

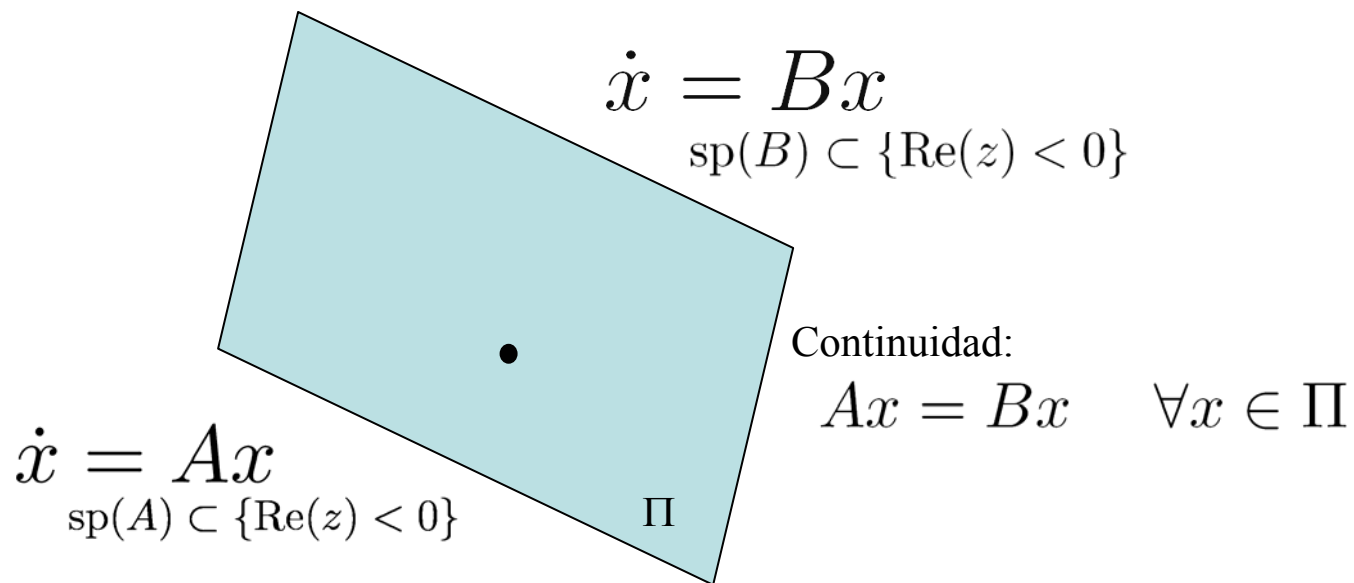
El acoplamiento continuo de sistemas lineales estables puede ser inestable

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Introducción



Con discontinuidad, el acoplamiento de sistemas lineales estables puede ser inestable. (M. Branicky, IEEE Trans. Automat. Contr., vol. 43, No. 4 (1998), pp. 475-482).

Introducción

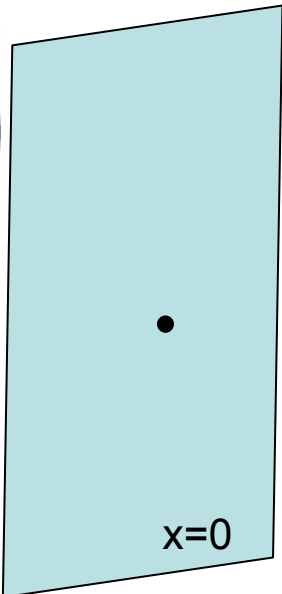
El acoplamiento continuo de sistemas lineales estables planos es siempre estable. (E. Freire, E. Ponce, F. Rodrigo, F. Torres,, I. J. of Bif. and Chaos Vol. 8, No 11 (1998), pp. 2073-2097.)

Estudiaremos el acoplamiento continuo de sistemas tridimensionales observables. En caso contrario, podemos reducir la dimensión.

Un cambio lineal de variables transforma los sistemas observables en forma canónica de Lienard.

