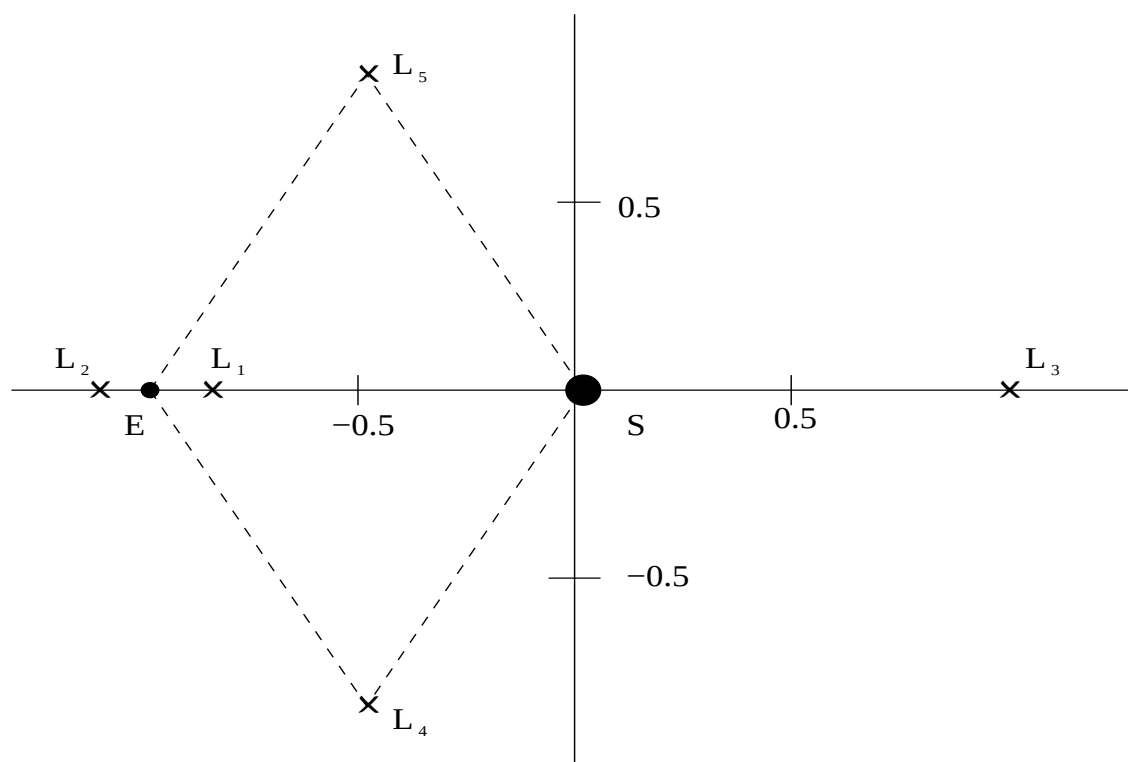


Aspectos Dinámicos del Diseño de Misiones Espaciales relacionadas con Orbitas de Libración

Josep J. Masdemont
Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya, UPC



LINEARIZED EQUATIONS

$$\left. \begin{aligned} \ddot{x} - 2 \dot{y} - (1 + 2c_2) x &= 0 \\ \ddot{y} + 2 \dot{x} + (c_2 - 1) y &= 0 \\ \ddot{z} + c_2 z &= 0 \end{aligned} \right\}$$

$$c_2 = \frac{1}{\gamma^3}(\mu + (1 - \mu)\frac{\gamma^3}{(1 \mp \gamma)^3}) \quad \text{for } L_1, L_2,$$

$$c_2 = \frac{1}{\gamma^3}(1 - \mu + \mu\frac{\gamma^3}{(1 + \gamma)^3}) \quad \text{for } L_3.$$

Solution of the Linear Equations

$$\left. \begin{aligned} x(t) &= A_1 e^{\lambda t} + A_2 e^{-\lambda t} + A_3 \cos \omega t + A_4 \sin \omega t \\ y(t) &= c A_1 e^{\lambda t} - c A_2 e^{-\lambda t} - \bar{k} A_4 \cos \omega t + \bar{k} A_3 \sin \omega t \\ z(t) &= A_5 \cos \nu t + A_6 \sin \nu t \end{aligned} \right\}$$

with $A_1, \dots, A_6$ arbitrary constants.

$$\omega = \sqrt{\frac{2 - c_2 + \sqrt{9c_2^2 - 8c_2}}{2}}, \quad \nu = \sqrt{c_2}, \quad c = \frac{\lambda^2 - 1 - 2c_2}{2\lambda},$$

$$\lambda = \sqrt{\frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}}, \quad \bar{k} = \frac{-(\omega^2 + 1 + 2c_2)}{2\omega}.$$

It is also convenient to look at the oscillatory solution of the linear part as having an amplitude and a phase,

$$\left. \begin{aligned} x(t) &= A_1 e^{\lambda t} + A_2 e^{-\lambda t} + A_x \cos(\omega t + \phi) \\ y(t) &= c A_1 e^{\lambda t} - c A_2 e^{-\lambda t} + \bar{k} A_x \sin(\omega t + \phi) \\ z(t) &= A_z \cos(\nu t + \psi) \end{aligned} \right\}$$

where the relations are $A_3 = A_x \cos \phi$, $A_4 = -A_x \sin \phi$, $A_5 = A_z \cos \psi$ and $A_6 = -A_z \sin \psi$.

Avoiding Unstable Motions

$$\left. \begin{aligned} x(t) &= A_1 e^{\lambda t} + A_2 e^{-\lambda t} + A_x \cos(\omega t + \phi) \\ y(t) &= c A_1 e^{\lambda t} - c A_2 e^{-\lambda t} + \bar{k} A_x \sin(\omega t + \phi) \\ z(t) &= A_z \cos(\nu t + \psi) \end{aligned} \right\}$$

- $A_1 = A_2 = 0$ gives linear Lissajous.
- $A_1 = 0, A_2 \neq 0$ defines **Stable Manifold**.
- $A_2 = 0, A_1 \neq 0$ defines **Unstable Manifold**.

To avoid unstable motions forward in time we require $A_1 = 0$ and to preserve this condition under maneuvers. It can be seen that this type of maneuvers must be orthogonal to $(\frac{\bar{k}}{d_2}, \frac{1}{d_1})$. This is,

$$(\Delta \dot{x}, \Delta \dot{y}) = \frac{\alpha}{\sqrt{c^2 + \bar{k}^2}} (d_2, -\bar{k}d_1), \quad \alpha \in \mathbb{R}$$

where $|\alpha|$, is the size of the maneuver.

Changing the In-Plane Amplitude

Let us assume that at a given time t_m we perform a maneuver $(\Delta\dot{x}, \Delta\dot{y})$. The central part (i.e. the libration terms) will change from an initial in-plane amplitude $A_x^{(i)}$ to a final one $A_x^{(f)}$ given by,

$$\begin{aligned} A_x^{(f)2} = & A_x^{(i)2} + \frac{c^2}{d_2^2}(\Delta\dot{x})^2 + \frac{1}{d_1^2}(\Delta\dot{y})^2 - 2\frac{c\lambda}{d_2^2}y(t_m)\Delta\dot{x} + \\ & + 2\frac{c^2}{d_2^2}\dot{x}(t_m^-)\Delta\dot{x} - 2\frac{c\lambda}{d_1^2}x(t_m)\Delta\dot{y} + 2\frac{1}{d_1^2}\dot{y}(t_m^-)\Delta\dot{y}. \end{aligned}$$

Assuming that the satellite is in a Lissajous orbit this expression reduces to,

$$A_x^{(f)2} = \left(\frac{c}{d_2}\Delta\dot{x} - \frac{1}{\bar{k}}y(t_m) \right)^2 + \left(\frac{1}{d_1}\Delta\dot{y} - x(t_m) \right)^2$$

We note that up to this point we have not still required the non scape condition.

Doing the maneuver in a Lissajous orbit in the direction of the non escape condition we obtain,

$$A_x^{(f)2} = \alpha^2 + \frac{2\alpha}{\sqrt{c^2 + \bar{k}^2}} \left(\frac{-c}{\bar{k}} y(t_m) + \bar{k} x(t_m) \right) + A_x^{(i)2}$$

Introducing the constant angle beta as

$$(\cos \beta, \sin \beta) = \left(\frac{c}{\sqrt{c^2 + \bar{k}^2}}, \frac{\bar{k}}{\sqrt{c^2 + \bar{k}^2}} \right),$$

the expression reduces to,

$$A_x^{(f)2} = \alpha^2 - 2\alpha A_x^{(i)} \sin(\omega t_m + \phi_i - \beta) + A_x^{(i)2}$$

where ϕ_i is the in-plane amplitude of the departure orbit.

Given an initial in-plane amplitude $A_x^{(i)}$ and a target one $A_x^{(f)}$ the magnitude of the maneuver needed for the transfer at epoch t_m is,

$$\alpha = A_x^{(i)} \sin(\omega t_m + \phi_i - \beta) \pm \sqrt{A_x^{(f)2} - A_x^{(i)2} \cos^2(\omega t_m + \phi_i - \beta)}$$

- $A_x^{(f)} \geq A_x^{(i)}$, the transfer maneuver is possible at any time.
- $A_x^{(f)} < A_x^{(i)}$, the transfer maneuver is possible only at some times.

Once the target amplitude $A_x^{(f)}$ is selected, we note the two basic possibilities that we have when selecting the maneuver.

- Select t_m in such a way that the Δv expended in changing the amplitude be a minimum. This corresponds to the minimum of $|\alpha|$.
- Select t_m in such a way that you arrive at the target orbit with a selected phase.

Optimal in-plane maneuvers

Assuming $A_x^{(f)} \neq A_x^{(i)}$, the local minima of $\alpha(t_m)$ is obtained performing the maneuver when t_m verifies,

$$\omega t_m + \phi_i = \beta + \frac{\pi}{2}, \quad \text{or}$$

$$\omega t_m + \phi_i = \beta + \frac{3\pi}{2}, \quad (\text{both mod } 2\pi).$$

This is, when the angle $\omega t_m + \phi_i$ is orthogonal to β , or equivalently, when the satellite on a Lissajous orbit crosses the plane $cx + \bar{k}y = 0$.

Changing the in-plane phase

Performing a maneuver at time t_m and using the amplitudes A_3 and A_4 we have,

$$A_3^{(f)} = A_3^{(i)} - \alpha(t_m) \sin(\omega t_m - \beta),$$

$$A_4^{(f)} = A_4^{(i)} + \alpha(t_m) \cos(\omega t_m - \beta),$$

Changing the in-plane phase maintaining the amplitude

We study the non trivial maneuver when $A_x^{(f)} = A_x^{(i)}$. It gives a jump in the stable manifold of the same torus. Essentially a phase shift given by,

$$\phi_f - \phi_i = -2(\omega t_m - \beta + \phi_i) \quad (\text{mod } 2\pi).$$

This fact will be used for the exclusion zone avoidance using in-plane maneuvers.

Changing the Out-of-Plane Amplitude

The discussion is similar to the in-plane ones but there is no scape direction. The maneuver to transfer from $A_z^{(i)}$ to $A_z^{(f)}$ is given by,

$$\frac{\Delta \dot{z}}{\nu} = A_z^{(i)} \sin(\nu t_m + \psi_i) \pm \sqrt{A_z^{(f)2} - A_z^{(i)2} \cos^2(\nu t_m + \psi_i)}$$

- $A_z^{(f)} \geq A_z^{(i)}$, the transfer maneuver is possible at any time.
- $A_z^{(f)} < A_z^{(i)}$, the transfer maneuver is possible only at some times.

Optimal out-of-plane maneuvers

When the transfer is possible, the maneuver to change the out-of-plane amplitude from $A_z^{(i)}$ to $A_z^{(f)}$ is optimal when t_m verifies,

$$\nu t_m + \psi_i = \frac{\pi}{2}, \quad \text{or}$$

$$\nu t_m + \psi_i = \frac{3\pi}{2}, \quad (\text{both mod } 2\pi).$$

This corresponds to the $z = 0$ crossing (natural in terms of energy).

