

Métodos analíticos para movimientos orbitales

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Orbital problem:

Main goal: Find $\boldsymbol{x} = \boldsymbol{x}(t)$ solution of the differential equation

$$\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t).$$

$\boldsymbol{x} = \boldsymbol{x}(t)$ is called **the orbit** or trajectory.

But, we can solve this problem **numerically, and efficiently!**

Why the need of **analytical methods?**

- Mathematicians need to be busy . . .

Why the need of analytical methods?

- A.M. give also high order
- A.M. are fastly evaluated at any point
(once the A.M. is obtained !)
- It is easy to play with initial conditions and parameters
- A.M. give not only numbers, but insight of the problem
- ...

Historical example: Orbital motion of the moon

Delaunay

20 years of mathematical work (without computers)

Two volumes (800 pages of formulas)

Precision: 300 km

Deprit, Henrard & Rom

2 years writing the software

Precision: 50 cm

2 mistakes in Delaunay work

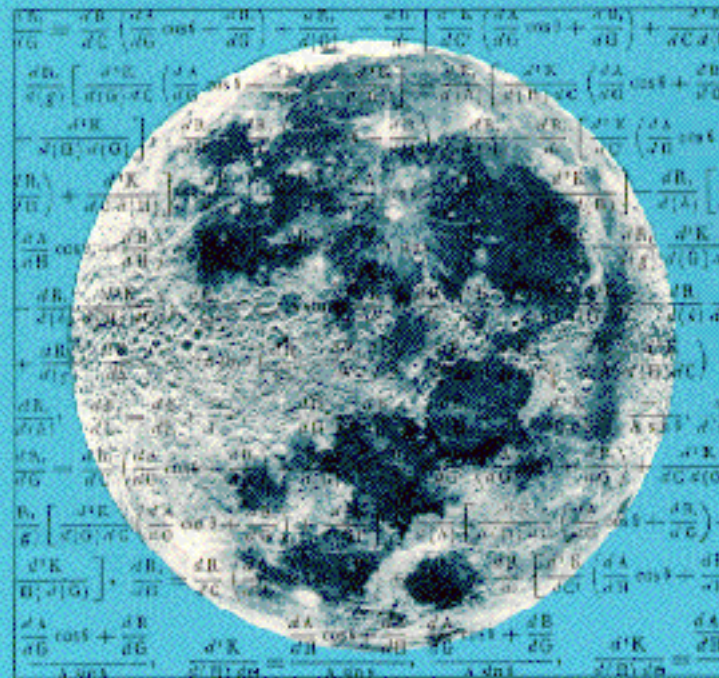
la fonction R ne contient plus aucun terme périodique; elle se trouve donc réduite à son terme non périodique seul, terme qui, en tenant compte des parties fournies par les opérations 129, 160, 349 et 415, a pour valeur

$$R = \frac{L}{2\pi}$$

$$\begin{aligned} & + n' \frac{n^2}{a^2} \left\{ \frac{1}{4} - \frac{1}{2} \gamma^2 + \frac{3}{8} e^2 + \frac{3}{8} e^2 + \frac{3}{2} \gamma^2 - \frac{9}{4} \gamma^2 e^2 - \frac{9}{4} \gamma^2 e^2 + \frac{9}{16} e^2 e^2 + \frac{15}{32} e^2 - \frac{33}{8} \gamma^2 e^2 \right. \\ & + \frac{9}{4} \gamma^2 e^2 + \frac{25}{16} \gamma^2 e^2 - \frac{27}{8} \gamma^2 e^2 e^2 - \frac{45}{16} \gamma^2 e^2 + \frac{45}{64} e^2 e^2 \\ & + \left(\frac{9}{16} \gamma^2 + \frac{225}{64} e^2 - \frac{27}{16} \gamma^2 - \frac{387}{32} \gamma^2 e^2 + \frac{98}{16} \gamma^2 e^2 - \frac{225}{128} e^2 + \frac{825}{64} e^2 e^2 + \frac{9}{8} \gamma^2 \right. \\ & \quad \left. + \frac{3877}{64} \gamma^2 e^2 - \frac{99}{16} \gamma^2 e^2 - \frac{1471}{256} \gamma^2 e^2 - \frac{1419}{32} \gamma^2 e^2 e^2 - \frac{225}{512} e^2 - \frac{825}{128} e^2 e^2 \right) \frac{n'}{a} \\ & - \left(\frac{31}{32} - \frac{33}{8} \gamma^2 - \frac{971}{32} e^2 + \frac{465}{64} e^2 + \frac{273}{64} \gamma^2 + \frac{5709}{64} \gamma^2 e^2 - \frac{117}{4} \gamma^2 e^2 + \frac{1989}{256} e^2 \right. \\ & \quad \left. - \frac{1925}{8} e^2 e^2 + \frac{2255}{128} e^2 \right) \frac{n^2}{a^2} \\ & - \left(\frac{255}{32} - \frac{31515}{1024} \gamma^2 - \frac{55115}{5120} e^2 + \frac{6085}{64} e^2 + \frac{20541}{512} \gamma^2 + \frac{987831}{2048} \gamma^2 e^2 \right. \\ & \quad \left. - \frac{218115}{512} \gamma^2 e^2 + \frac{1622985}{16384} e^2 - \frac{5069635}{2048} e^2 e^2 \right) \frac{n^3}{a^3} \\ & - \left(\frac{5515}{192} - \frac{299779}{3072} \gamma^2 - \frac{6380963}{12288} e^2 + \frac{16285}{24} e^2 \right) \frac{n^4}{a^4} \\ & - \left(\frac{28811}{288} - \frac{11381827}{294912} \gamma^2 - \frac{1681201051}{1179648} e^2 + \frac{1302609}{384} e^2 \right) \frac{n^5}{a^5} \\ & - \frac{2814775}{31684} \frac{n^6}{a^6} - \frac{129268199}{665552} \frac{n^7}{a^7} \\ & + \left[\frac{9}{64} - \frac{45}{16} \gamma^2 + \frac{45}{64} e^2 + \frac{15}{128} e^2 \right. \\ & \quad \left. + \left(\frac{225}{512} - \frac{1915}{256} \gamma^2 + \frac{7425}{1024} e^2 + \frac{225}{64} e^2 \right) \frac{n'}{a} + \frac{869}{512} \frac{n^2}{a^2} - \frac{10391}{8192} \frac{n^3}{a^3} \right] \frac{a^2}{n^2} \end{aligned}$$

FUTURE DIRECTIONS FOR RESEARCH IN SYMBOLIC COMPUTATION

Report of a Workshop on Symbolic and Algebraic Computation
Anthony C. Hearn, Workshop Chair



SIAM Reports on Issues in the Mathematical Sciences

Abril,
1988

Washington

Who is the force function $\mathbf{F}$?

In our case, $\mathbf{F}$ comes from a potential function $U = U(\mathbf{x}, t)$,

such that

$$\mathbf{F} = -\nabla_{\mathbf{x}} U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right).$$

Who is the potential function U ? **The geopotential**

Usually expanded into series, and in terms of the Spherical Polar variables (r, λ, β) .

$$U = -\frac{\mu}{r} \sum_{n \geq 0} \left(\frac{\alpha}{r}\right)^n \sum_{0 \leq m \leq n} (C_n^m \cos m\lambda + S_n^m \sin m\lambda) P_n^m(\cos \beta),$$

with

α the terrestrial radius.

μ the Gaussian constant.

P_n^m the associated Legendre polynomials.

C_n^m, S_n^m the harmonic coefficients.

Several representations in spherical harmonics:

$$U = -\frac{\mu}{r} \left[1 + \sum_{n \geq 1} \left(\frac{\alpha}{r} \right)^n \left\{ J_n P_n(\cos \beta) + \sum_{1 \leq m \leq n} (C_n^m \cos m\lambda + S_n^m \sin m\lambda) P_n^m(\cos \beta) \right\} \right]$$

If the origin of the reference frame is located at the center of masses,

$$U = -\frac{\mu}{r} \left[1 + \sum_{n \geq 2} \left(\frac{\alpha}{r} \right)^n \left\{ J_n P_n(\cos \beta) + \sum_{1 \leq m \leq n} (C_n^m \cos m\lambda + S_n^m \sin m\lambda) P_n^m(\cos \beta) \right\} \right]$$

J_n zonal harmonics

C_n^m, S_n^m tesseral harmonics

$\lambda = \lambda(t)$, then the independent variable t appears explicitly !

