned}$$

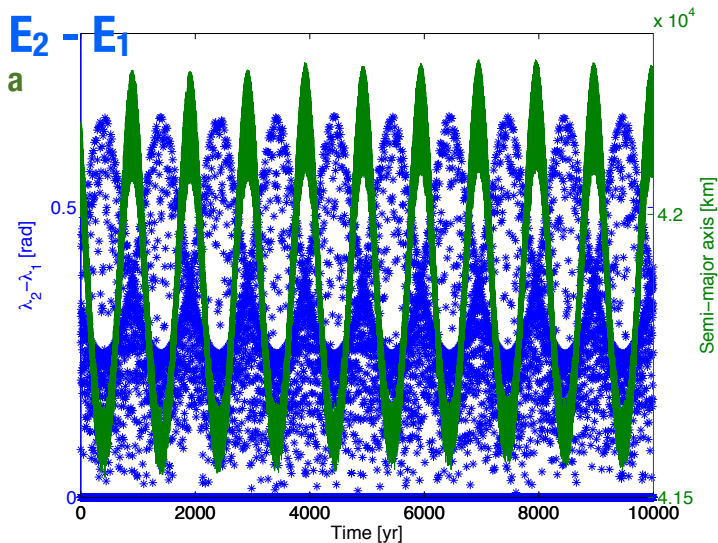
but not for L or a .

Very long periodic motion of the semi-major axis

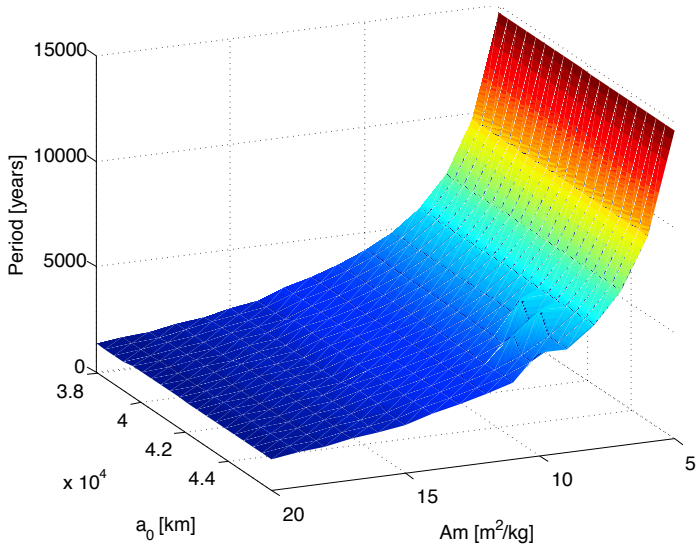
$$\begin{aligned}\langle \dot{L} \rangle &= \left\langle \frac{\partial \mathcal{H}}{\partial M} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial M} dM \\&= \frac{1}{2\pi} \left[\int_0^{E_1} \frac{\partial \mathcal{H}}{\partial M} (1 - e \cos E) dE + \int_{E_2}^{2\pi} \frac{\partial \mathcal{H}}{\partial M} (1 - e \cos E) dE \right] \\&= \frac{1}{\pi} C_r P_r \frac{A}{m} \bar{a} \left[\bar{\xi} \sin \frac{S}{2} - \bar{\eta} \sqrt{1 - \bar{e}^2} \cos \frac{S}{2} \right] \sin \frac{D}{2} \\ \langle \dot{a} \rangle &= \bar{a}^{3/2} \frac{2}{\pi \sqrt{\mu}} C_r P_r \frac{A}{m} \left[\bar{\xi} \sin \frac{S}{2} - \bar{\eta} \sqrt{1 - \bar{e}^2} \cos \frac{S}{2} \right] \sin \frac{D}{2}\end{aligned}$$

Synchronism between a and $E_2 - E_1$

$A/m = 25 \text{ m}^2/\text{kg}$ - period $\simeq 1200$ years - $\Delta a \simeq 600$ km