Forma canónica de Lienard para sistemas tridimensionales

$$\dot{v} = A^{\pm} v$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} t^- & -1 & 0 \\ m^- & 0 & -1 \\ d^- & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$


(6 parámetros)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} t^+ & -1 & 0 \\ m^+ & 0 & -1 \\ d^+ & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

En nuestro caso:

Autovalores de la matriz A^-

$$\lambda^-, \alpha^- \pm i\beta^-$$

$$\lambda^- < 0, \alpha^- < 0$$

(6 parámetros)

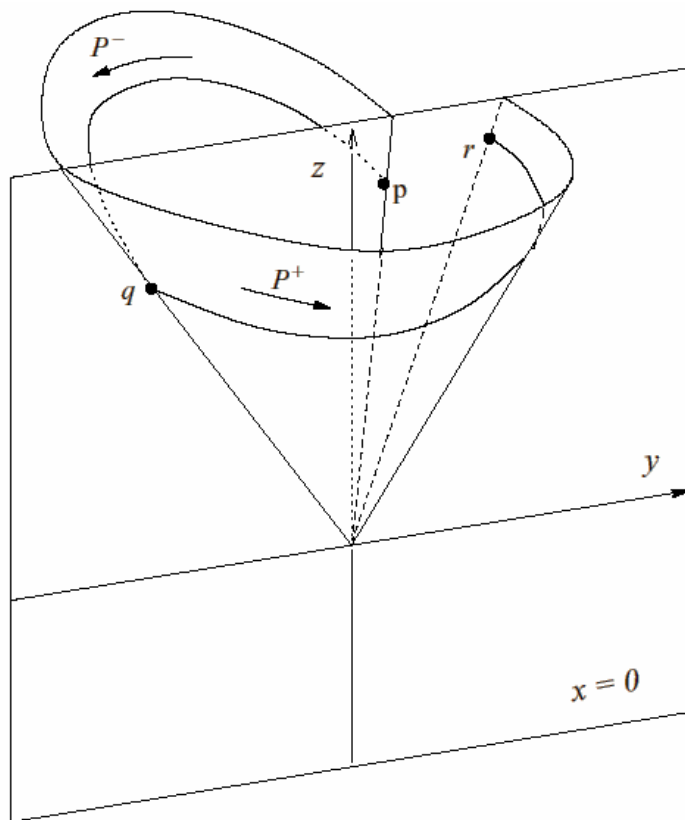
Autovalores de la matriz A^+

$$\lambda^+, \alpha^+ \pm i\beta^+$$

$$\lambda^+ < 0, \alpha^+ < 0$$

- Consecuencia geométrica de homogeneidad. Conos invariantes.
- No existencia de conos invariantes implica estabilidad.
- Comportamiento en presencia de conos invariantes. Interpretación geométrica.
- Semiaplicaciones de Poincaré. Existencia de Conos Invariantes.
- Casos de estabilidad.
- Casos de inestabilidad. Esquema de la prueba.

Homogeneidad. Conos Invariantes



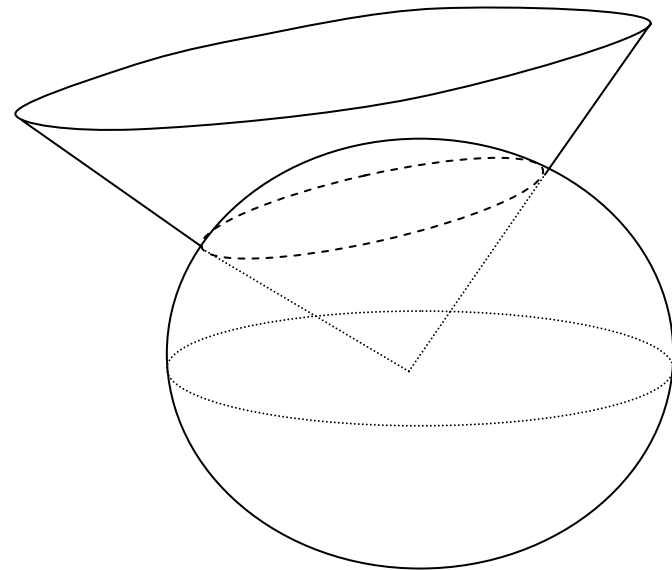
- El flujo del sistema transforma semirrectas contenidas en $x=0$ que pasan por el origen en semirrectas del mismo tipo.
- Una *semirrecta invariante* proporciona un **cono invariante** “bizonal”.