Solution:

Formulate the problem in a synodic frame rotating with the Earth.

Consequences:

- 1) t does not appear.
- 2) A new term appears in the Kinetic energy

$$-w \Omega$$

which originates difficulties in the tesseral case.

Assuming the Earth is of revolution, we only have

$$U = -\frac{\mu}{r} \left[1 + \sum_{n \geq 2} \left(\frac{\alpha}{r} \right)^n J_n P_n(\cos \beta) \right].$$

$J_2 \approx 10^{-3}$, $J_n < 10^{-6}$, $r > \alpha$. Thus,

$$U = U_0 + U_1 + U_2 + \dots, \quad \text{with} \quad U_0 \gg U_1 \gg U_2 \gg \dots$$

When $U = U_0 = -\frac{\mu}{r}$, **the Kepler Problem**

When $U = U_0 + U_1 = -\frac{\mu}{r} \left[1 + \left(\frac{\alpha}{r} \right)^2 J_2 P_2(\cos \beta) \right],$

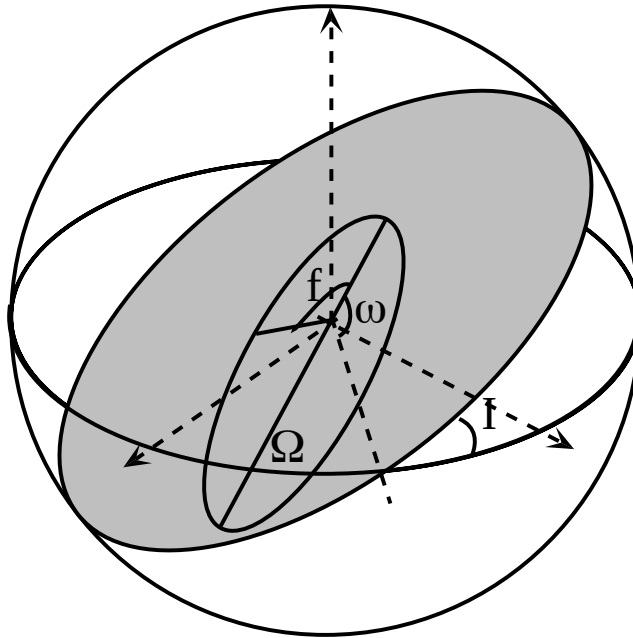
the Main Problem

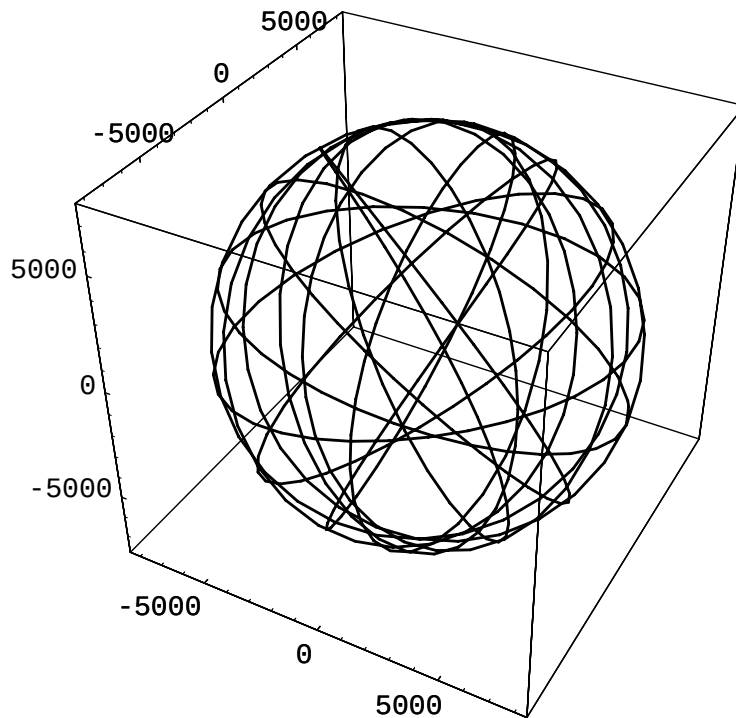
Perturbed Kepler Problem

Orbital elements for the Kepler motion — $a, e, I, \omega, \Omega, T$

Orbital elements for Perturbed Kepler motions

— $a(t), e(t), I(t), \omega(t), \Omega(t), T(t)$





Canonical transformations.

Example: The harmonic oscillator Cartesian variables

$$\mathcal{H} = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\omega^2(p_1^2 + p_2^2).$$

Equations of motion

$$\begin{aligned}\frac{dP_1}{dt} &= -\frac{\partial \mathcal{H}}{\partial p_1} = -\omega^2 p_1, & \frac{dp_1}{dt} &= \frac{\partial \mathcal{H}}{\partial P_1} = P_1, \\ \frac{dP_2}{dt} &= -\frac{\partial \mathcal{H}}{\partial p_2} = -\omega^2 p_2, & \frac{dp_2}{dt} &= \frac{\partial \mathcal{H}}{\partial P_2} = P_2.\end{aligned}$$

The harmonic oscillator Poincaré's variables

$$\begin{aligned}\varphi^\# : \mathbb{R}^2 \times [0, 2\pi) \times [0, 2\pi) &\longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \\ (I_1, I_2, \varphi_1, \varphi_2) &\longmapsto (P_1, P_2, p_1, p_2) \\ P_i = \sqrt{2\omega I_i} \cos \varphi_i, & \quad p_i = \sqrt{\frac{2I_i}{\omega}} \sin \varphi_i,\end{aligned}$$

is canonical, and gives

$$\mathcal{H} = \omega(I_1 + I_2)$$

Equations of motion:

$$\dot{I}_i = 0, \implies I_i = \text{constant}, \quad \dot{\varphi}_i = \omega, \implies \varphi_i = \omega t.$$

The harmonic oscillator Lissajous' variables (Deprit 1991)

$$\begin{aligned} \varphi^\# : \mathbb{R}^2 \times [0, 2\pi) \times [0, 2\pi) &\longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \\ (L, G, \ell, g) &\longmapsto (P_1, P_2, p_1, p_2) \end{aligned}$$

$$p_1 = \sqrt{\frac{L+G}{2\omega}} \cos(\ell+g) - \sqrt{\frac{L-G}{2\omega}} \cos(\ell-g),$$

$$p_2 = \sqrt{\frac{L+G}{2\omega}} \sin(\ell+g) + \sqrt{\frac{L-G}{2\omega}} \sin(\ell-g),$$

$$P_1 = \omega \frac{\partial x}{\partial \ell} = -\sqrt{\frac{\omega(L+G)}{2}} \sin(\ell+g) + \sqrt{\frac{\omega(L-G)}{2}} \sin(\ell-g),$$

$$P_2 = \omega \frac{\partial y}{\partial \ell} = \sqrt{\frac{\omega(L+G)}{2}} \cos(\ell+g) + \sqrt{\frac{\omega(L-G)}{2}} \cos(\ell-g).$$

is canonical

Lissajous transformation gives

$$\mathcal{H} = \omega L$$

Equations of motion:

$$\dot{L} = 0, \implies L = \text{constant}, \quad \dot{\ell} = \omega, \implies \ell = \omega t,$$

$$\dot{G} = 0, \implies G = \text{constant}, \quad \dot{g} = 0, \implies g = \text{constant}.$$

The Lie derivative of a function F is simply

$$\mathcal{L}_{\mathcal{H}} F = -\omega \frac{\partial F}{\partial \ell}$$

Canonical variables for the Kepler motion. — Cartesian

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{\mu}{r}$$

with $r = \sqrt{x^2 + y^2 + z^2}$.

Equations of motion

$$\dot{X} = -\mu \frac{x}{r^3}, \quad \dot{x} = X, \quad \dots$$

System of 6 equations with 6 unknowns. (3 degrees of freedom).