Changing phase without changing amplitude

The non trivial maneuver at time t_m produces a shift in the out-of-plane phase given by,

$$\psi_f - \psi_i = -2(\nu t_m + \psi_i) \quad (\text{mod } 2\pi)$$

Effective Phases

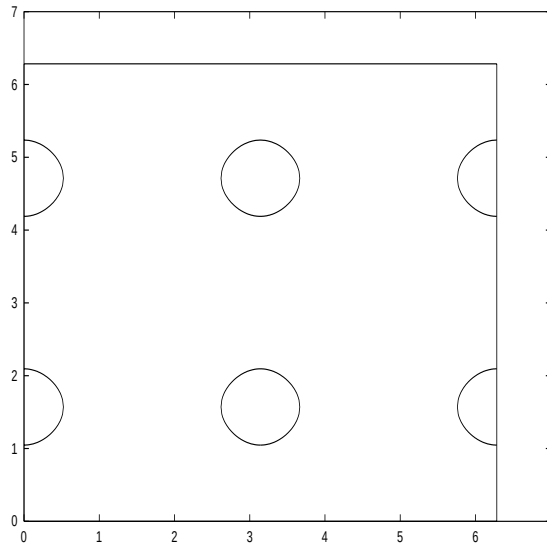
We consider the angular variables of the torus,

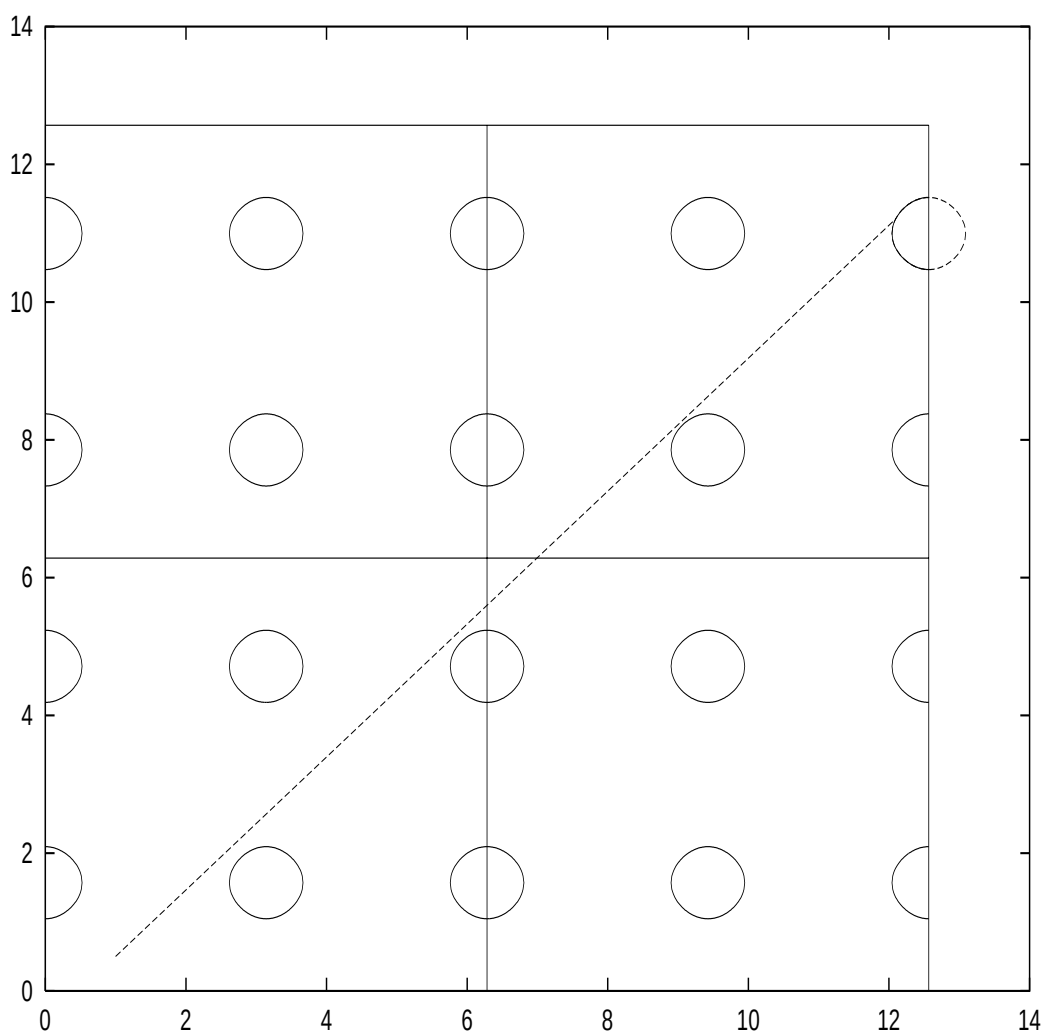
$$\Phi = \omega t + \phi, \quad \Psi = \nu t + \psi, \quad (\text{mod } 2\pi).$$

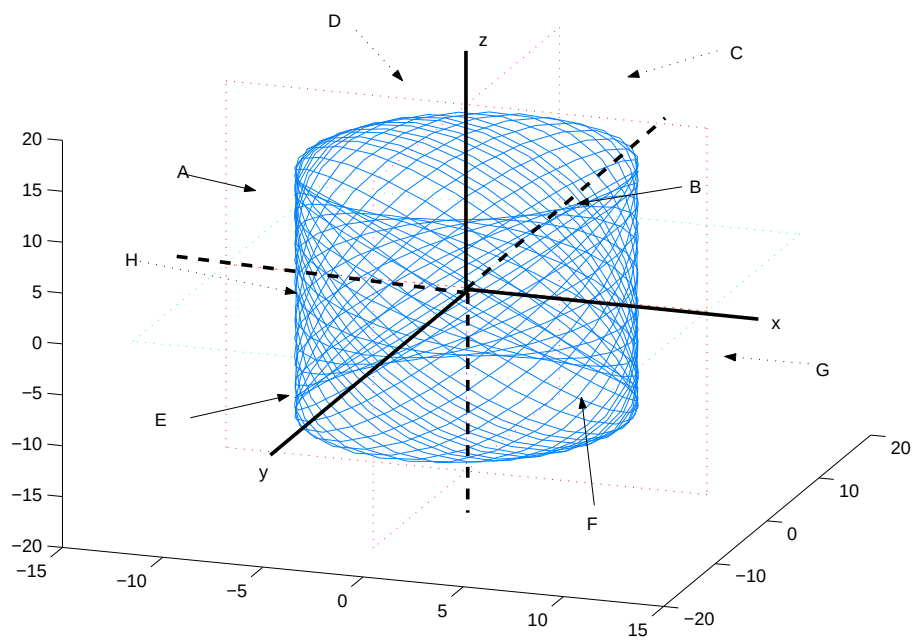
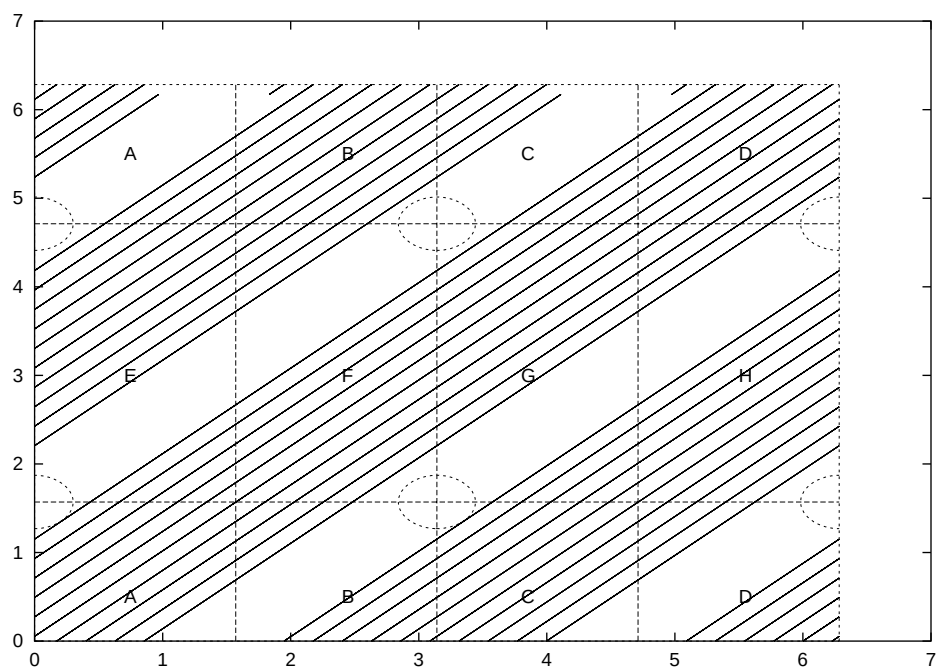
And the exclusion zone,