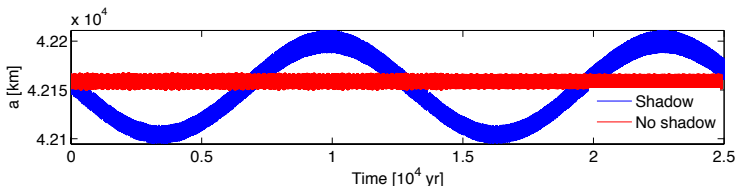
Very long period decreasing with the coefficient A/m



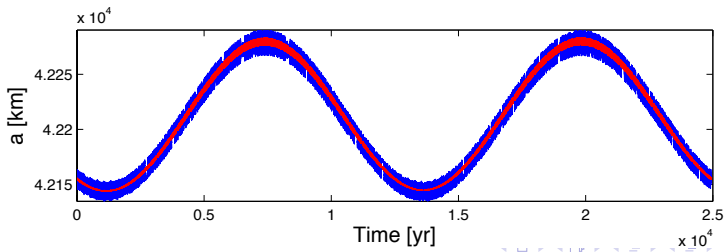
Comparisons

Coefficient $A/m = 5 \text{ m}^2/\text{kg}$ - period $\simeq 13\,000$ years

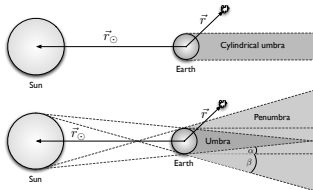
Numerical integration of the simplified system with shadow / without shadow



Symplectic numerical integration with shadow / Simplified system with shadow



Smooth Shadowing effects



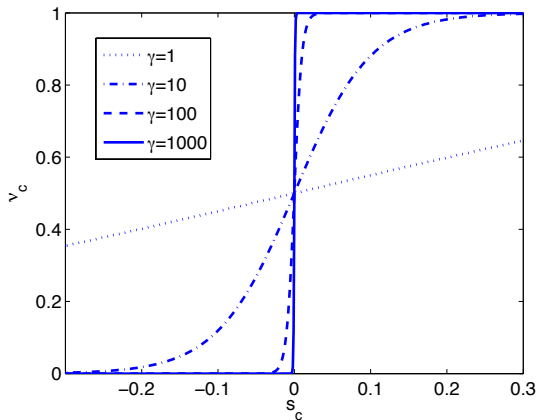
$$s_C(\vec{r}) = \sqrt{r^2 - R_{\oplus}^2} + \frac{\vec{r} \cdot \vec{r}_{\odot}}{r_{\odot}} \leq 0$$

switch on/off (numerically ?)
adapted averaging methods

$$\nu_C = \frac{1}{2}(1 + \tanh(\gamma s_C(\vec{r}))) \simeq \begin{cases} 0 & \text{in cylindrical umbra} \\ 1 & \text{otherwise} \end{cases}$$

Penumbra/umbra function for the conical case (ν_p) with 2 parameters

Gamma parameter



Cone

$$\alpha = \operatorname{atan} \frac{R_{\odot} - R_{\oplus}}{\|\vec{r} - \vec{r}_{\odot}\|} \quad \text{and} \quad \beta = \operatorname{atan} \frac{R_{\odot} + R_{\oplus}}{\|\vec{r} - \vec{r}_{\odot}\|}$$

with $R_{\odot}$ the radius of the Sun. Extending relation (3.7), it follows that space debris are in the umbra cone when

$$s_u(\vec{r}) := \frac{\vec{r} \cdot \vec{r}_{\odot}}{r_{\odot}} + \cos \alpha \left[\sqrt{r^2 - R_{\oplus}^2 \cos^2 \alpha} + R_{\oplus} \sin \alpha \right] \leq 0$$

and in the penumbra cone when

$$s_p(\vec{r}) := \frac{\vec{r} \cdot \vec{r}_{\odot}}{r_{\odot}} + \cos \beta \left[\sqrt{r^2 - R_{\oplus}^2 \cos^2 \beta} - R_{\oplus} \sin \beta \right] \leq 0.$$