Not a big deal !

Canonical variables for the Kepler motion.

— Polar-nodal variables (Whittaker var. or Hill var.)

$$\begin{aligned}r, & \quad R = \dot{r} \\ \vartheta = \omega + f, & \quad \Theta = \|\mathbf{x} \times \mathbf{X}\| \\ \nu = \Omega, & \quad N = \|\mathbf{x} \times \mathbf{X}\| \cos I.\end{aligned}$$

$$\mathcal{H} = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r}.$$

$$\mathcal{H} = \mathcal{H}(R, \Theta, -, r, -, -)$$

Canonical variables for the Kepler motion.

— Delaunay variables

$$\ell = M, \quad L = \mu a$$

$$g = \omega, \quad G = \|\mathbf{x} \times \mathbf{X}\|$$

$$h = \Omega, \quad H = \|\mathbf{x} \times \mathbf{X}\| \cos I.$$

$$\mathcal{H} = -\frac{\mu}{2L^2}.$$

$$\mathcal{H} = \mathcal{H}(L, -, -, -, -, -)$$

Infinitesimal contact transformations (Sophus Lie, 1880)

Let us consider the function $\mathcal{W}(\mathbf{q}, \mathbf{Q})$, and ε a parameter.

An i.c.t. $\chi : (\mathbf{q}, \mathbf{Q}, \varepsilon) \longmapsto (\mathbf{p}, \mathbf{P})$ is defined by

$$\mathbf{p} = \mathbf{q} + \varepsilon \nabla_{\mathbf{Q}} \mathcal{W} + \mathcal{O}(\varepsilon^2) = \mathbf{q} + \varepsilon(\mathbf{q}; \mathcal{W}) + \mathcal{O}(\varepsilon^2),$$

$$\mathbf{P} = \mathbf{Q} - \varepsilon \nabla_{\mathbf{q}} \mathcal{W} + \mathcal{O}(\varepsilon^2) = \mathbf{Q} + \varepsilon(\mathbf{Q}; \mathcal{W}) + \mathcal{O}(\varepsilon^2).$$

$\mathcal{W}(\mathbf{q}, \mathbf{Q})$ is called the generator.

Note that $\mathbf{p}(\mathbf{q}, \mathbf{Q}, \varepsilon = 0) = \mathbf{q}$ and $\mathbf{P}(\mathbf{q}, \mathbf{Q}, \varepsilon = 0) = \mathbf{Q}$.

The i.c.t. is a solution of the Hamiltonian system

$$\frac{d\mathbf{p}}{d\varepsilon} = \nabla_{\mathbf{P}} \mathcal{W}, \quad \frac{d\mathbf{P}}{d\varepsilon} = -\nabla_{\mathbf{p}} \mathcal{W},$$

Infinitesimal contact transformations (Sophus Lie, 1880)

Let us consider a function $F(\mathbf{p}, \mathbf{P})$. Then, by Taylor,

$$\chi^\# F(\mathbf{p}(\mathbf{q}, \mathbf{Q}, \varepsilon), \mathbf{P}(\mathbf{q}, \mathbf{Q}, \varepsilon)) = F|_{\varepsilon=0} + \varepsilon \frac{dF}{d\varepsilon}|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2).$$

but

$$\frac{dF}{d\varepsilon} = \frac{\partial F}{\partial \varepsilon} + (F; \mathcal{W}),$$

hence,

$$\chi^\# F = F(\mathbf{q}, \mathbf{Q}) + \varepsilon (F(\mathbf{q}, \mathbf{Q}); \mathcal{W}) + \mathcal{O}(\varepsilon^2).$$

Infinitesimal contact transformations

Example: Perturbed oscillators $\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1$

$$\mathcal{H}_0 = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\omega^2(p_1^2 + p_2^2),$$

$$\mathcal{H}_1 = 2\alpha p_1^2 p_2^2 + \beta(p_1^4 + p_2^4).$$

In Lissajous variables

$$\mathcal{H}_0 = \omega L$$

$$\mathcal{H}_1 = \begin{cases} \frac{1}{4}(\alpha + 3\beta)(d^4 + 4d^2 s^2 + s^4) - \frac{3}{2}(\alpha - \beta)d^2 s^2 \cos 4g \\ - (\alpha + 3\beta) \left[sd(d^2 + s^2) \cos 2\ell + \frac{1}{2}d^2 s^2 \cos 4\ell \right] \\ + (\alpha - \beta) \left[sd(s^2 \cos(4g + 2\ell) + d^2 \cos(4g - 2\ell)) \right. \\ \left. - \frac{1}{4}s^4 \cos(4g + 4\ell) - \frac{1}{4}d^4 \cos(4g - 4\ell) \right] \end{cases}$$

Infinitesimal contact transformations

$$\mathcal{H}' = \langle \mathcal{H} \rangle_\ell$$

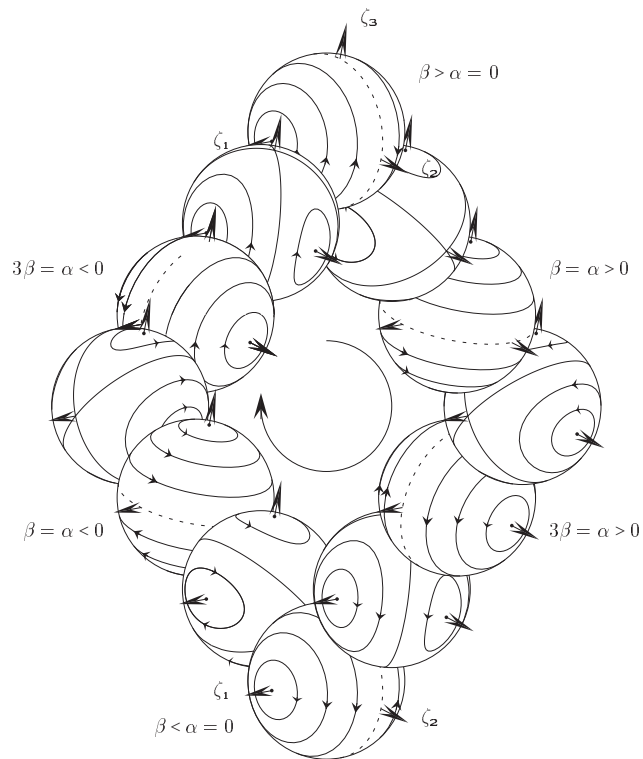
$$= \omega L'$$

$$+ \epsilon \left[\frac{1}{4}(\alpha + 3\beta)(d'^4 + 4d'^2 s'^2 + s'^4) - \frac{3}{2}(\alpha - \beta)s'^2 d'^2 \cos 4g' \right]$$

NORMALIZATION

$$\left. \begin{aligned} (\mathcal{H}_0; \mathcal{W}) &= \mathcal{H}'_1 - \mathcal{H}_1, \\ (\mathcal{H}_0; \mathcal{W}) &= -\omega \frac{\partial \mathcal{W}}{\partial \ell}, \end{aligned} \right\} \Rightarrow \mathcal{W} = -\omega \int (\mathcal{H}'_1 - \mathcal{H}_1) d\ell$$

Normalized Phase flow



Lie transformations (André DEPRIT, 1969)

A Lie transform is an extension of an Infinit. Contact Transform.

$\chi : (\mathbf{q}, \mathbf{Q}, \varepsilon) \longmapsto (\mathbf{p}, \mathbf{P})$, solution of

$$\frac{d\mathbf{p}}{d\varepsilon} = \nabla_{\mathbf{P}}\mathcal{W}, \quad \frac{d\mathbf{P}}{d\varepsilon} = -\nabla_{\mathbf{p}}\mathcal{W},$$

with the initial conditions

$$\mathbf{p}(\mathbf{q}, \mathbf{Q}, \varepsilon = 0) = \mathbf{q}, \quad \mathbf{P}(\mathbf{q}, \mathbf{Q}, \varepsilon = 0) = \mathbf{Q},$$

The generator $\mathcal{W}(\mathbf{p}, \mathbf{P}, \varepsilon)$ being

$$\mathcal{W} \equiv \mathcal{W}(\mathbf{p}, \mathbf{P}, \varepsilon) = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} \mathcal{W}_{n+1}(\mathbf{p}, \mathbf{P}).$$

Lie transformations

Given a function

$$F(\mathbf{p}, \mathbf{P}, \varepsilon) = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} F_{n,0}(\mathbf{p}, \mathbf{P})$$

What is the action of the Lie transformation onto it?