Sistema sobre la esfera unidad

$$\dot{v} = A^{\pm} v = \begin{cases} A^{-} v, & \text{si } x \leq 0 \\ A^{+} v, & \text{si } x > 0 \end{cases}$$

$$\Downarrow u = \frac{v}{\|v\|} \quad r = \|v\|$$

$$\begin{cases} \dot{u} = A^{\pm} u - \langle A^{\pm} u, u \rangle u, & u \in S^2 \\ \dot{r} = \langle A^{\pm} u, u \rangle r. \end{cases}$$



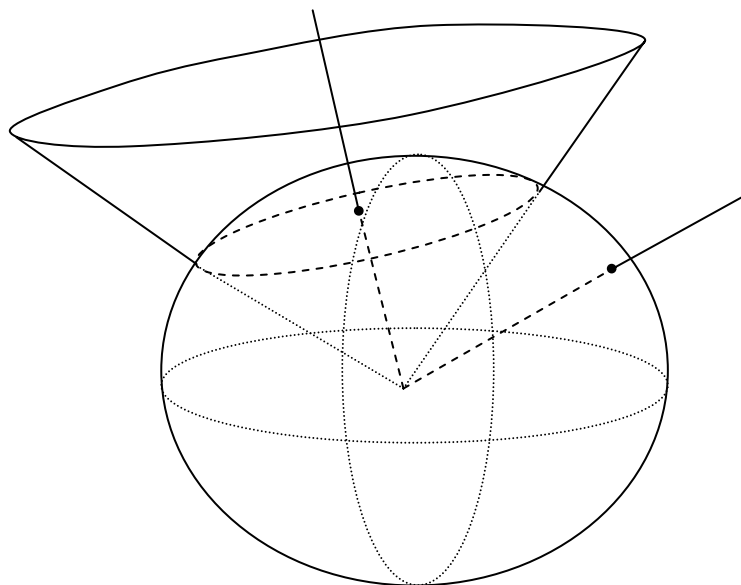
Conos Invariantes $\ni$ Órbitas Periódicas de $\dot{u} = A^{\pm} u - \langle A^{\pm} u, u \rangle u, \quad u \in S^2$

Un primer caso de existencia de conos invariantes

Puntos de equilibrio del sistema en la esfera $\dot{u} = A^\pm u - \langle A^\pm u, u \rangle u, \quad u \in S^2$