$$y^2 + z^2 < R^2,$$

$$\bar{k}^2 A_x^2 \sin^2 \Phi + A_z^2 \cos^2 \Psi = R^2$$

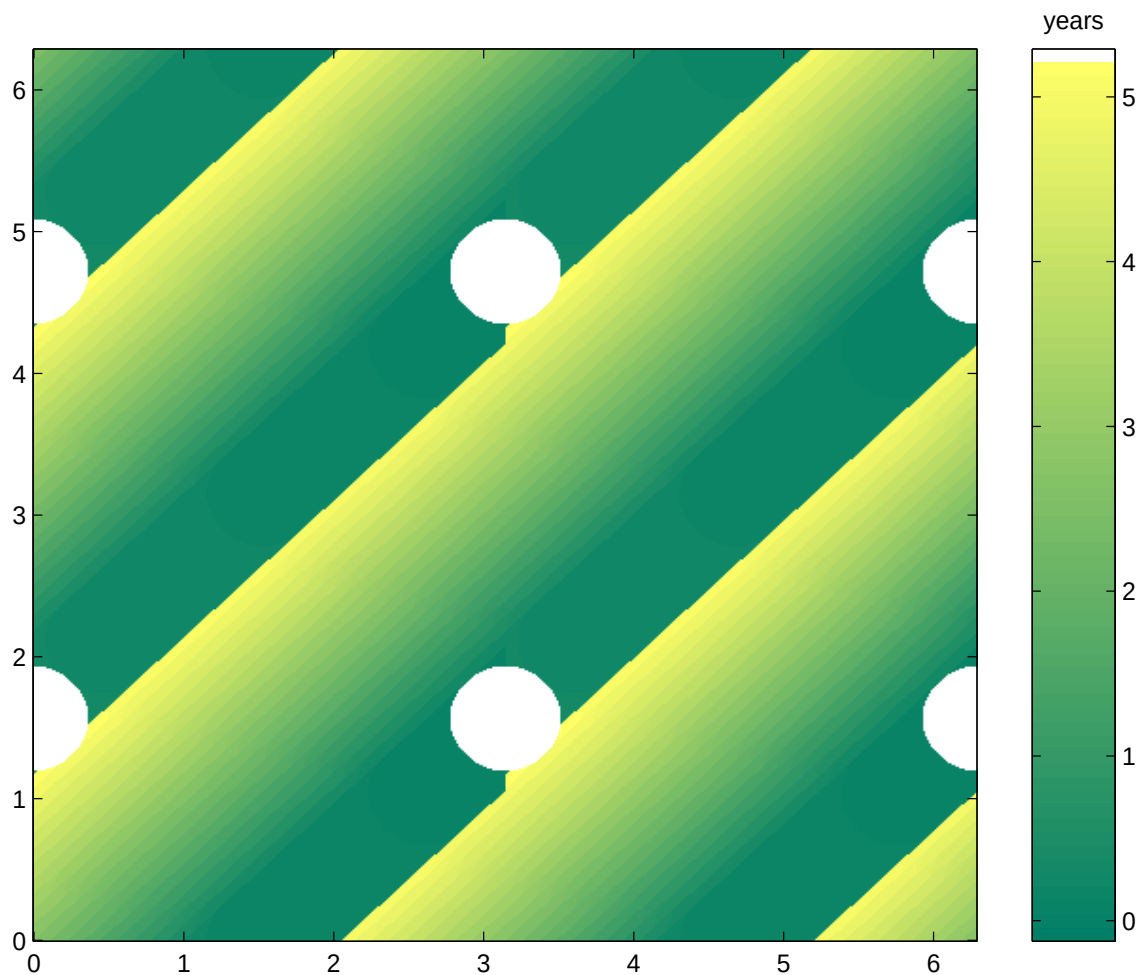




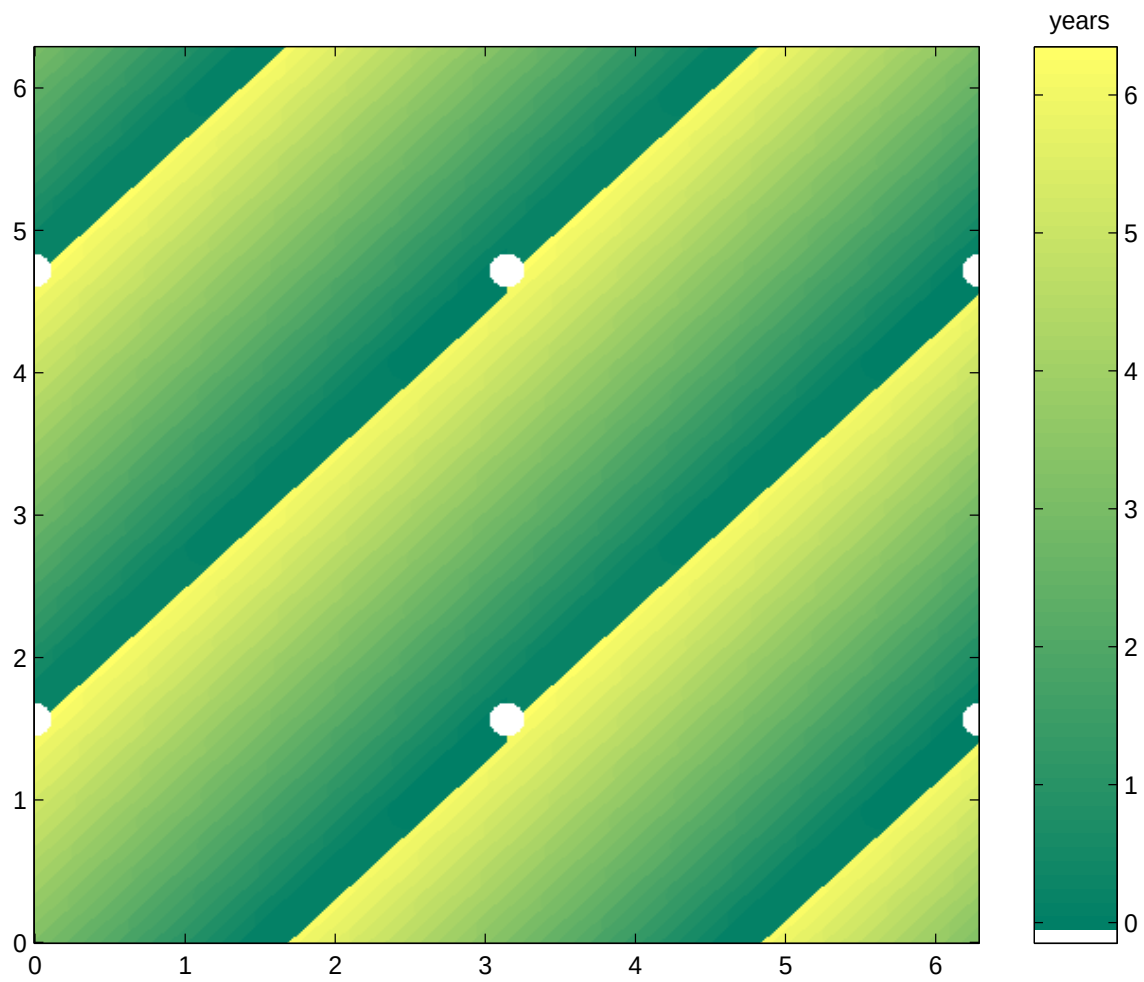


	ω	ν	ν/ω	usual R (km)	angle from Earth
L ₁	2.086	2.015	0.966	90000	$\simeq 3.5$ deg radius
L ₂	2.057	1.985	0.965	14000	$\simeq 0.54$ deg radius

Initial phases classification according to the time when they first hit an exclusion zone.

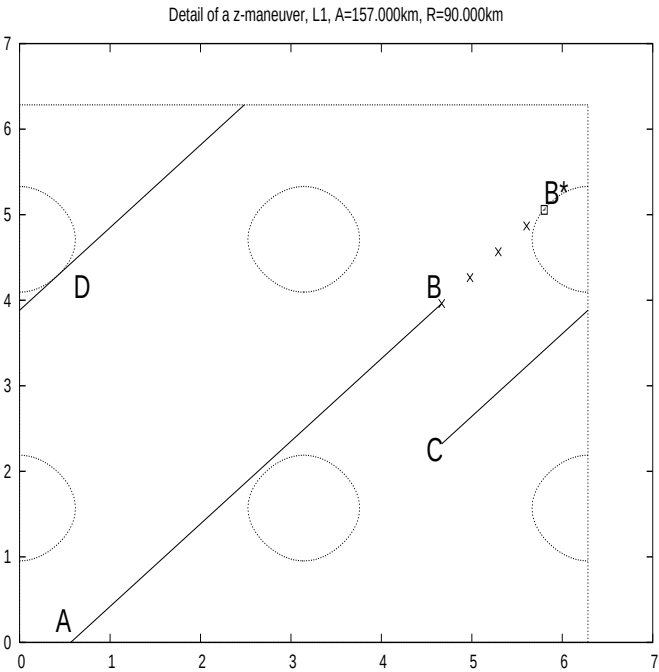
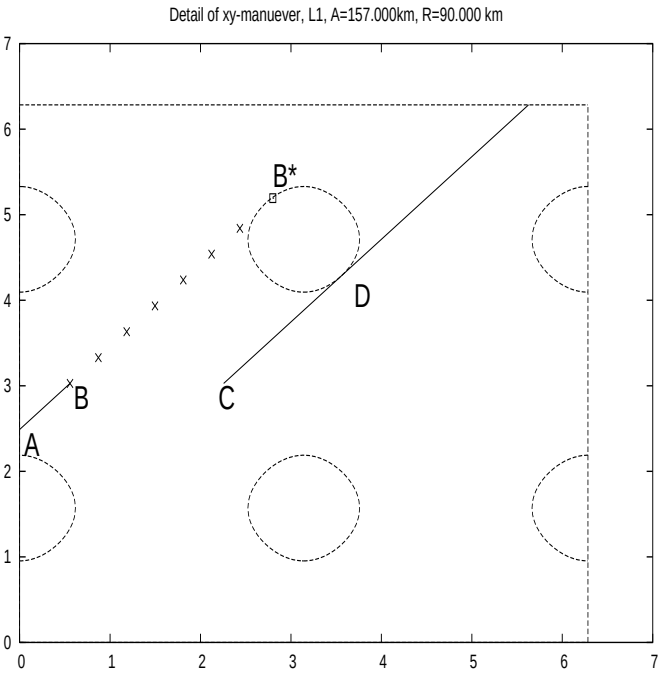


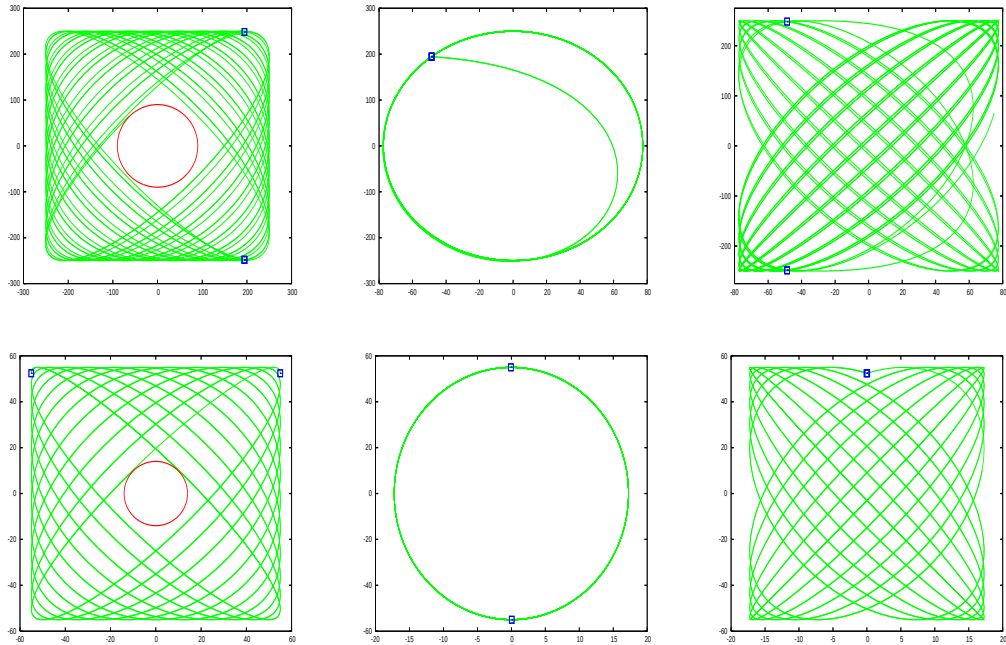
L₁ case $A_y = A_z = 250000$ km, $R = 90000$ km.



L₂ case $A_y = A_z = 120000$ km, $R = 14000$ km.

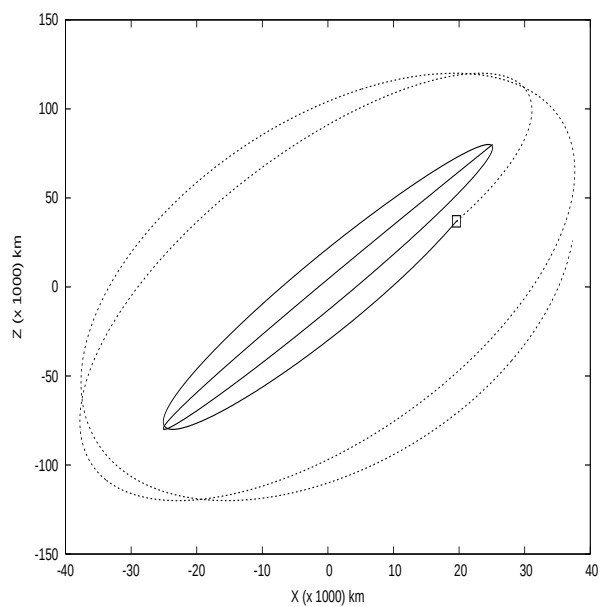
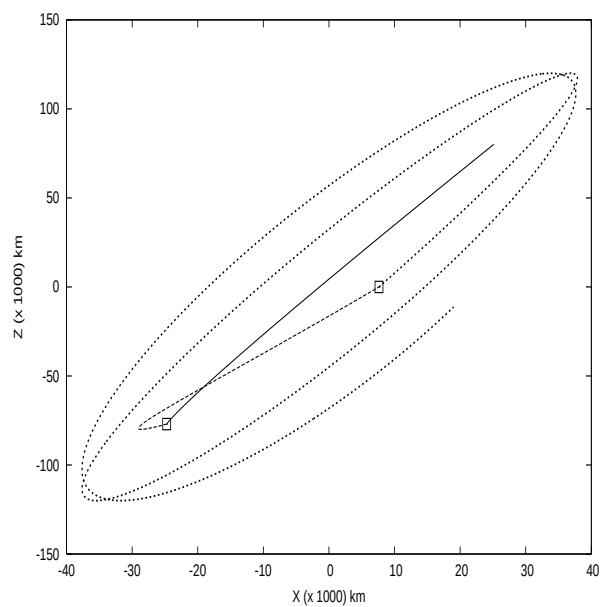
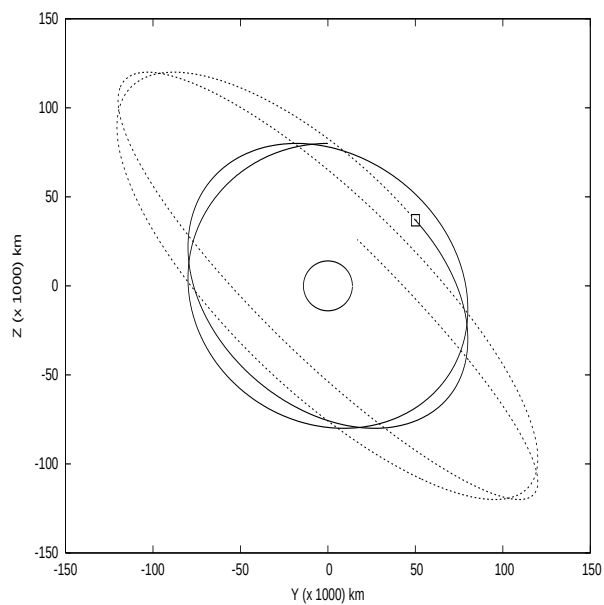
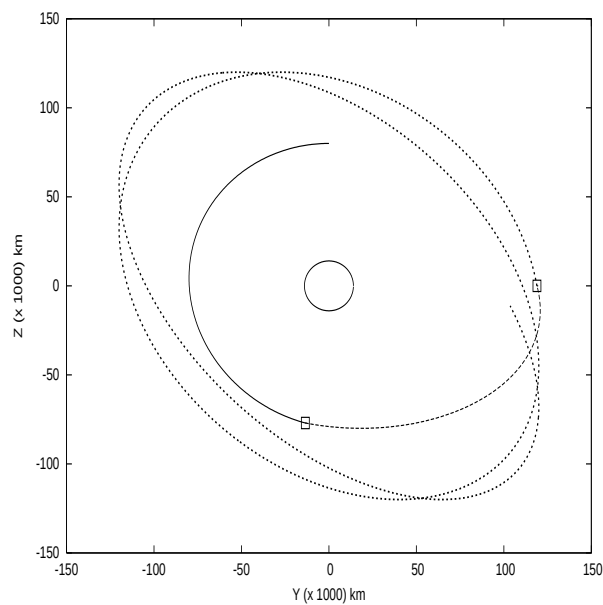
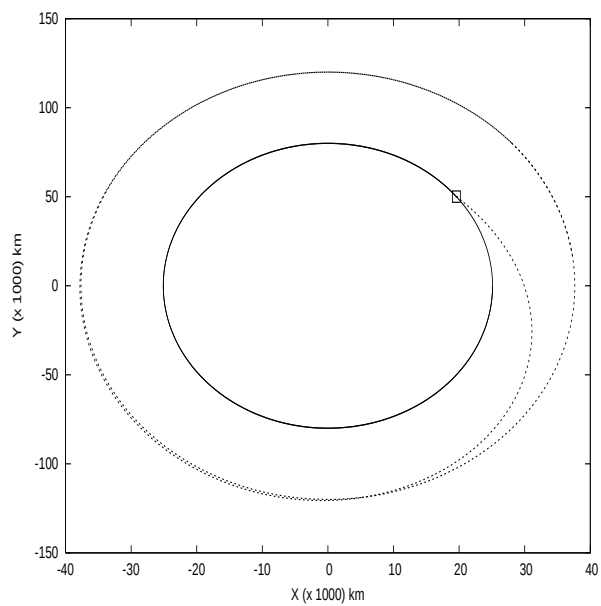
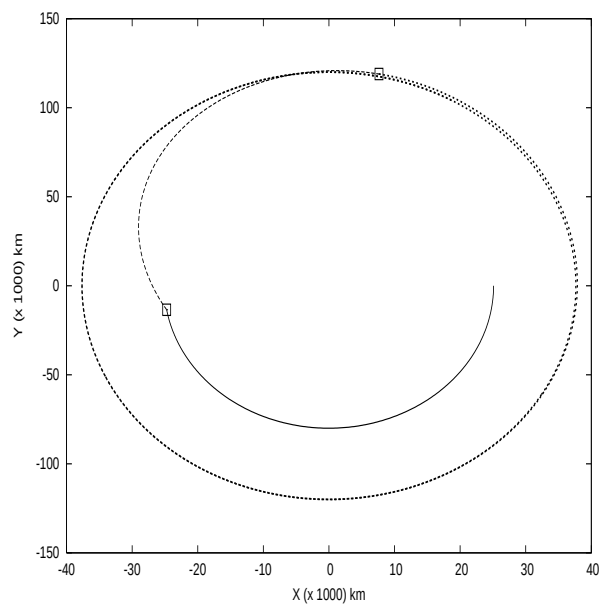
LOEWE Strategies