$\chi^\# F(\mathbf{q}, \mathbf{Q}, \varepsilon)$ is a Taylor series

$$\chi^\# F = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} \left. \frac{d^n F}{d\varepsilon^n} \right|_{\varepsilon=0} = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} F_{0,n}(\mathbf{q}, \mathbf{Q}).$$

Lie transformations

Problem.— The functions $F_{n,0}$ and $\mathcal{W}_n$ given, determine $F_{0,n}$

We have to compute the derivatives

$$\frac{d^n F}{d\varepsilon^n}$$

Liouville theorem:

$$\frac{dF}{d\varepsilon} = \frac{\partial F}{\partial \varepsilon} + (F; \mathcal{W})$$

where $\frac{\partial F}{\partial \epsilon} = \sum_{n \geq 0} \frac{\epsilon^n}{n!} F_{n+1,0}$ and

$$\begin{aligned}
 (F; \mathcal{W}) &= \sum_{j \geq 0} \frac{\epsilon^j}{j!} (F_{j,0}; \mathcal{W}) = \sum_{j \geq 0} \frac{\epsilon^j}{j!} \sum_{k \geq 0} \frac{\epsilon^k}{k!} (F_{j,0}; \mathcal{W}_{k+1}) \\
 &= \sum_{j \geq 0} \sum_{k \geq 0} \frac{\epsilon^{j+k}}{j! k!} (F_{j,0}; \mathcal{W}_{k+1}) \\
 &= \sum_{n \geq 0} \frac{\epsilon^n}{n!} \sum_{0 \leq m \leq n} \binom{n}{m} (F_{n-m,0}; \mathcal{W}_{m+1}).
 \end{aligned}$$

consequently,

$$\frac{dF}{d\epsilon} = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \left[F_{n+1,0} + \sum_{0 \leq m \leq n} \binom{n}{m} (F_{n-m,0}; \mathcal{W}_{m+1}) \right],$$

or

$$\frac{dF}{d\epsilon} = \sum_{n \geq 0} \frac{\epsilon^n}{n!} F_{n,1}.$$

In general,

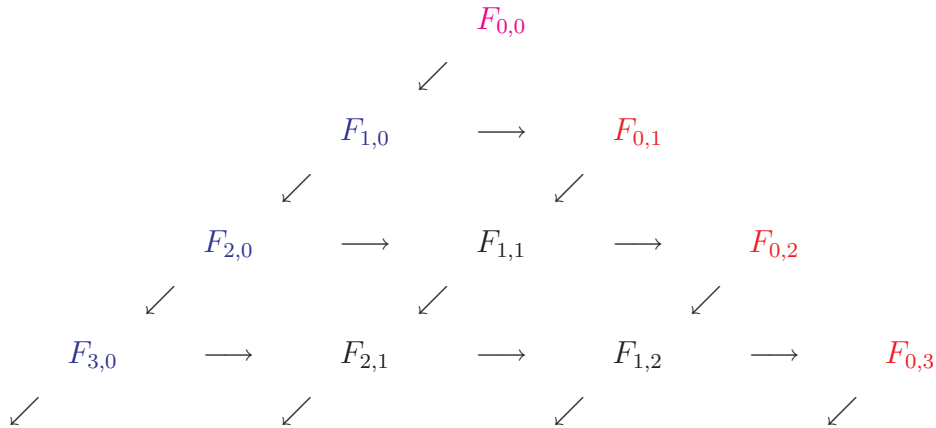
$$\frac{d^k F}{d\epsilon^k} = \sum_{n \geq 0} \frac{\epsilon^n}{n!} F_{n,k},$$

$$F_{i,j} = F_{i+1,j-1} + \sum_{0 \leq m \leq i} \binom{i}{m} (F_{i-m,j-1}; \mathcal{W}_{m+1})$$

For example,

$$F_{2,1} = F_{3,0} + (F_{2,0}; \mathcal{W}_1) + 2(F_{1,0}; \mathcal{W}_2) + (F_{0,0}; \mathcal{W}_3).$$

With this, we may follow the **Lie–Triangle** :



For the computing of each term, we need the previous ones in the row, and the elements above the diagonal of this one.

Homologic equation

$$\mathcal{L}_0 \mathcal{W}_n = (F_{0,0}; \mathcal{W}_n) = F_{0,n} - \tilde{F}_{n,0}$$

- $F_{0,n}$ our choice
- $\tilde{F}_{0,n}$ computed
- $\mathcal{W}_n$ computed by integration

Lie— transformations Example

$$\mathcal{H} = \frac{1}{2} (X^2 + \omega^2 x^2) + \frac{1}{2} (Y^2 + \omega^2 y^2) + \epsilon \omega^2 (\alpha x^3 + \beta xy^2).$$

In the Lissajous coordinates, the Hamiltonian takes the form:

$$\mathcal{H} = \omega L$$

$$\begin{aligned} & + \epsilon \omega^2 \{ s d^2 \left[-\frac{3}{4}\beta + \frac{3}{4}\alpha \right] \cos(\ell - 3g) - [s^2 d(\frac{1}{2}\beta + \frac{3}{2}\alpha) + d^3(\frac{1}{4}\beta + \frac{3}{4}\alpha)] \cos(\ell - g) \\ & + [s^3(\frac{1}{4}\beta + \frac{3}{4}\alpha) + s d^2(\frac{1}{2}\beta + \frac{3}{2}\alpha)] \cos(\ell + g) + s^2 d[\frac{3}{4}\beta - \frac{3}{4}\alpha] \cos(\ell + 3g) \\ & + d^3[\frac{1}{4}\beta - \frac{1}{4}\alpha] \cos(3\ell - 3g) + s d^2[\frac{1}{4}\beta + \frac{3}{4}\alpha] \cos(3\ell - g) \\ & + s^2 d[-\frac{1}{4}\beta - \frac{3}{4}\alpha] \cos(3\ell + g) + s^3[-\frac{1}{4}\beta + \frac{1}{4}\alpha] \cos(3\ell + 3g) \}. \end{aligned}$$

Lie— transformations First order Normalization

$$\mathcal{H}_{0,1} = \langle \mathcal{H} \rangle_\ell = 0.$$

$$\left. \begin{aligned} (\mathcal{H}_{0,0}; \mathcal{W}_1) &= \mathcal{H}_{0,1} - \mathcal{H}_{1,0}, \\ (\mathcal{H}_{0,0}; \mathcal{W}_1) &= -\omega \frac{\partial \mathcal{W}_1}{\partial \ell}, \end{aligned} \right\} \Rightarrow \mathcal{W}_1 = \omega \int \mathcal{H}_{1,0} d\ell$$

Second order ?