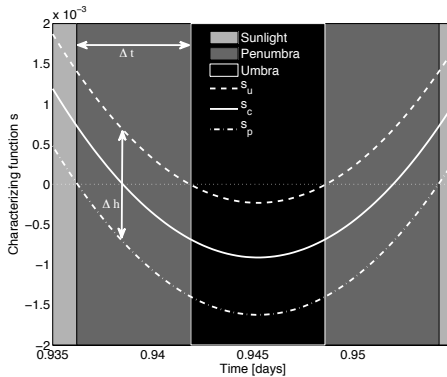
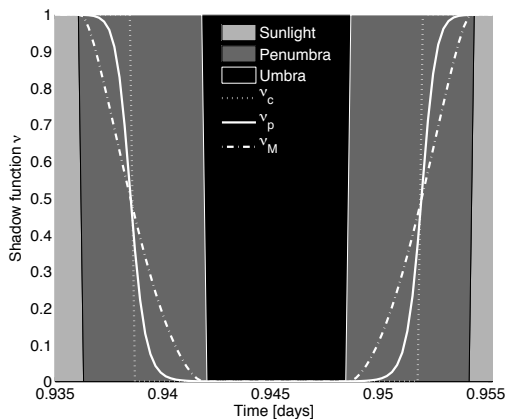


FIGURE 3.4 • Evolution of the functions s_c , s_u and s_p during a shadow crossing on a geostationary orbit. The penumbra and umbra cones are crossed respectively when s_p is negative and s_u is negative. The time spent in the penumbra transition is noted Δt and the difference between s_u and s_p at the entrance of the cylindrical shadow is denoted by Δh .

Smooth Shadowing effects



ν_M proposed by Montenbruck and Ghill (2005)

- Earth + J_2 + J_{22} + Moon + Sun + SRP
- $A/m = 5m^2/kg$
- shadowing effects
- Cylindrical or conical
- Chaotic motion

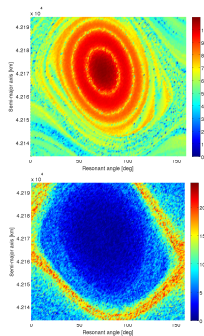


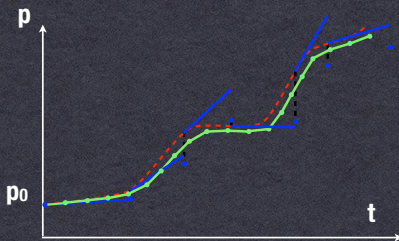
FIGURE 5.15 • Stability analysis of the two-dimensional plane (σ_m, α) represented using MEGNO values at 30 yr (GSI) with Earth's shadow. Other initial conditions and integrator are chosen as in Fig. 5.12 with time steps equal to 0.01 day/2 π . The model includes the central body attraction, the gravitational up to degree and order 2, luni-solar perturbations and SRP with the AMR equal to 5 m^2/kg with cylindrical (top) and conical (bottom) Earth's shadow.

Symplectic : Introduction

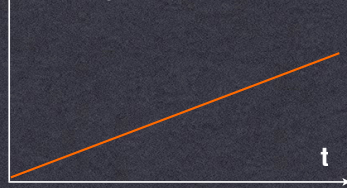
- Solar system or extrasolar systems : symplectic integrators for long integration
- SyMBA (Duncan et al., 1998)
- Artificial satellites : short mission times : classical numerical integration (RK4 to ABM10)
- Resonant motion : presence of a slow angle
- Space debris : presence for thousands of years
- Symplecticity
- CM : Yoshida (1990) or Laskar and Robutel (2001)

Symplectic integrator

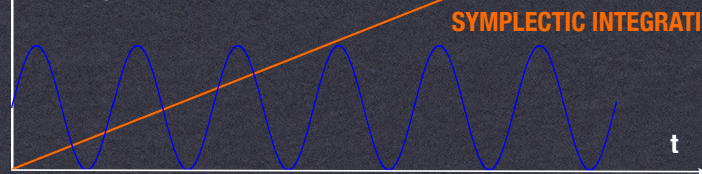
CLASSICAL NUMERICAL INTEGRATION



$$E = H - H_0$$



$$E = H - H_0$$



SYMPLECTIC INTEGRATION

Symplectic integrator for space debris : Hubaux

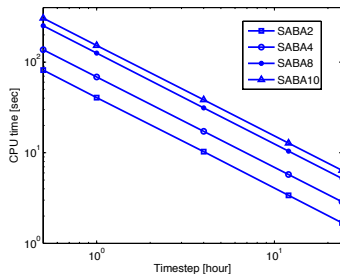
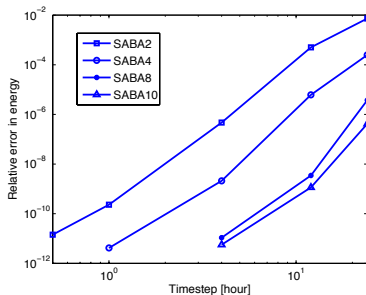
Two separate parts in the Hamiltonian : A and B : 1/2 step with one, 1 step with the second, 1/2 step with one.