**Autovalores de la
linealización***

$$(\alpha^- - \lambda^-) \pm i\beta^-$$



**Autovalores de la
linealización**

$$(\alpha^+ - \lambda^+) \pm i\beta^+$$

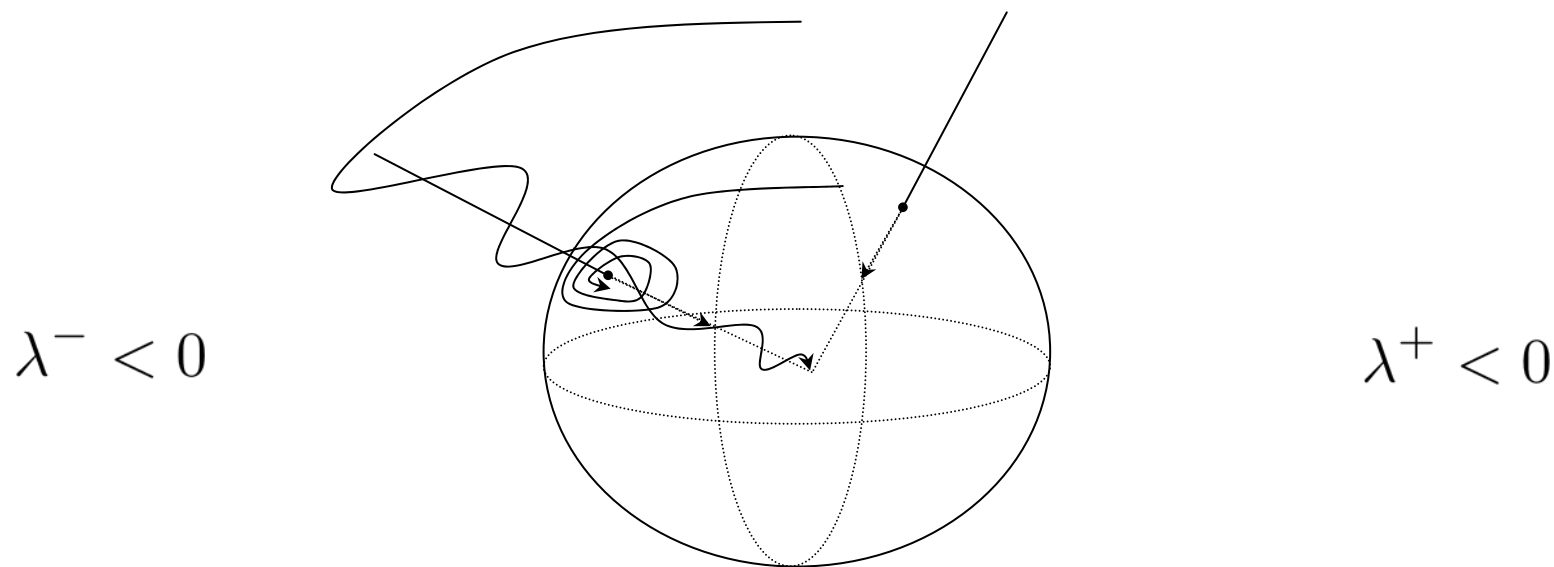
Si $(\alpha^- - \lambda^-) \cdot (\alpha^+ - \lambda^+) > 0 \implies$ Existe un cono invariante bizonal

[*] K.P. Hadeler, Journal Diff. Equat., 95 (1992), pp. 183-202.

V. Carmona, E. Freire, E. Ponce and F. Torres, *Bifurcation of Invariant Cones in Piecewise Linear Homogeneous Systems*, to appear in International Journal of Bifurcation and Chaos.

Un primer caso de estabilidad

Puntos de equilibrio del sistema en la esfera $\dot{u} = A^\pm u - \langle A^\pm u, u \rangle u$, $u \in S^2$