First row, example of one-sided cycle. Trajectory around L_1 with $A = 250000$ km, $R = 90000$ km (xy -maneuvers). Second row, example of two-sided cycle. Trajectory about L_2 with $A = 55000$ km, $R=14000$ km. (z -maneuvers). The maneuvers are marked with a small box.

Animaciones



Objects in the Center Manifold

The "more relevant" objects that appear in the central manifold of the collinear equilibrium points of the RTBP are:

- The Lyapunov planar periodic orbit (outer curve),
- One **vertical periodic orbit**,
- Two symmetric periodic **halo orbits**,
- A Cantor family 2-D tori around the vertical periodic orbits (**Lissajous orbits**),
- Two Cantor families of 2-D tori around the halo orbits (**quasihalo orbits**).

All these object can be determined either using **numerical** procedures or in a **semi-analytical** way, computing formal series aproximations using the Lindstedt-Poincaré method.

The **numerical procedures** are based in the use of Newton's method for the computation of fixed points (periodic orbits) or invariant curves (2-D tori).

Formal series solutions

Lissajous orbits: two frequencies and two amplitudes.

$$\sum_{i,j=1}^{\infty} \left(\sum_{|k| \leq i, |m| \leq j} a_{ijkm} \frac{\cos}{\sin} (k\theta_1 + m\theta_2) \right) \alpha^i \beta^j.$$

Halo orbits: one frequency and one amplitude.

$$\sum_{i,j=1}^{\infty} \left(\sum_{|k| \leq i+j} a_{ijk} \frac{\cos}{\sin} (k\omega\tau) \right) \alpha^i \beta^j,$$

$$\Delta(\alpha, \beta) = 0.$$

Quasihalo orbits: two frequencies and one amplitude.

$$\sum_{i=1}^{\infty} \left(\sum_{|k| \leq K, |m| \leq i} a_{ijm} \frac{\cos}{\sin} (k\theta_1 + m\theta_2) \right) \gamma^i.$$

Where $\theta_1 = \omega\tau + \phi_1$, $\theta_2 = \nu\tau + \phi_1$,

$$\omega = \sum_{i,j=0}^{\infty} \omega_{ij} \alpha^i \beta^j, \quad \nu = \sum_{i,j=0}^{\infty} \nu_{ij} \alpha^i \beta^j.$$

Stable and Unstable Manifolds

Numerical computation

For the **periodic orbits** (halo and vertical), the eigenvectors of the monodromy matrix can be used to get local (linear) approximations of their stable and unstable manifolds. The numerical globalization is easy.

For **quasihalo orbits** with not very large values of γ , the stable/unstable **eigenvectors of the variational matrix computed over one revolution**, approximate quite well the s/u directions.

When we have the direction of the s/u manifold, at the beginning of a revolution, it can be obtained at any point, transporting the vector by means of the differential of the flow.

Semianalytical Computations of Invariant Manifolds for Lissajous orbits

$$\left\{ \begin{array}{l} \ddot{x} - 2 \dot{y} - (1 + 2c_2) x = \frac{\partial}{\partial x} \sum_{n \geq 3} c_n \rho^n P_n \left(\frac{x}{\rho} \right), \\ \ddot{y} + 2 \dot{x} + (c_2 - 1) y = \frac{\partial}{\partial y} \sum_{n \geq 3} c_n \rho^n P_n \left(\frac{x}{\rho} \right), \\ \ddot{z} + c_2 z = \frac{\partial}{\partial z} \sum_{n \geq 3} c_n \rho^n P_n \left(\frac{x}{\rho} \right), \end{array} \right.$$

$$\begin{aligned} x(t) &= \alpha_1 e^{\lambda_0 t} + \alpha_2 e^{-\lambda_0 t} + \alpha_3 \cos(\omega_0 t + \phi_1), \\ y(t) &= \bar{k}_2 \alpha_1 e^{\lambda_0 t} - \bar{k}_2 \alpha_2 e^{-\lambda_0 t} + \bar{k}_1 \alpha_3 \sin(\omega_0 t + \phi_1), \\ z(t) &= \alpha_4 \cos(\nu_0 t + \phi_2). \end{aligned}$$

$$x(t) = \sum e^{(i-j)\theta_3} \left[x_{ijk m}^{pq} \cos(p\theta_1 + q\theta_2) + \bar{x}_{ijk m}^{pq} \sin(p\theta_1 + q\theta_2) \right] \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m$$

$$y(t) = \sum e^{(i-j)\theta_3} \left[y_{ijk m}^{pq} \cos(p\theta_1 + q\theta_2) + \bar{y}_{ijk m}^{pq} \sin(p\theta_1 + q\theta_2) \right] \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m$$

$$z(t) = \sum e^{(i-j)\theta_3} \left[z_{ijk m}^{pq} \cos(p\theta_1 + q\theta_2) + \bar{z}_{ijk m}^{pq} \sin(p\theta_1 + q\theta_2) \right] \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m$$

where $\theta_1 = \omega t + \phi_1$, $\theta_2 = \nu t + \phi_2$, $\theta_3 = \lambda t$ and,

$$\omega = \sum \omega_{ijk m} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m,$$

$$\nu = \sum \nu_{ijk m} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m,$$

$$\lambda = \sum \lambda_{ijk m} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m.$$

Zero Values and Symmetries

Lissajous case

Facts to save computer storage and CPU time.
Considering a series truncated at order (N_1, N_2) .

- We consider always terms with $i, j, k, m > 0$ and $p > 0$.
- Always $i + j \leq N_1$, $k + m \leq N_2$ (and $i + j + k + m \leq N_2$ for a type I series).
- Always $p \leq k$ and $p \equiv k \pmod{2}$.
- Always $|q| \leq m$ and $q \equiv m \pmod{2}$ and in case that $p = 0$, only terms with $q \geq 0$ must be kept.

- Coefficients x_{ijkm}^{pq} , $\bar{x}_{ijkm}^{pq}$, y_{ijkm}^{pq} and $\bar{y}_{ijkm}^{pq}$ are zero when m is odd.
- Coefficients z_{ijkm}^{pq} and $\bar{z}_{ijkm}^{pq}$ are zero when m is even.

- For any i, j, k, m, p, q we have,

$$x_{ijkm}^{pq} = x_{jikm}^{pq} \quad \bar{x}_{ijkm}^{pq} = -\bar{x}_{jikm}^{pq}$$

$$y_{ijkm}^{pq} = -y_{jikm}^{pq} \quad \bar{y}_{ijkm}^{pq} = \bar{y}_{jikm}^{pq}$$

$$z_{ijkm}^{pq} = z_{jikm}^{pq} \quad \bar{z}_{ijkm}^{pq} = -\bar{z}_{jikm}^{pq}$$

and in particular, $\bar{x}_{iikm}^{pq} = y_{iikm}^{pq} = \bar{z}_{iikm}^{pq} = 0$

- Terms of the frequency series can be different from zero only when $i = j$ besides k and m are even.

Semianalytical Computations of Invariant Manifolds for Halo orbits

$$\begin{aligned}
 x(t) &= \alpha_1 e^{\lambda_0 t} + \alpha_2 e^{-\lambda_0 t} + \alpha_3 \cos(\omega_0 t + \phi) \\
 y(t) &= \bar{k}_2 \alpha_1 e^{\lambda_0 t} - \bar{k}_2 \alpha_2 e^{-\lambda_0 t} + \bar{k}_1 \alpha_3 \sin(\omega_0 t + \phi) \\
 z(t) &= \alpha_4 \cos(\omega_0 t + \phi)
 \end{aligned}$$

$$\left\{ \begin{array}{lcl}
 \ddot{x} - 2 \dot{y} - (1 + 2c_2) x & = & \frac{\partial}{\partial x} \sum_{n \geq 3} c_n \rho^n P_n \left(\frac{x}{\rho} \right), \\
 \ddot{y} + 2 \dot{x} + (c_2 - 1) y & = & \frac{\partial}{\partial y} \sum_{n \geq 3} c_n \rho^n P_n \left(\frac{x}{\rho} \right), \\
 \ddot{z} + c_2 z & = & \frac{\partial}{\partial z} \sum_{n \geq 3} c_n \rho^n P_n \left(\frac{x}{\rho} \right) + \Delta z,
 \end{array} \right.$$

$$x(t) = \sum e^{(i-j)\theta_3} \left[x_{ijkm}^p \cos(p\theta_1) + \bar{x}_{ijkm}^p \sin(p\theta_1) \right] \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m$$

$$y(t) = \sum e^{(i-j)\theta_3} \left[y_{ijkm}^p \cos(p\theta_1) + \bar{y}_{ijkm}^p \sin(p\theta_1) \right] \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m$$

$$z(t) = \sum e^{(i-j)\theta_3} \left[z_{ijkm}^p \cos(p\theta_1) + \bar{z}_{ijkm}^p \sin(p\theta_1) \right] \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m$$

where $\theta_1 = \omega t + \phi_1$, $\theta_3 = \lambda t$ and,

$$\omega = \sum \omega_{ijkm} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m,$$

$$\lambda = \sum \lambda_{ijkm} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m.$$

$$\Delta = \sum d_{ijkm} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^m = 0.$$

Zero Values and Symmetries

Halo case

Facts to save computer storage and CPU time.
Considering a series truncated at order (N_1, N_2) .