Most classical : SABA, SBAB (Laskar and Robutel)

Time dependent perturbation - 500 years - ephemerides Sun

Different orders : 2, 4, 6, 8

$$H(\vec{v}, \Lambda, \vec{r}, \theta) = H_K(\vec{v}, \vec{r}) + H_{Rot}(\Lambda) + H_{geo}(\vec{r}, \theta) + H_{3B}(\vec{r}, \theta) + H_{SRP}(\vec{r}, \theta) = A(\vec{v}, \vec{r}, \Lambda) + B(\vec{r}, \theta)$$



Use of several symplectic integrators

$$\dot{\vec{x}} = L_{\mathcal{H}}\vec{x} = \{\mathcal{H}, \vec{x}\} = \sum_{j=1}^N \left(\frac{\partial \mathcal{H}}{\partial p_j} \frac{\partial \vec{x}}{\partial q_j} - \frac{\partial \mathcal{H}}{\partial q_j} \frac{\partial \vec{x}}{\partial p_j} \right).$$

$$\vec{x}(t) = e^{\tau L_{\mathcal{H}}} \vec{x}(t_0) = \sum_{j=0}^{\infty} \frac{\tau^j}{j!} L_{\mathcal{H}}^j \vec{x}(t_0)$$

$$\begin{aligned} H(p,q) &= A(p,q) + B(p,q) \\ \Phi &= A + B + O(\tau^{n+1}) \end{aligned}$$

Yoshida 1990
 S_4, S_6

$$\begin{aligned} H(p,q) &= A(p,q) + \varepsilon B(p,q) \\ \Phi &= A + \varepsilon B + O(\tau^{2n} \varepsilon + \tau^2 \varepsilon^2) \end{aligned}$$

Laskar and Robutel 2001
 $SABA_4, SABA_6, SABA_8, SABA_{10}$

$$\text{SABA}_{2n}(\tau)$$

$$= e^{c_1 \tau L_A} e^{d_1 \tau L_{\varepsilon B}} \dots e^{c_n \tau L_A} e^{d_n \tau L_{\varepsilon B}} e^{c_{n+1} \tau L_A} e^{d_n \tau L_{\varepsilon B}} e^{c_n \tau L_A} \dots e^{d_1 \tau L_{\varepsilon B}} e^{c_1 \tau L_A}$$

$$\text{SABA}_{2n+1}(\tau)$$

$$= e^{c_1 \tau L_A} e^{d_1 \tau L_{\varepsilon B}} \dots e^{c_{n+1} \tau L_A} e^{d_{n+1} \tau L_{\varepsilon B}} e^{c_{n+1} \tau L_A} \dots e^{d_1 \tau L_{\varepsilon B}} e^{c_1 \tau L_A}$$

$$\text{SBAB}_{2n}(\tau)$$

$$= e^{d_1 \tau L_{\varepsilon B}} e^{c_2 \tau L_A} e^{d_2 \tau L_{\varepsilon B}} \dots e^{c_{n+1} \tau L_A} e^{d_{n+1} \tau L_{\varepsilon B}} e^{c_{n+1} \tau L_A} \dots e^{d_2 \tau L_{\varepsilon B}} e^{c_2 \tau L_A} e^{d_1 \tau L_{\varepsilon B}}$$

$$\text{SBAB}_{2n+1}(\tau)$$

$$= e^{d_1 \tau L_{\varepsilon B}} e^{c_2 \tau L_A} \dots e^{d_{n+1} \tau L_{\varepsilon B}} e^{c_{n+2} \tau L_A} e^{d_{n+1} \tau L_{\varepsilon B}} \dots e^{c_2 \tau L_A} e^{d_1 \tau L_{\varepsilon B}}$$