Lie— transformations Second order Normalization

From the Lie triangle,

$$\begin{aligned}\mathcal{H}_{0,2} &= \mathcal{H}_{2,0} + (\mathcal{H}_{1,0}, \mathcal{W}_1) + (\mathcal{H}_{0,1}, \mathcal{W}_1) + (\mathcal{H}_{0,0}, \mathcal{W}_2) \\ &= \widetilde{\mathcal{H}}_{2,0} + (\mathcal{H}_{0,0}, \mathcal{W}_2).\end{aligned}$$

(Homologic equation)

We select $\mathcal{H}_{0,2} = \langle \widetilde{\mathcal{H}}_{2,0} \rangle_\ell$

Then,

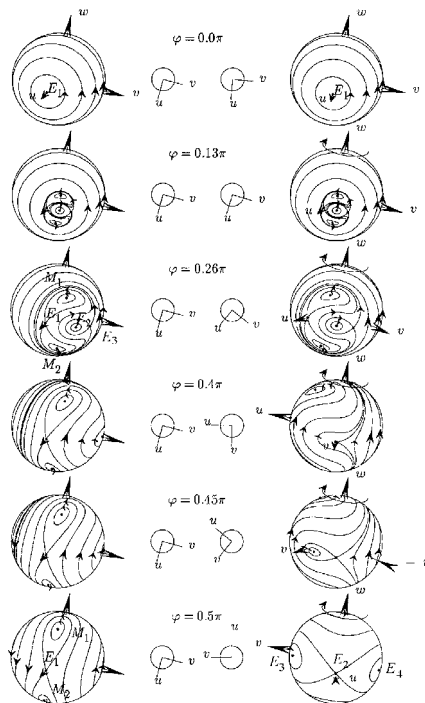
$$\mathcal{W}_2 = -\omega \int (\mathcal{H}_{0,2} - \widetilde{\mathcal{H}}_{2,0}) d\ell$$

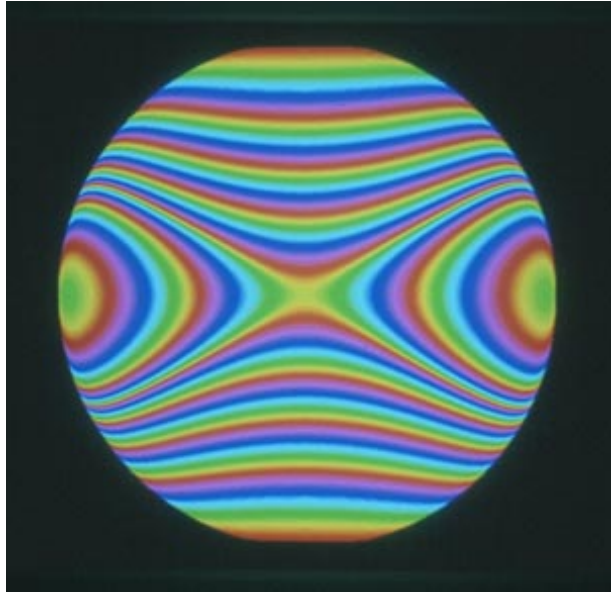
Lie— transformations Fourth order Normalization

$$\mathcal{H}' = \omega L$$

$$\begin{aligned}
 & +\epsilon^2 L^2 \left\{ e^2 \left(-\frac{11}{32} \beta^2 + \frac{9}{16} \alpha \beta - \frac{15}{32} \alpha^2 \right) - \frac{1}{48} \beta^2 - \frac{7}{8} \alpha \beta - \frac{15}{16} \alpha^2 \right. \\
 & \quad \left. + e \left(-\frac{5}{24} \beta^2 + \frac{15}{8} \alpha^2 \right) \cos 2g + e^2 \left(\frac{5}{32} \beta^2 + \frac{5}{16} \alpha \beta - \frac{15}{32} \alpha^2 \right) \cos 4g \right\} \\
 & +\epsilon^4 \frac{L^3}{\omega} \left\{ e^2 \left(-\frac{111}{256} \beta^4 - \frac{659}{384} \alpha \beta^3 + \frac{175}{128} \alpha^2 \beta^2 + \frac{483}{128} \alpha^3 \beta - \frac{2115}{256} \alpha^4 \right) \right. \\
 & \quad - \frac{79}{3456} \beta^4 + \frac{107}{576} \alpha \beta^3 - \frac{163}{64} \alpha^2 \beta^2 - \frac{401}{64} \alpha^3 \beta - \frac{705}{128} \alpha^4 \\
 & \quad + \left(e^3 \left(-\frac{1015}{4608} \beta^4 + \frac{1285}{2304} \alpha \beta^3 + \frac{195}{128} \alpha^2 \beta^2 - \frac{1285}{256} \alpha^3 \beta + \frac{2115}{512} \alpha^4 \right) \right. \\
 & \quad \left. + e \left(-\frac{61}{3456} \beta^4 - \frac{401}{576} \alpha \beta^3 - \frac{161}{96} \alpha^2 \beta^2 + \frac{401}{64} \alpha^3 \beta + \frac{2115}{128} \alpha^4 \right) \right) \cos 2g \\
 & \quad + e^2 \left(\frac{97}{256} \beta^4 + \frac{1091}{1152} \alpha \beta^3 + \frac{151}{128} \alpha^2 \beta^2 + \frac{319}{128} \alpha^3 \beta - \frac{2115}{256} \alpha^4 \right) \cos 4g \\
 & \quad \left. + e^3 \left(\frac{739}{4608} \beta^4 - \frac{1025}{2304} \alpha \beta^3 + \frac{59}{384} \alpha^2 \beta^2 - \frac{319}{256} \alpha^3 \beta + \frac{705}{512} \alpha^4 \right) \cos 6g \right\} \\
 & + \mathcal{O}(\epsilon^5).
 \end{aligned}$$

Normalized Phase flow





Algebraic and symbolic tools

General purpose system

Mathematica, Maple, etc.

Specialized systems (non-lineal dynamics)

Poisson series processor

$$\mathcal{S}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{i} \in \mathcal{I}, \mathbf{j} \in \mathcal{J}} C_{\mathbf{i}}^{\mathbf{j}} x_0^{i_0} \dots x_{n-1}^{i_{n-1}} \begin{pmatrix} \text{sen} \\ \text{cos} \end{pmatrix} (j_0 y_0 + \dots + j_{m-1} y_{m-1}),$$
$$C_{\mathbf{i}}^{\mathbf{j}} \in \mathbb{R}$$

Specialized systems:

software based on the properties of particular **mathematical objects**

- algebraic structure
- symbolic representation
- computational representation
- algorithms

Main problem of the artificial satellite

$$\mathcal{H} = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + J_2 \frac{\mu}{r} \left(\frac{\alpha}{r} \right)^2 P_2(\sin \theta \sin I)$$

Taking

$$\epsilon = J_2 \approx 10^{-3}$$

the main problem may be formulated as an asymptotic expansion

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1$$

In polar–nodal variables $(r, \theta, \nu, R, \Theta, N)$

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1 + \sum_{j>1}^m \frac{\epsilon^j}{j!} \mathcal{H}_j$$

- $\mathcal{H}_0 = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r}$
- $\mathcal{H}_1 = \mathcal{M}_0 \frac{1}{r^3} + \mathcal{M}_1 \frac{1}{r^3} \cos 2\theta$
- $\mathcal{H}_j = 0$

where $\mathcal{M}_i = \mathcal{M}_i(s) = \mathcal{M}_i(\Theta, N)$

Lie operator:
$$\mathcal{L}_0 = R \frac{\partial}{\partial r} - \left(\frac{\mu}{r^2} - \frac{\Theta}{r^3} \right) \frac{\partial}{\partial R} + \frac{\theta}{r^2} \frac{\partial}{\partial \theta}$$

In Delaunay variables (ℓ, g, h, L, G, H)

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1 + \sum_{j>1}^m \frac{\epsilon^j}{j!} \mathcal{H}_j$$

- $\mathcal{H}_0 = -\frac{\mu^2}{2L^2}$
- $\mathcal{H}_1 = \mathcal{M}_0 \frac{a^3}{r^3} + \mathcal{M}_1 \frac{a^3}{r^3} \cos(2g + 2f)$
- $\mathcal{H}_j = 0$