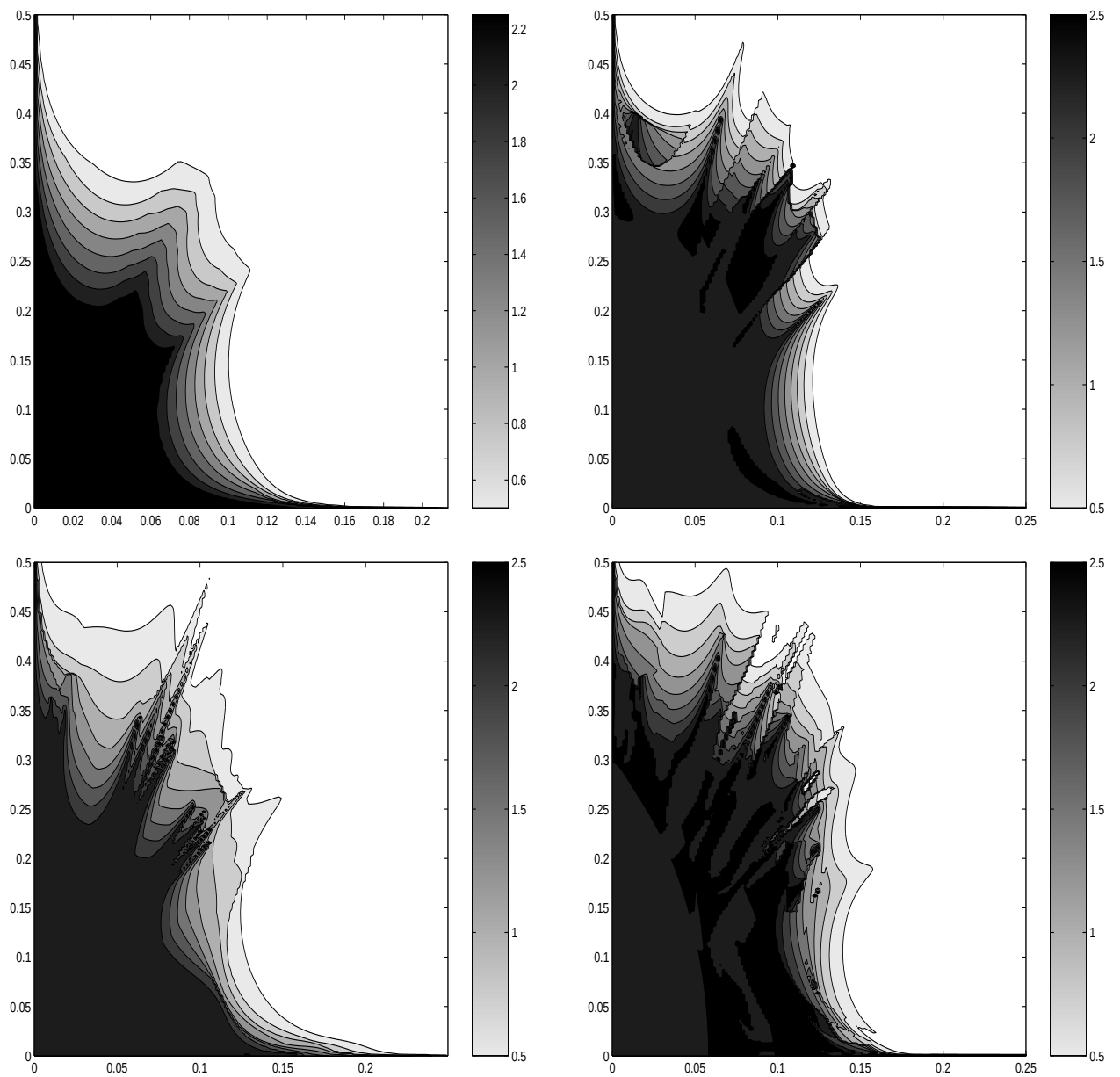
- We consider always terms with $i, j, k, m > 0$ and $p > 0$.
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- Always $p \leq k + m$ and $p \equiv k + m \pmod{2}$.
- Coefficients x_{ijkm}^p , $\bar{x}_{ijkm}^p$, y_{ijkm}^p and $\bar{y}_{ijkm}^p$ are zero when m is odd.
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- For any i, j, k, m, p we have,

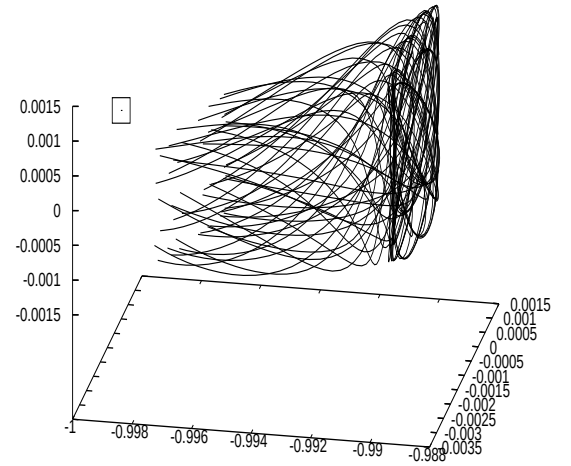
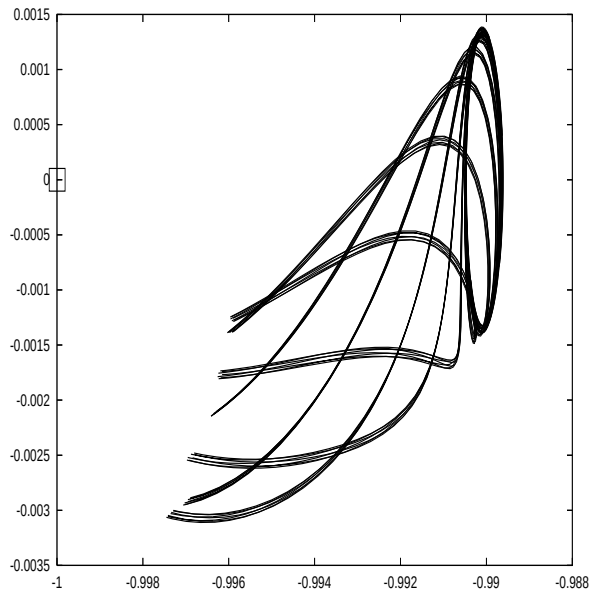
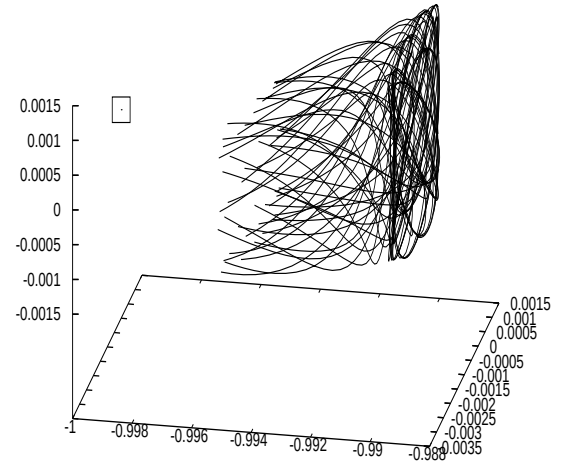
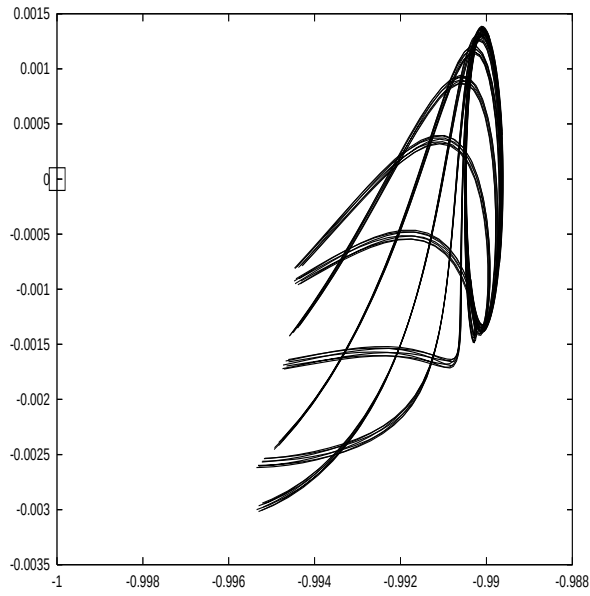
$$\begin{array}{ll}
 x_{ijkm}^p = x_{jikm}^p & \bar{x}_{ijkm}^p = -\bar{x}_{jikm}^p \\
 y_{ijkm}^p = -y_{jikm}^p & \bar{y}_{ijkm}^p = \bar{y}_{jikm}^p \\
 z_{ijkm}^p = z_{jikm}^p & \bar{z}_{ijkm}^p = -\bar{z}_{jikm}^p
 \end{array}$$

In particular, $\bar{x}_{iikm}^p = y_{iikm}^p = \bar{z}_{iikm}^p = 0$.

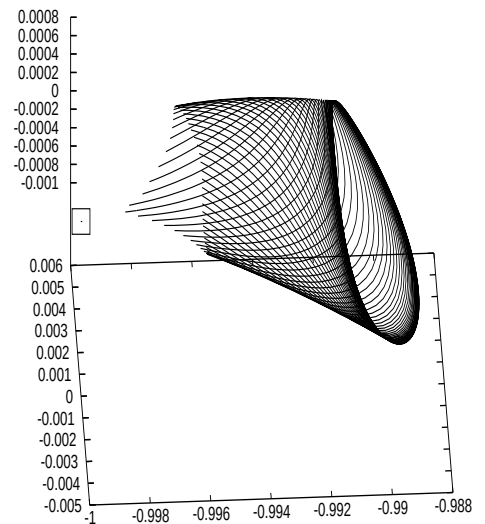
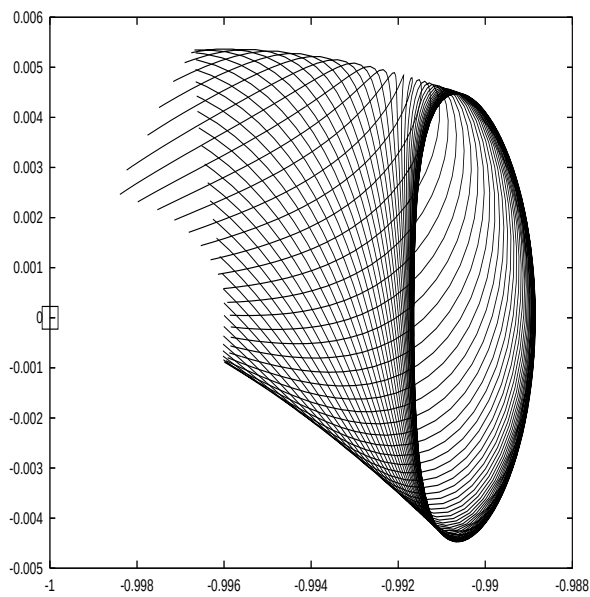
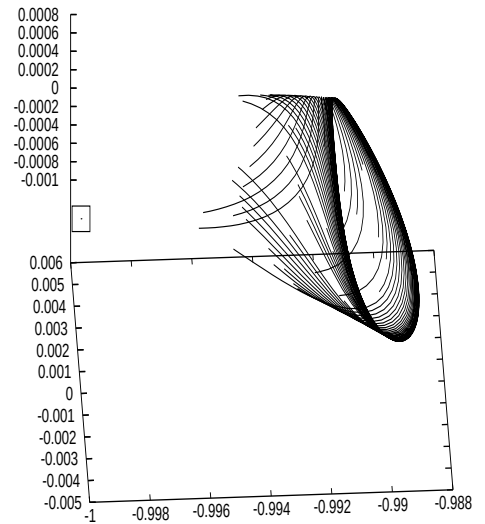
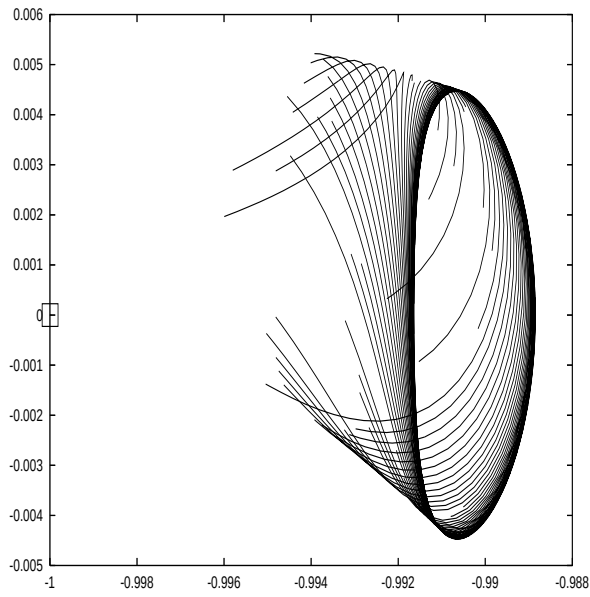
- Terms of frequency series can be different from zero only when $i = j$ besides k and m are even.