Symplectic integrator

$$\text{SABA}_2(\tau) = e^{c_1 \tau L_A} e^{d_1 \tau L_{\varepsilon B}} e^{c_2 \tau L_A} e^{d_1 \tau L_{\varepsilon B}} e^{c_1 \tau L_A}$$

$$\vec{x}_{k+1} = \text{SABA}_2(\tau) \vec{x}_k$$

$$t_{k+1} = t_k + \tau$$

$$\mathcal{H}(\vec{p}, \vec{q}) = A(\vec{p}) + \varepsilon B(\vec{q}).$$

$$\vec{x}_{k_1} = e^{c_1 \tau L_A} \vec{x}_k$$

$$\vec{x}_{k_2} = e^{d_1 \tau L_{\varepsilon B}} \vec{x}_{k_1}$$

$$\vec{x}_{k_3} = e^{c_2 \tau L_A} \vec{x}_{k_2}$$

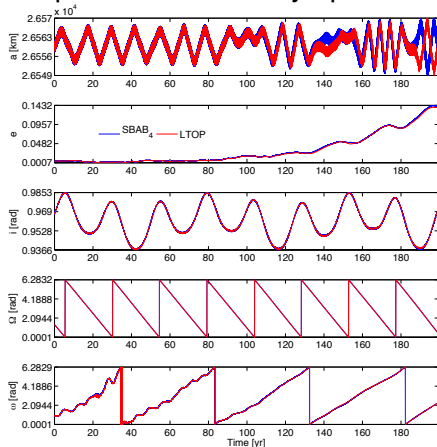
$$\vec{x}_{k_4} = e^{d_1 \tau L_{\varepsilon B}} \vec{x}_{k_3}$$

$$\vec{x}_{k+1} = \vec{x}_{k_5} = e^{c_1 \tau L_A} \vec{x}_{k_4}$$

$$t_{k+1} = t_k + \tau.$$

Symplectic integrator

Compared with a test symplectic integrator of ESOC : LTOP

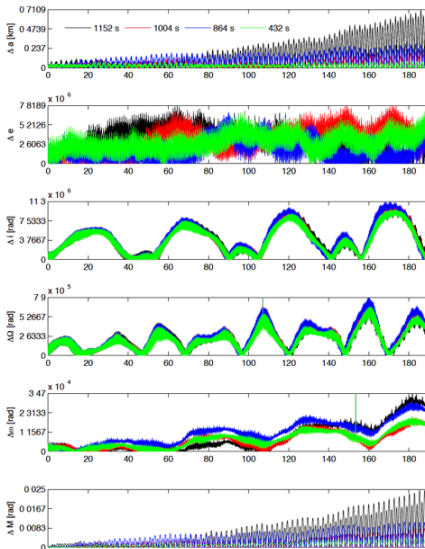


Symplectic integrator

Compared to the classical NIMASTEP integrator

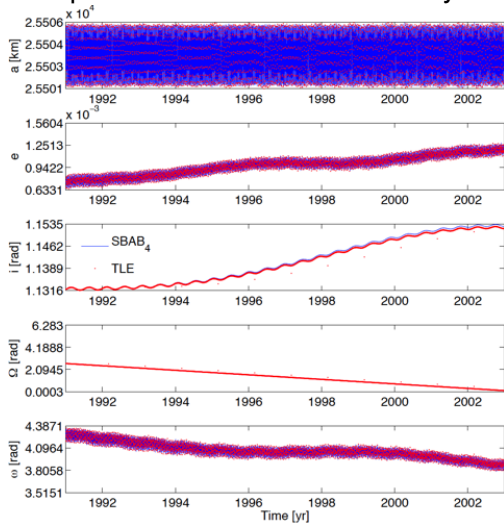
ABM 10 Adams - Bashforth - Moulton

Stepsize for $SABA_4$: 4 hours = 14400 sec



Symplectic integrator

Compared to a real motion for 10 years



Conclusions

- Symplectic integrators : powerful tool
- More complex to program
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Thanks for your welcome and attention !
Do not hesitate to contact me.