where $\mathcal{M}_i = \mathcal{M}_i(a^3, s) = \mathcal{M}_i(L, G, H)$

Lie operator: $\mathcal{L}_0 = n \frac{\partial}{\partial \ell}$

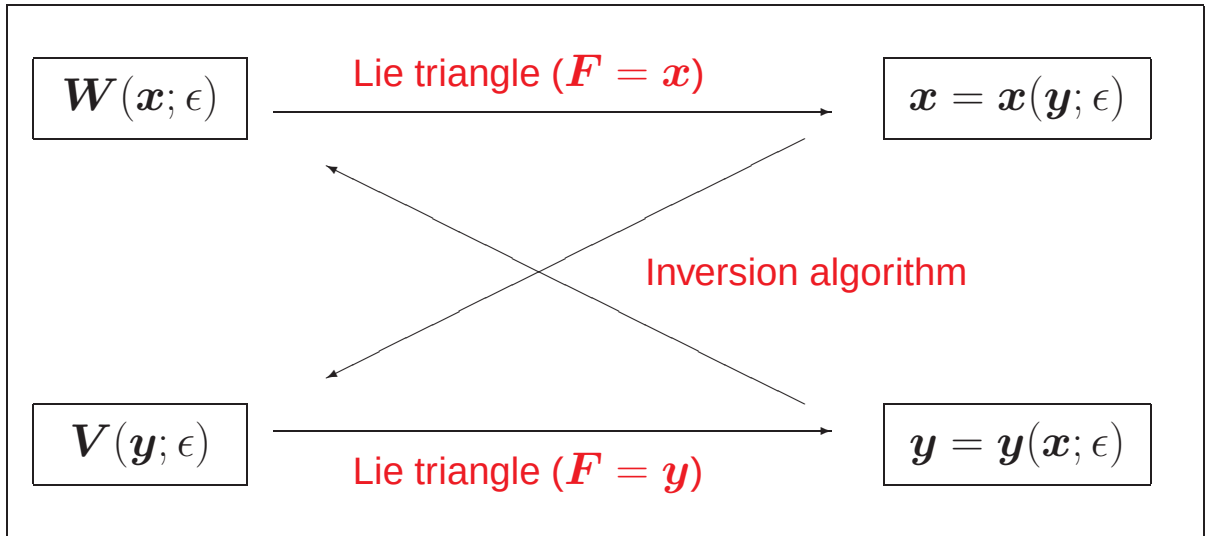
r, f, E in Delaunay variables (ℓ, g, h, L, G, H)

$$\frac{a^3}{r^3}, \quad \frac{a^3}{r^3} \cos(2g + 2f)$$

$$\frac{a}{r} = \frac{1 + e \cos f}{\sqrt{1 - e^2}}, \quad \frac{r}{a} = 1 - e \cos E$$

expansions in powers of the eccentricity

- $E = E(e, \ell)$
- $f = f(e, \ell)$



Example: Delaunay normalization

- Original Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \epsilon \mathcal{H}_1 + \sum_{j \geq 1}^m \frac{\epsilon^j}{j!} \mathcal{H}_j = \sum_{j \geq 0}^m \frac{\epsilon^j}{j!} \mathcal{H}_{j,0}$$

- Canonical transformation of generator

$$\sum_{j \geq 0}^m \frac{\epsilon^j}{j!} \mathcal{W}_{j+1}$$

- Transformed Hamiltonian

$$\mathcal{H} = \sum_{j \geq 0}^m \frac{\epsilon^j}{j!} \mathcal{H}_{0,j}$$

Homologic equation

$$\mathcal{L}_0 \mathcal{W}_n = n \frac{\partial \mathcal{W}_n}{\partial \ell} = \mathcal{H}_{0,n} - \tilde{\mathcal{H}}_{n,0}$$

- $\tilde{\mathcal{H}}_{0,n}$ computed with a Poisson series processor
- $\mathcal{H}_{0,n} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mathcal{H}}_{0,n} d\ell$
- $\mathcal{W}_n = \int (\tilde{\mathcal{H}}_{0,n} - \tilde{\mathcal{H}}_{0,n}) d\ell$

$$\int F(\ell) d\ell, \quad \int F(r, f, E) d\ell$$

- $\frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} d\ell = \frac{L^3}{G^3}$
- $\int \left(\frac{a^3}{r^3} - \frac{L^3}{G^3} \right) d\ell = \frac{L^3}{G^3} (f - \ell + e \sin f)$

Integrals with the equation of the center $\phi = (f - \ell)$ in the integrand introduce special functions like the dilogarithm

$$\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

First order solution of the main problem

Generator

$$\mathcal{W} = \frac{\mu}{G^3} \left(\mathcal{M}_0 \phi + \mathcal{M}_0 e \sin f + \frac{\mathcal{M}_1}{2} e \sin(f + 2g) \right. \\ \left. \frac{\mathcal{M}_1}{2} \sin(2f + 2g) + \frac{\mathcal{M}_1}{6} e \sin(3f + 2g) \right)$$

New Hamiltonian

$$\mathcal{H} = -\frac{\mu}{2L^2} + \epsilon \mathcal{M}_0 \frac{L^3}{G^3}$$

Integrable

$$\mathcal{H}(-, -, -, L, G, H)$$

Greater order (qualitative results): $\mathcal{H}(-, g, -, L, G, H)$

Averaged Hamiltonian: L, H constants, with $0 \leq |H| \leq L$

$$\mathcal{H} = \mathcal{H}(g, G) = \mathcal{H}(\xi_1, \xi_2, \xi_3)$$

$$\xi_1 = LG \operatorname{se} \cos g$$

$$\xi_2 = LG \operatorname{se} \sin g$$

$$\xi_3 = G^2 - \frac{L^2 + H^2}{2}$$

Phase space

- plane: (g, G)
- sphere: $\xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{(L^2 - H^2)^2}{4}$





Normalization

Lie derivative ($\mathcal{L}_0$) is a semi-simple operator:

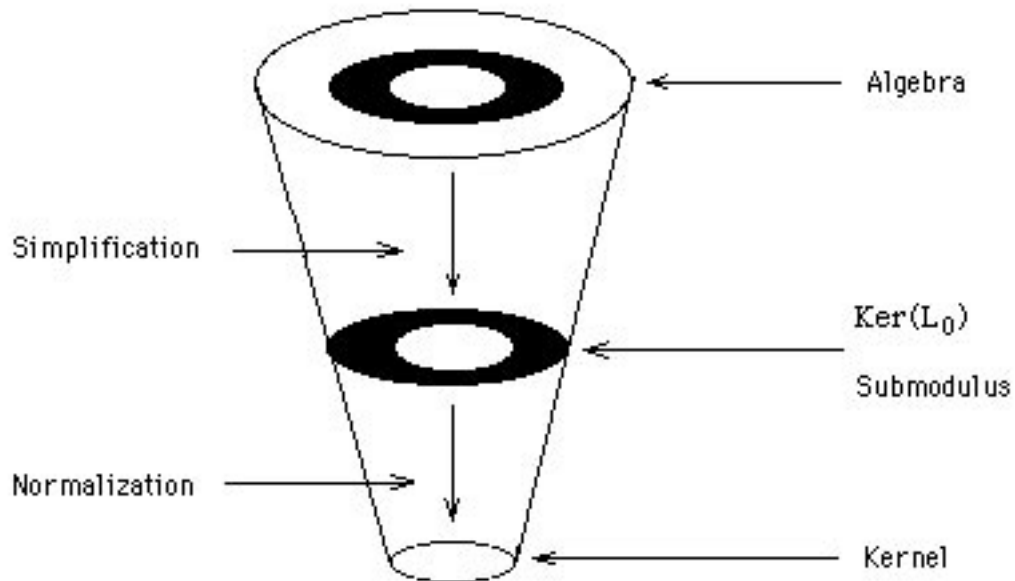
$$\mathcal{P} = \ker \mathcal{L}_0 \oplus \operatorname{im} \mathcal{L}_0$$

Obtention of $\mathcal{K}_n(\mathcal{H}_{0,n})$, $\mathcal{W}_n$:

$$(\mathcal{H}_0; \mathcal{W}_n) + \mathcal{A}_n = \mathcal{K}_n, \Leftrightarrow \mathcal{L}_0 \mathcal{W}_n = \mathcal{A}_n - \mathcal{K}_n$$

$$\left. \begin{array}{l} \mathcal{K}_n = \mathcal{A}_n^k \in \ker \mathcal{L}_0 \\ \mathcal{L}_0 \mathcal{W}_n = \mathcal{A}_n^i \in \operatorname{im} \mathcal{L}_0 \end{array} \right\} \mathcal{A}_n^k + \mathcal{A}_n^i = \mathcal{A}_n$$

Simplification



Hamiltonian in polar–nodal variables

Homological equation

$$R \frac{\partial \mathcal{W}_n}{\partial r} - \left(\frac{\mu}{r^2} - \frac{\Theta}{r^3} \right) \frac{\partial \mathcal{W}_n}{\partial R} + \frac{\Theta}{r^2} \frac{\partial \mathcal{W}_n}{\partial \theta} = \tilde{\mathcal{H}}_{n,0} - \mathcal{H}_{0,n}$$