Columns order 9,15. Rows min and max agreement for all phases. 10^{-6} test.



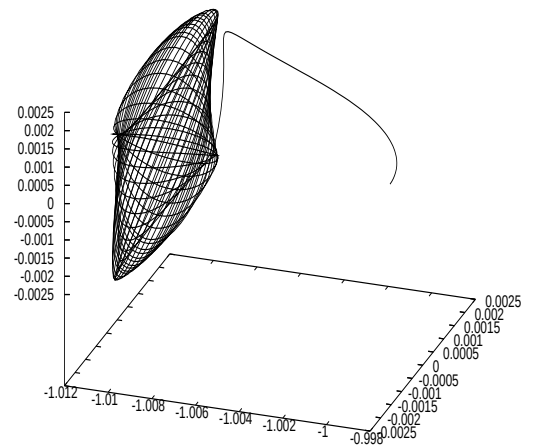
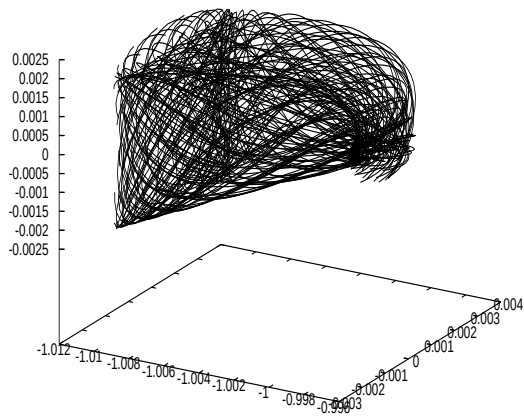
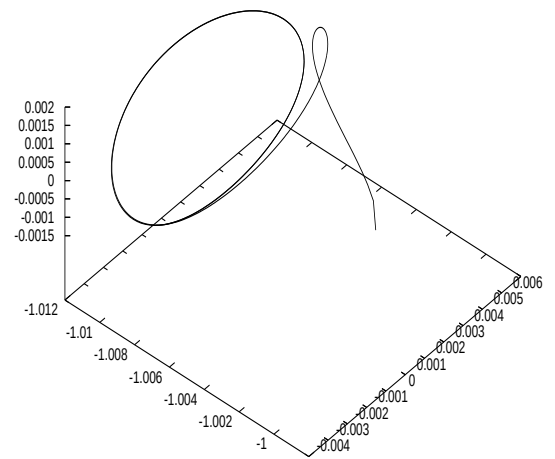
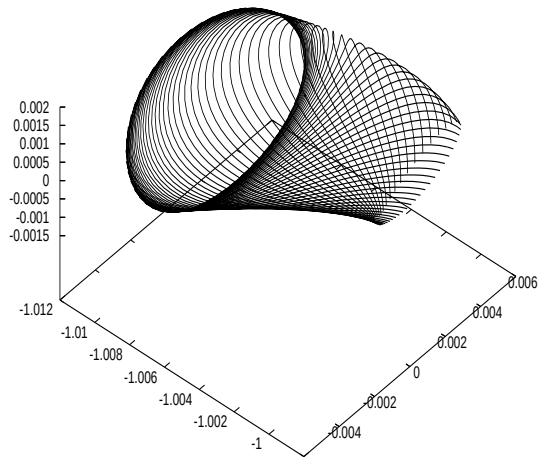
Rows, order 9,15. $\alpha_3 = 0.042$, $\alpha_4 = 0.13$. 10^{-6} test for WS.



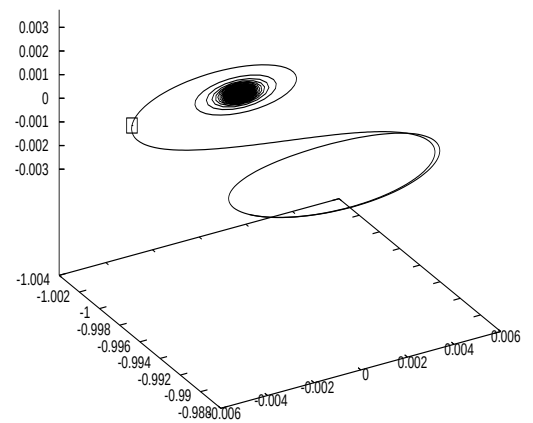
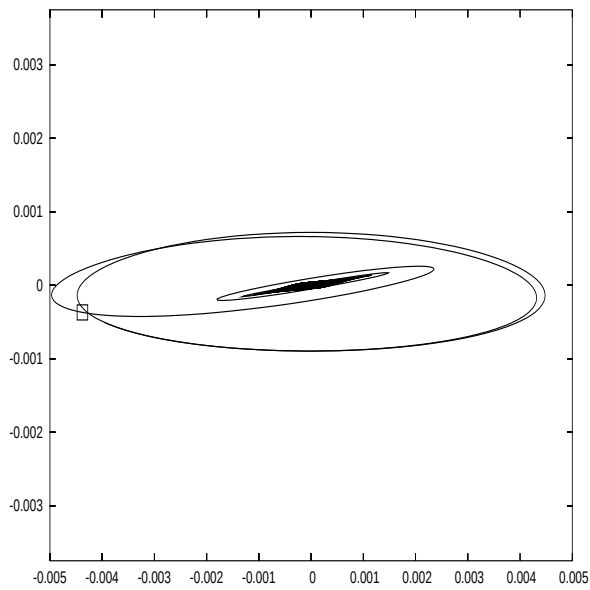
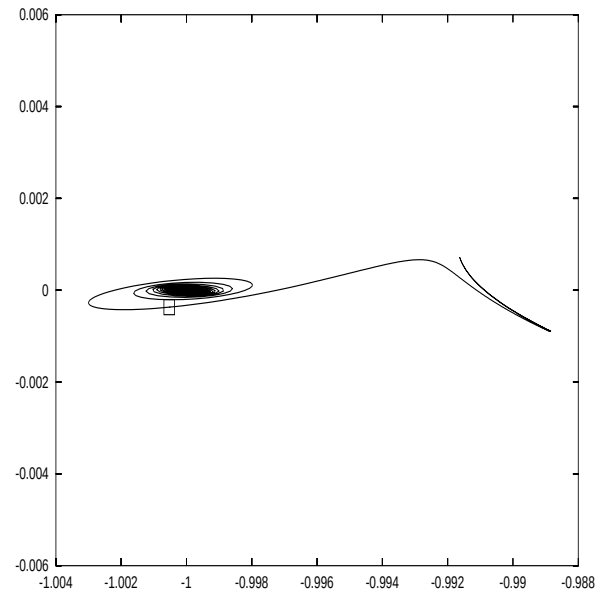
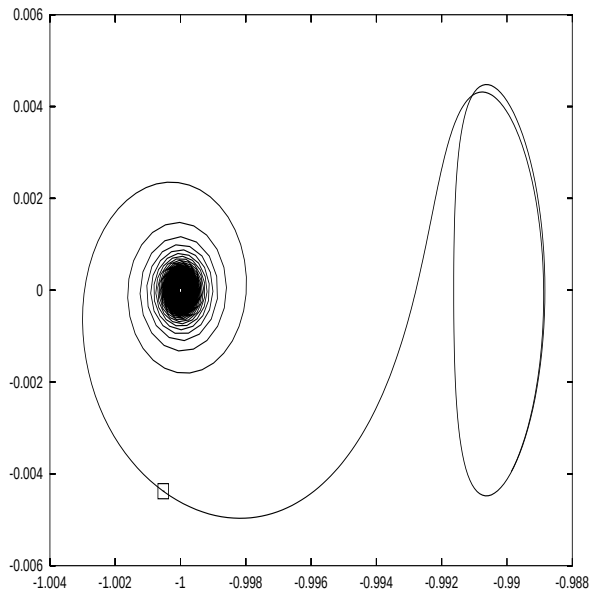
Rows, order 9,15. $\alpha_4 = 0.08$. 10^{-6} test for WU.

The use of the Invariant Manifolds

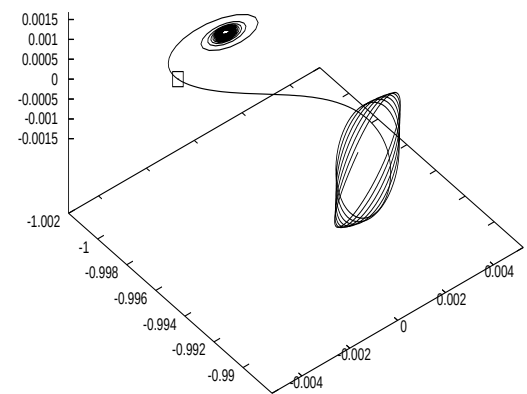
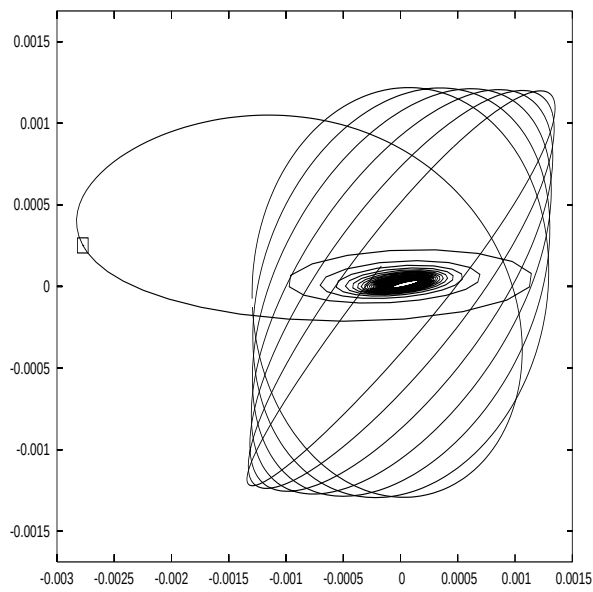
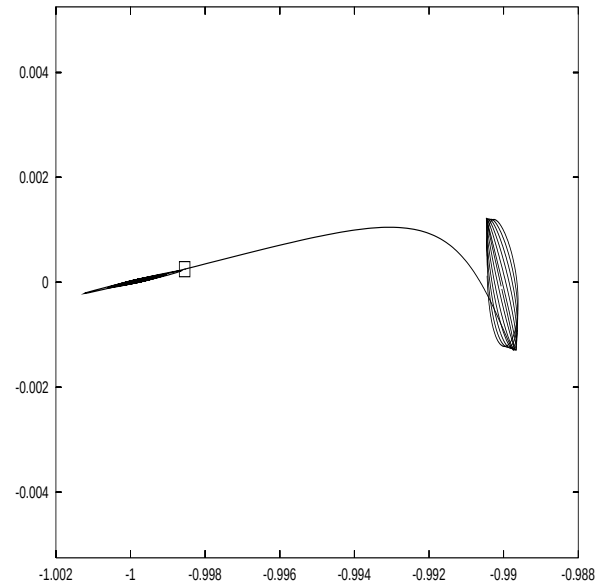
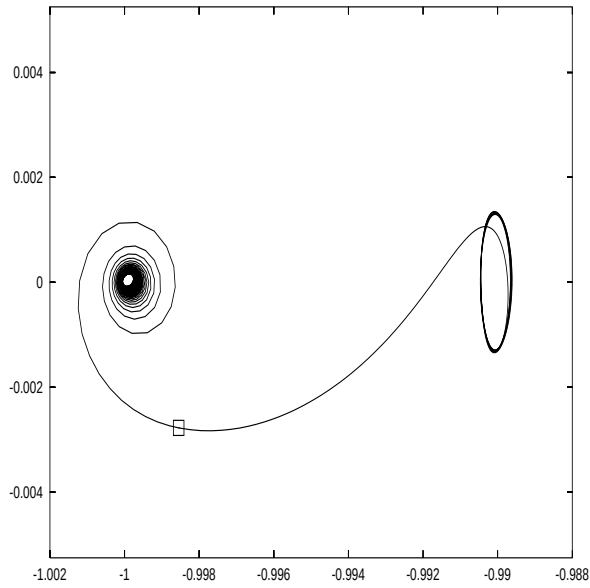
- Transfer from the vicinity of the Earth
- Station keeping and eclipse avoidance
- Transfer between libration point orbits