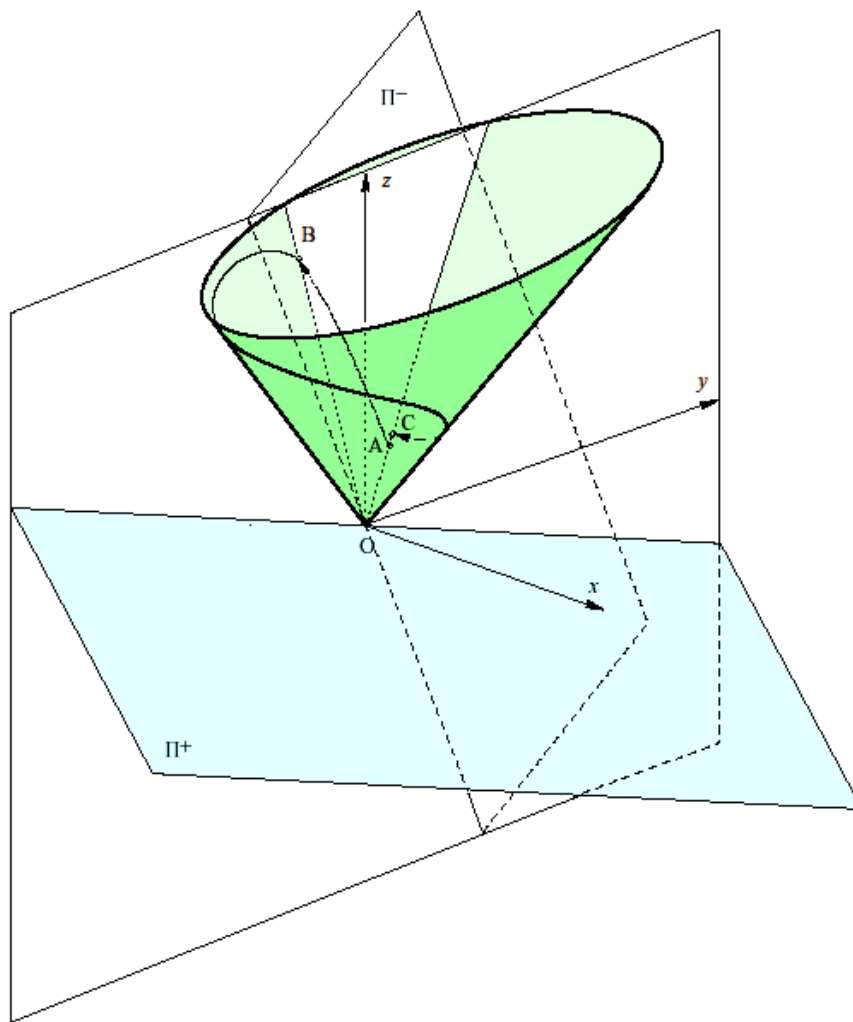


Si no existen conos invariantes bizonales $\implies$ El origen es globalmente asintóticamente estable

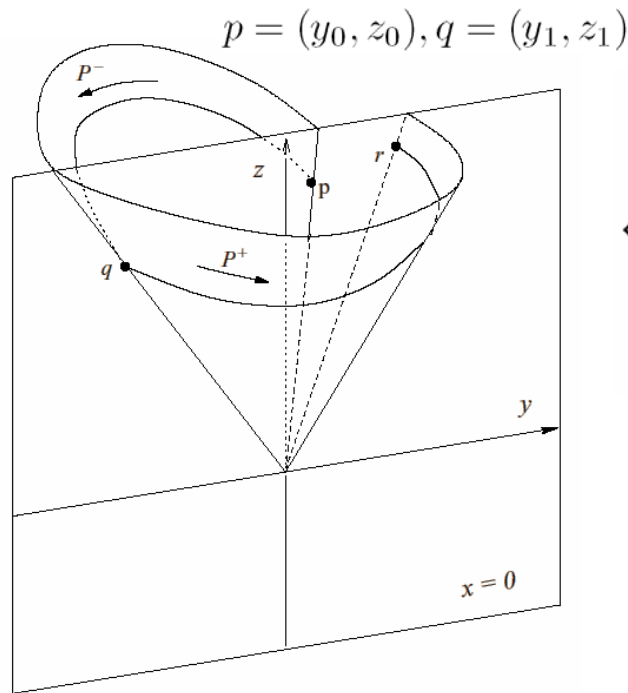
Semiplanos focales

$$(\lambda^-)^2 x - \lambda^- y + z = 0, \quad x \leq 0$$

$$(\lambda^+)^2 x - \lambda^+ y + z = 0, \quad x \geq 0$$



Semiaplicaciones de Poincaré.