Elimination of the parallax

Goal: reduce the factors

$$\frac{\mu}{r} \left(\frac{p}{r} \right)^{2n} \quad \text{to} \quad \frac{\mu}{p} \left(\frac{p}{r} \right)^2,$$

while eliminating the explicit appearance of θ

Poisson algebra (including the Hamiltonian of the satellite)

$$\mathcal{F} = \left\{ F = \sum_{j \geq 0} (C_j \cos j \theta + S_j \sin j \theta), \quad C_j, S_j \in \ker(\mathcal{L}_0) \right\}$$

$$\frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + \epsilon \left(\mathcal{M}_0 \frac{1}{r^3} + \mathcal{M}_1 \frac{1}{r^3} \cos 2\theta \right) \in \mathcal{F}$$

$1/r$ and R belong to $\mathcal{F}$

$$\frac{1}{r} = \frac{1}{p} + \frac{C}{p} \cos \theta + \frac{S}{p} \sin \theta$$

$$R = \frac{C\Theta}{p} \sin \theta - \frac{S\Theta}{p} \cos \theta$$

where C and S are the state functions

$$C = e \cos g, \quad S = e \sin g, \quad C, S \in \ker(\mathcal{L}_0)$$

Elimination of the parallax

$$\mathcal{L}_0 \left[\sum_{j \geq 0} \frac{1}{j} (C_j \sin j\theta - S_j \cos j\theta) \right] = \frac{\Theta}{r^2} \sum_{j \geq 0} (C_j \cos j\theta + S_j \sin j\theta)$$

Elimination of the parallax

PROPOSITION: Given a function

$$F(r, \theta, R, \Theta) = \sum_{j \geq 0} (C_j \cos j\theta + S_j \sin j\theta), \quad C_j, S_j \in \ker(L_0),$$

the PDE in $\mathcal{W}$, F' :

$$L_0(\mathcal{W}) + F' = \frac{\Theta}{r^2} F$$

is satisfied by choosing

$$\begin{aligned} F' &= \frac{\Theta}{r^2} C_0 \\ \mathcal{W} &= \sum_{j \geq 1} \frac{1}{j} (C_j \sin j\theta - S_j \cos j\theta) \end{aligned}$$

N.B. $\mathcal{W}$ is not unique.

$$\mathcal{L}_0(\mathcal{W}_n) = \frac{\Theta}{r^2} C_0 + \frac{\Theta}{r^2} F$$

$$\begin{aligned} \bullet \quad F &= \frac{r^2}{\Theta} \tilde{\mathcal{H}}_{n,0} \\ &= \sum_{j \geq 0} (C_j \cos j\theta + S_j \sin j\theta) \end{aligned}$$

$$\bullet \quad \mathcal{H}_{0,n} = \frac{\Theta}{r^2} C_0$$

$$\bullet \quad \mathcal{W}_n = \sum_{j \geq 0} \frac{1}{j} (C_j \sin j\theta - S_j \cos j\theta)$$

Hamiltonian after elimination of the parallax

$$\begin{aligned}
 \mathcal{H} = & \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + \\
 & \epsilon \frac{\Theta^2}{r^2} \left(\frac{\alpha}{p} \right)^2 \left(\frac{1}{2} - \frac{3}{4} \sin^2 i \right) + \\
 & \frac{\epsilon^2}{2!} \frac{\Theta^2}{r^2} \left(\frac{\alpha}{p} \right)^4 \left[-\frac{5}{4} - \frac{3}{8} C^2 - \frac{3}{8} S^2 + \left(\frac{21}{8} + \right. \right. \\
 & \left. \left. \frac{27}{16} C^2 - \frac{15}{16} S^2 \right) \sin^2 i + \left(-\frac{21}{16} - \frac{75}{64} C^2 + \frac{105}{64} S^2 \right) \sin^4 i \right]
 \end{aligned}$$

First order does not depend on g .

Second order does $(C(g), S(g))$

Elimination of the perigee (g) $(\vartheta = g + f)$

$$\mathcal{W}_n = \mathcal{W}_n^*(C, S, \Theta, \theta) + \tilde{\mathcal{W}}_n(C, S, \Theta, -)$$

★ with $\mathcal{W}_n^*$ we eliminate ϑ

★ with $\tilde{\mathcal{W}}_n$ we eliminate at order $n + 1$ terms containing only g

Elimination of the perigee (g)

Determining $\mathcal{W}_n^*$? As usual.

$$\mathcal{L}_0 \mathcal{W}_n = R \frac{\partial \mathcal{W}_n}{\partial r} - \left(\frac{\mu}{r^2} - \frac{\Theta}{r^3} \right) \frac{\partial \mathcal{W}_n}{\partial R} + \frac{\Theta}{r^2} \frac{\partial \mathcal{W}_n}{\partial \theta} = \frac{\Theta}{r^2} \frac{\partial \mathcal{W}_n^*}{\partial \theta}$$

$$\bullet \quad \frac{\Theta}{r^2} \frac{\partial \mathcal{W}_n^*}{\partial \theta} = \tilde{\mathcal{H}}_{n,0} - \mathcal{H}_{0,n} \implies \mathcal{W}_n^* = \int \frac{r^2}{\Theta} (\tilde{\mathcal{H}}_{n,0} - \mathcal{H}_{0,n}) d\theta$$

- $\tilde{\mathcal{W}}_n$? It is not determined at order n , but rather *is deferred until the next order* ($n+1$)

First order (elimination of the perigee)

$$\tilde{\mathcal{H}}_{1,0} = \mathcal{H}_{1,0} = \frac{\mu r_{\oplus}^2}{4\Theta^2 r^2} (-2 + 3s_i^2)$$

does not depend on g

- $\mathcal{H}_{0,1} = \tilde{\mathcal{H}}_{1,0}$
- $\mathcal{W}_1^* = \int \frac{r^2}{\Theta} (\tilde{\mathcal{H}}_{1,0} - \mathcal{H}_{0,1}) d\theta = 0$

Greater order (elimination of the perigee)

Homological equation: $\mathcal{L}_0 \mathcal{W}_n^* = \tilde{\mathcal{H}}_{n,0} + \mathcal{H}_{0,n} + 2(\mathcal{H}_{1,0}; \tilde{\mathcal{W}}_{n-1})$

- $\tilde{\mathcal{H}}_{n,0} = \tilde{\mathcal{H}}_{n,0}^\theta + \tilde{\mathcal{H}}_{n,0}^*$
- $\mathcal{H}_{0,n} = \langle \tilde{\mathcal{H}}_{n,0}^* \rangle_g = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mathcal{H}}_{n,0}^*(C, S) dg$
- $2(\mathcal{H}_{1,0}; \tilde{\mathcal{W}}_{n-1}) = \mathcal{F}_1(\theta) - \frac{3\mu r_\oplus^2}{2\Theta^3 r^2} (4 - 5s_i^2) \frac{\partial \tilde{\mathcal{W}}_{n-1}}{\partial g}$

$$\mathcal{L}_0 \mathcal{W}_n^* = \tilde{\mathcal{H}}_{n,0}^\theta + \mathcal{F}_1(\theta) + \mathcal{F}_2(g) - \frac{3\mu r_\oplus^2}{2\Theta^3 r^2} (4 - 5s_i^2) \frac{\partial \tilde{\mathcal{W}}_{n-1}}{\partial g}$$

$$\mathcal{L}_0 \mathcal{W}_n^* = \tilde{\mathcal{H}}_{n,0}^\theta + \mathcal{F}_1(\theta) + \mathcal{F}_2(g) - \frac{3\mu r_\oplus^2}{2\Theta^3 r^2} (4 - 5s_i^2) \frac{\partial \tilde{\mathcal{W}}_{n-1}}{\partial g}$$

- $$\mathcal{F}_2(g) - \frac{3\mu r_\oplus^2}{2\Theta^3 r^2} (4 - 5s_i^2) \frac{\partial \tilde{\mathcal{W}}_{n-1}}{\partial g} = 0$$

$$\tilde{\mathcal{W}}_{n-1} = \frac{2\Theta^3 r^2}{3\mu r_\oplus^2} \frac{1}{(4 - 5s_i^2)} \int \mathcal{F}_2(g) dg$$