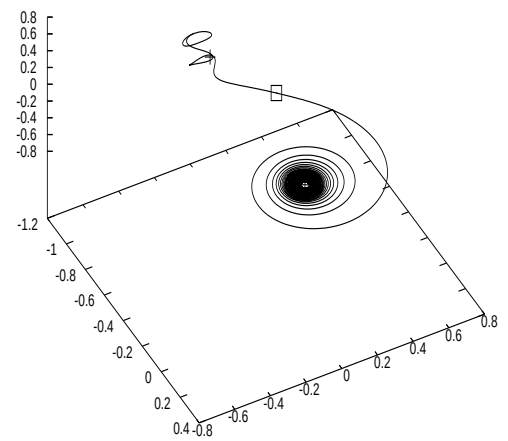
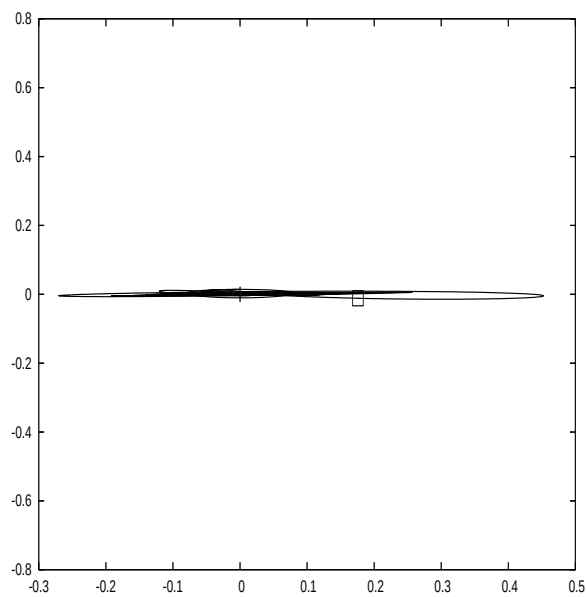
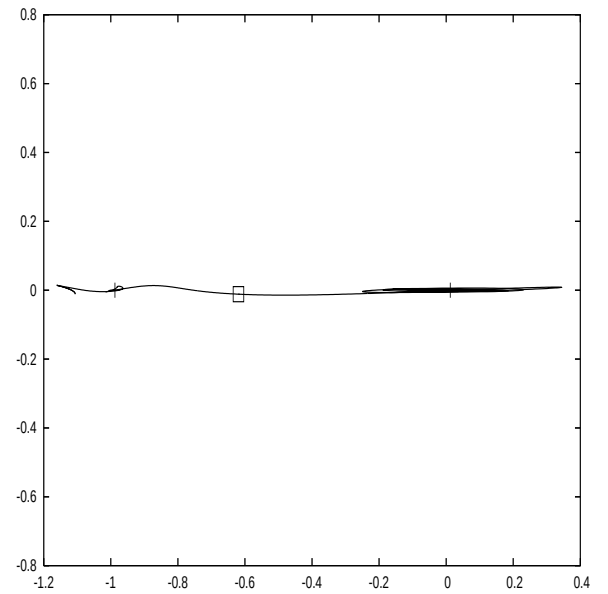
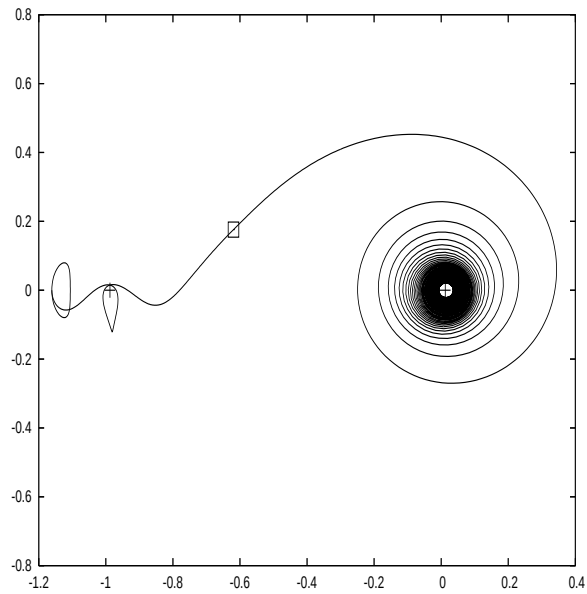
Transfer to L2



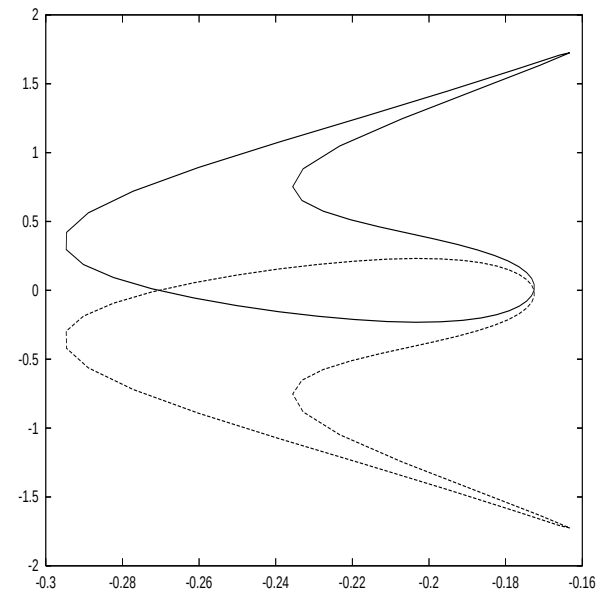
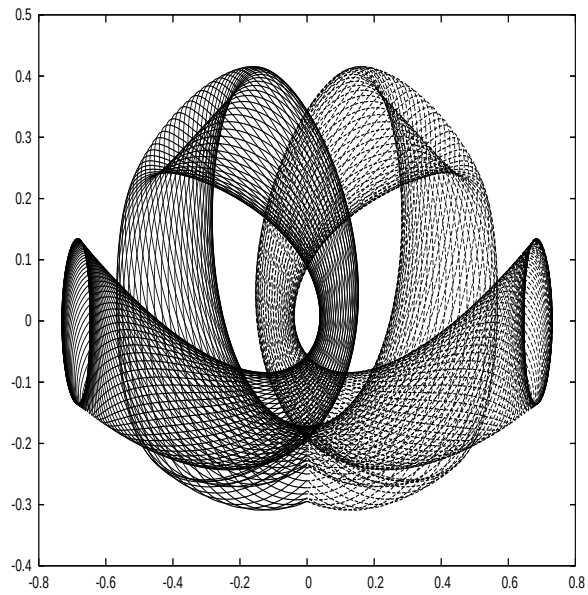
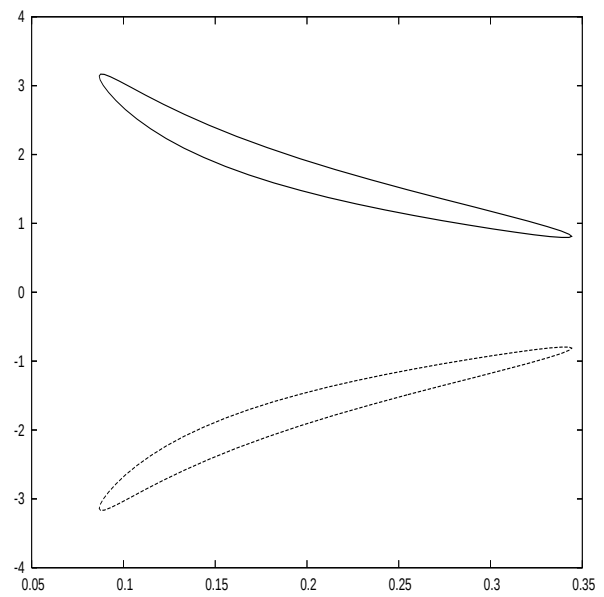
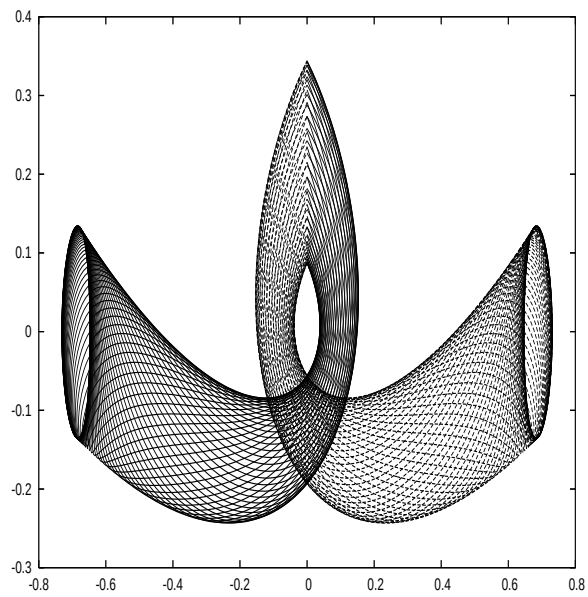
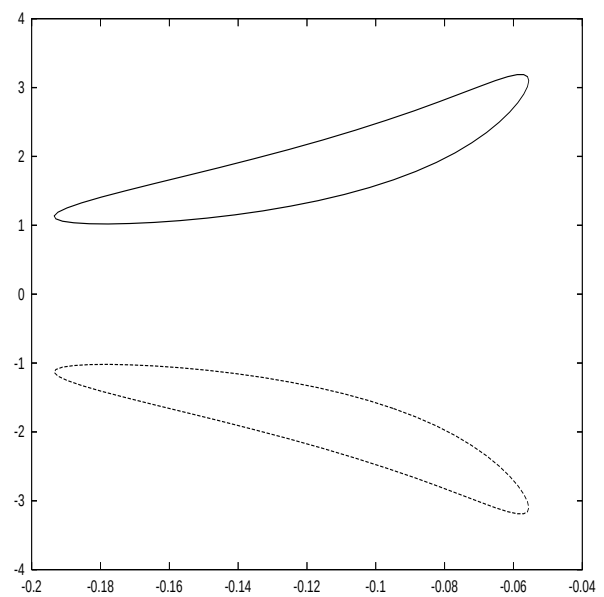
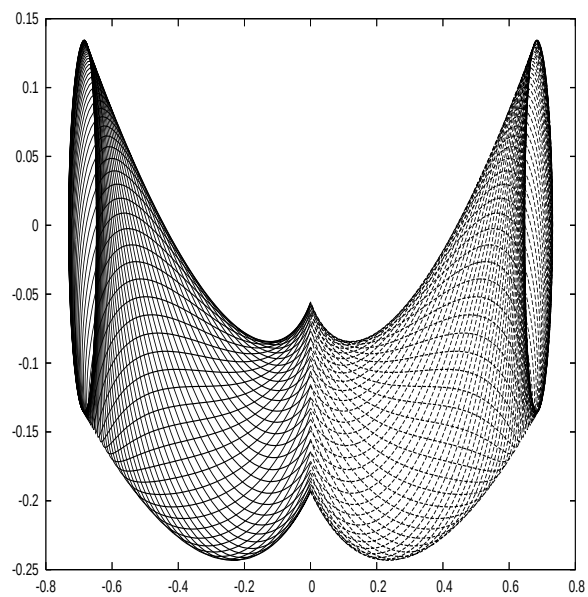
Low thrust transfer to a halo orbit about L_1 Sun-Earth

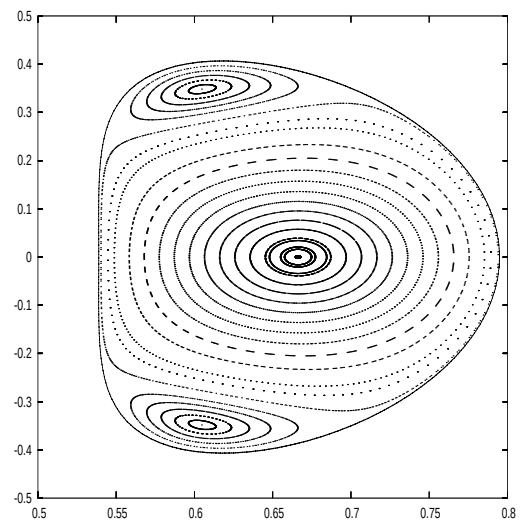
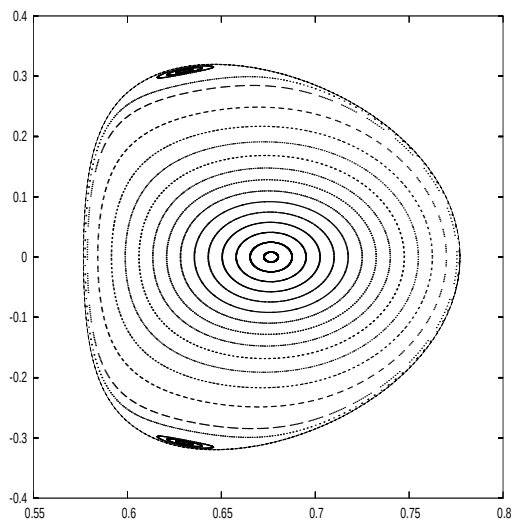
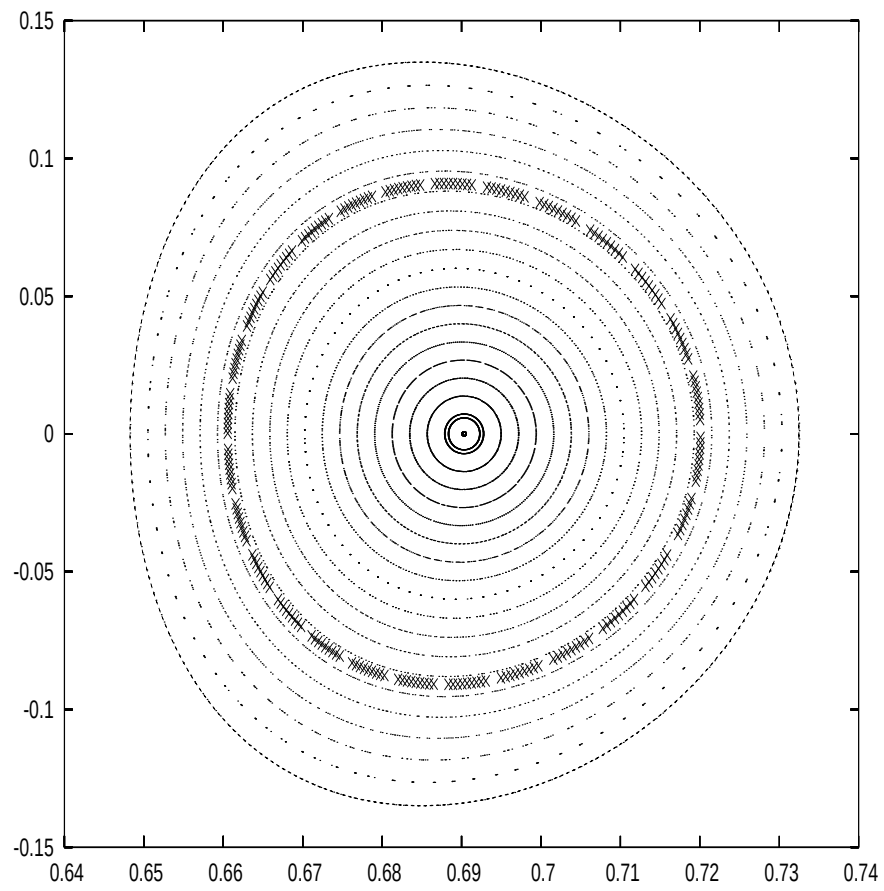


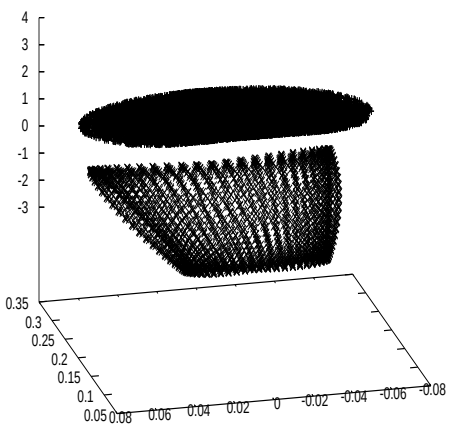
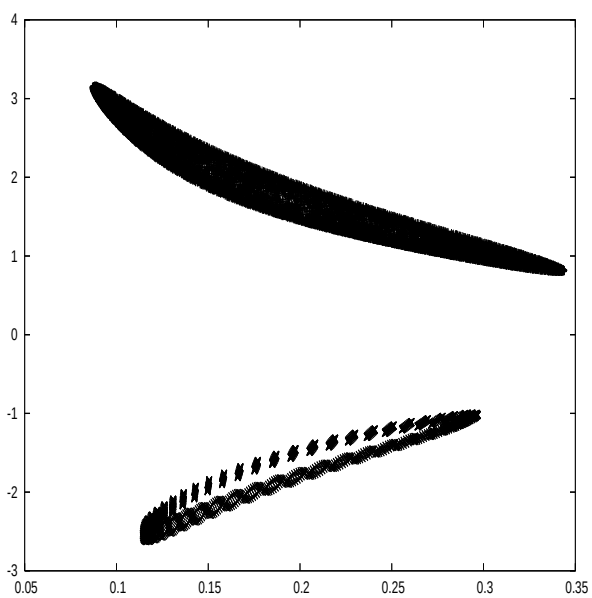
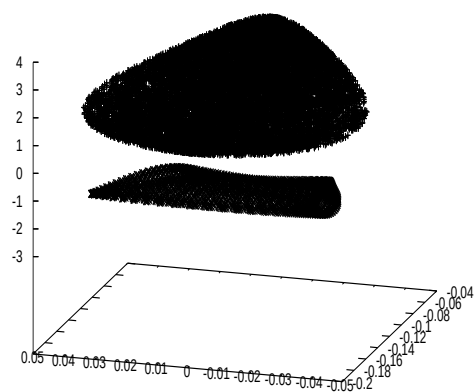
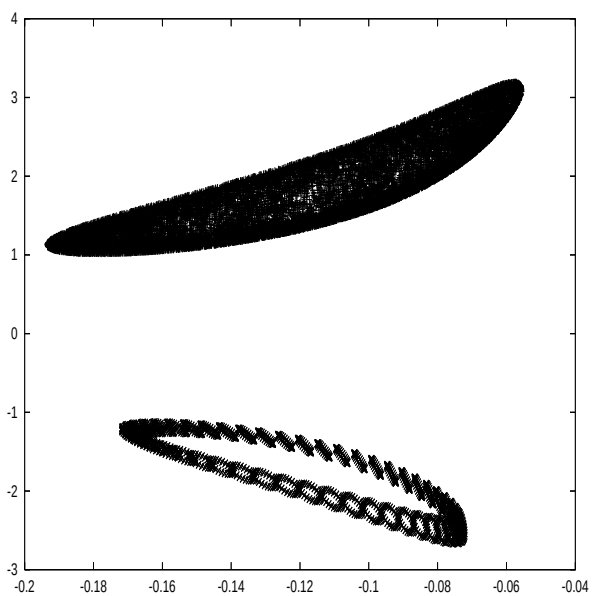
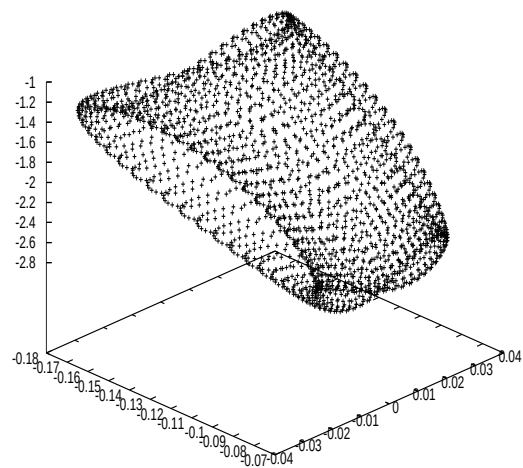
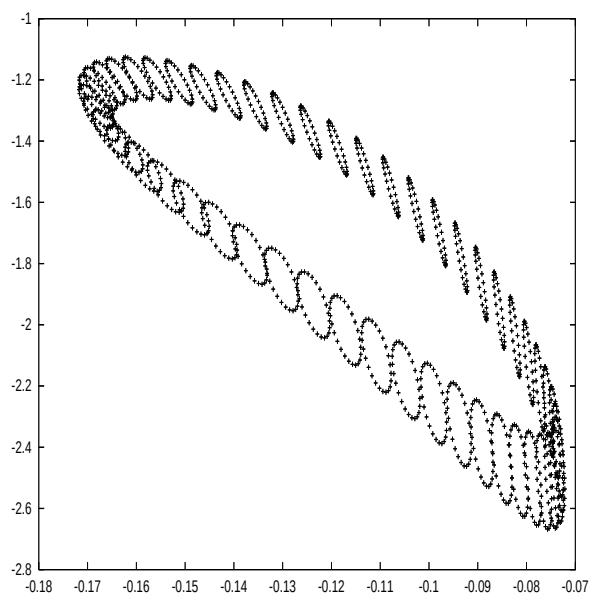
Low thrust transfer to a Lissajous orbit about L_1 Sun-Earth

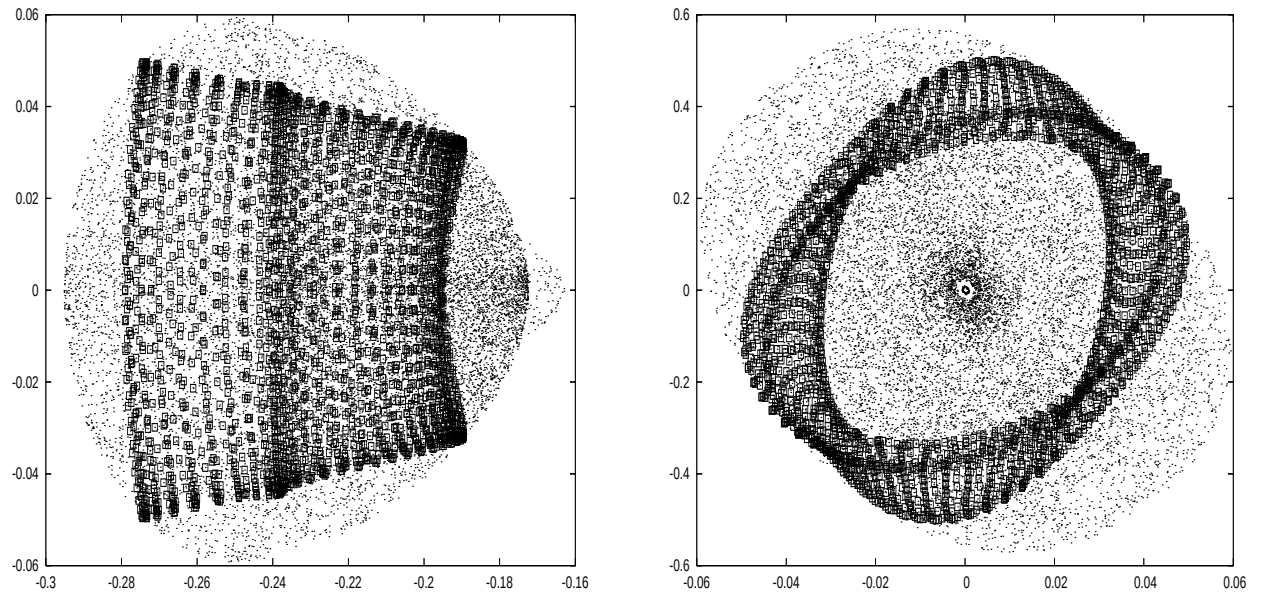


Low thrust transfer to a halo orbit about L_2 Earth-Moon

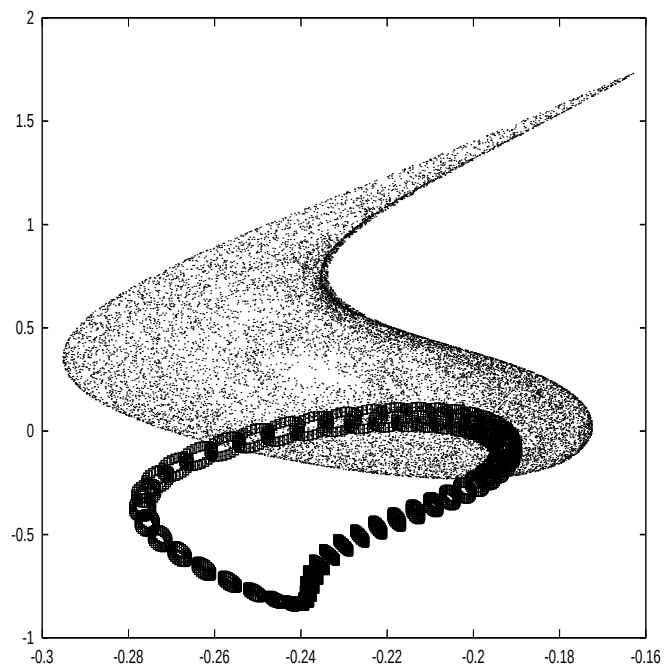








Third encounters, yz and $z\dot{z}$ projections.



Third encounters, $y\dot{y}$ projection.