- $$\mathcal{L}_0 \mathcal{W}_n^* = \tilde{\mathcal{H}}_{n,0}^\theta + \mathcal{F}_1(\theta)$$

$$\mathcal{W}_n^* = \int \frac{r^2}{\Theta} \left(\tilde{\mathcal{H}}_{n,0}^\theta + \mathcal{F}_1(\theta) \right) d\theta$$

Hamiltonian after elimination of the perigee

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + \epsilon \frac{\Theta^2}{r^2} \left(\frac{\alpha}{p} \right)^2 \left(\frac{1}{2} - \frac{3}{4} \sin^2 i \right) + \\ & \frac{\epsilon^2}{2!} \left[\frac{\Theta^2}{r^2} \left(\frac{\alpha}{p} \right)^4 \left(-\frac{13}{8} + 3 \sin^2 i - \frac{69}{64} \sin^4 i \right) + \right. \\ & \left. \frac{\Theta^2}{r^2} \left(\frac{\alpha}{p} \right)^4 \eta^2 \left(\frac{3}{8} - \frac{3}{8} \sin^2 i - \frac{15}{64} \sin^4 i \right) \right], \end{aligned}$$

$$\mathcal{H} = \mathcal{H}(r, -, -, R, \Theta, N)$$

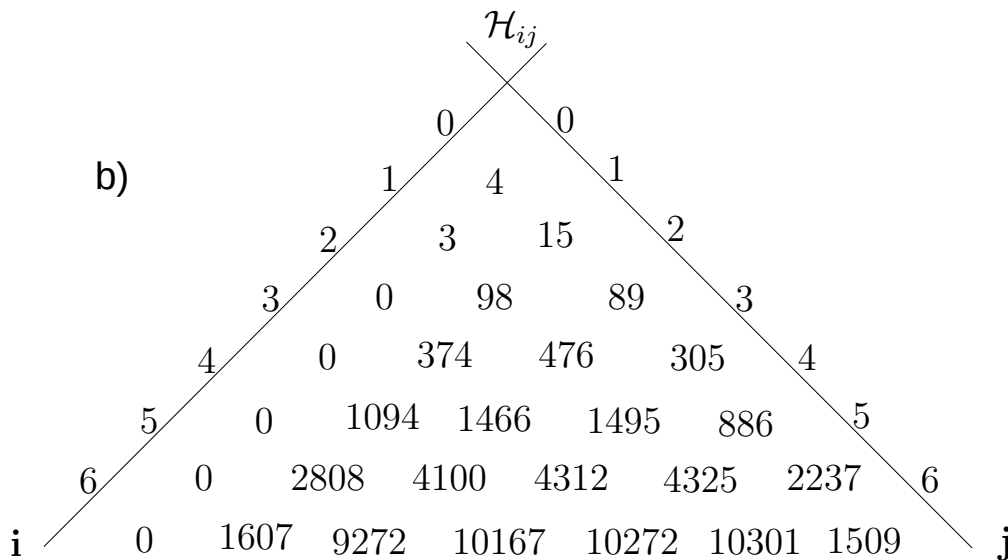
Advantages of simplifications: automatization and shorter series

	Elimination of the parallax		Second short period transformation		Brouwer-Kozai (Von Zeipel)	
Order	Hamiltonian	Generator	Hamiltonian	Generator	Hamiltonian	Generator
1	2	7	2	2	2	7
2	8	32	13	16	15	213
3	11	112	28	125	28	2076
4	26	264	72	612	?	?

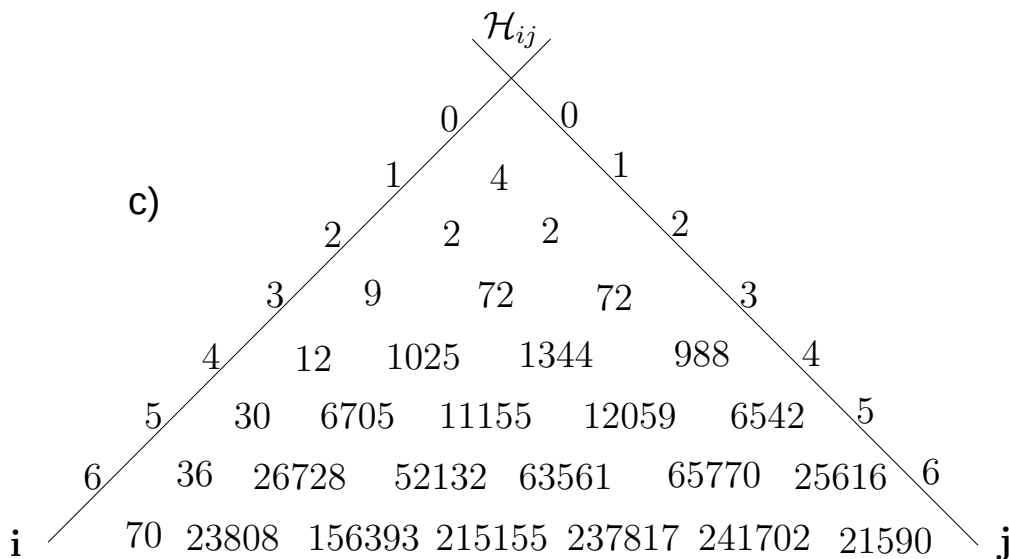
Number of terms of the new Hamiltonians and generators for several orders and transformations:

order	0	1	2	3	4	5	6
After elimination of the parallax							
$\mathcal{H}_{0,i}$	4	2	9	12	30	36	70
W_i	0	7	32	112	264	643	1340
After elimination of the perigee							
$\mathcal{H}_{0,i}$	4	2	6	20	35	68	106
W_i	0	2	34	289	1038	2984	5242
After Delaunay normalization							
$\mathcal{H}_{0,i}$	2	2	9	36	111	991	2682
W_i	0	2	15	104	474	7092	23687

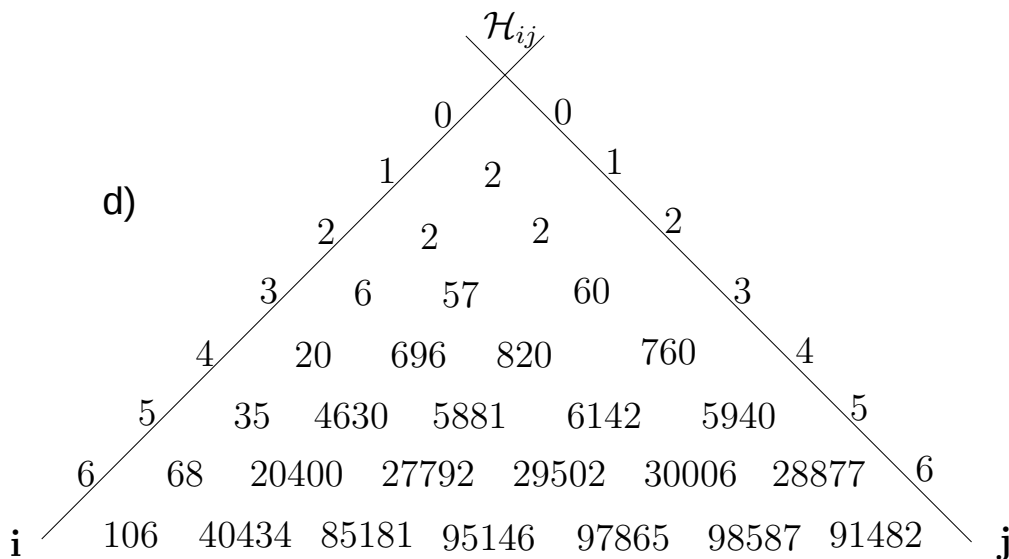
Number of terms of Lie triangle in the elimination of the parallax



Number of terms of Lie triangle in the elimination of the perigee



Number of terms of Lie triangle in the Delaunay's normalization



Time in minutes for generating analytical theory of order n