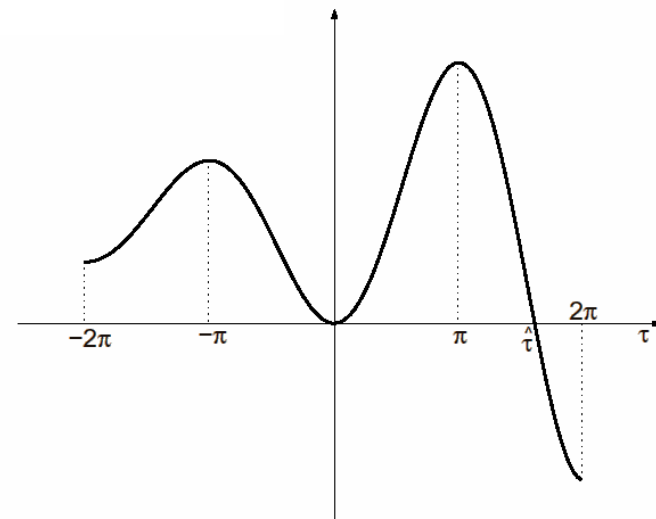


$$\begin{cases} \frac{z_0}{y_0} = \lambda^- + \beta^- \left[(\gamma^-)^2 + 1 \right] \frac{e^{\gamma^- \tau^-} \sin \tau^-}{\varphi_{\gamma^-}(\tau^-)}, \\ \frac{z_1}{y_1} = \lambda^- - \beta^- \left[(\gamma^-)^2 + 1 \right] \frac{e^{-\gamma^- \tau^-} \sin \tau^-}{\varphi_{-\gamma^-}(\tau^-)}, \end{cases}$$

$$\frac{y_1}{y_0} = \frac{\varphi_{-\gamma^-}(\tau^-)}{\varphi_{\gamma^-}(\tau^-)} e^{\left(\gamma^- + \frac{\alpha^-}{\beta^-} \right) \tau^-}$$

endo $\gamma^- = \frac{\alpha^- - \lambda^-}{\beta^-}$

$$\varphi_{\gamma}(\tau) = 1 - e^{\gamma \tau} (\cos \tau - \gamma \sin \tau)$$



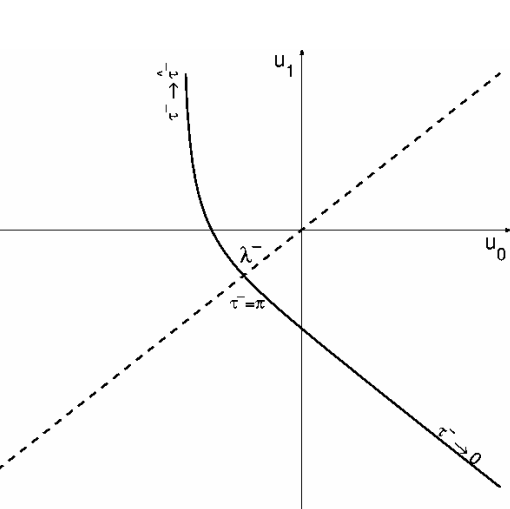
Llibre, A. Teruel. *Existence of Poincaré Maps in Piecewise Linear Differential Systems in $\mathbb{R}^n$.*
Internat. J. Bifur. Chaos Appl. Sci. Engrg.

Semiaplicaciones de Poincaré.

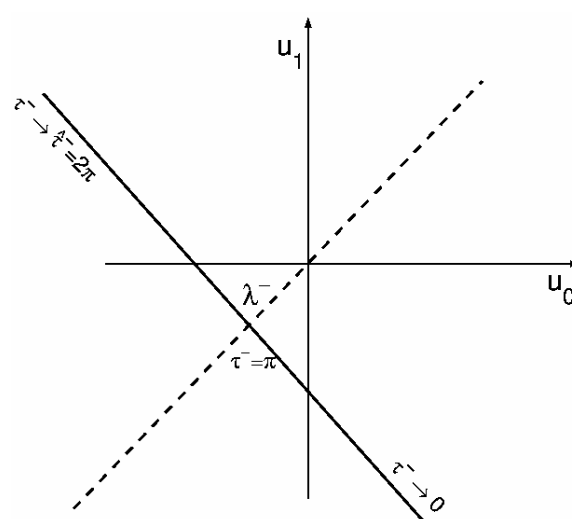
$$u_0(\tau^-) = \frac{z_0}{y_0} = \lambda^- + \beta^- [(\gamma^-)^2 + 1] \frac{e^{\gamma^- \tau^-} \sin(\tau^-)}{\varphi_{\gamma^-}(\tau^-)}$$

$\downarrow S^-$

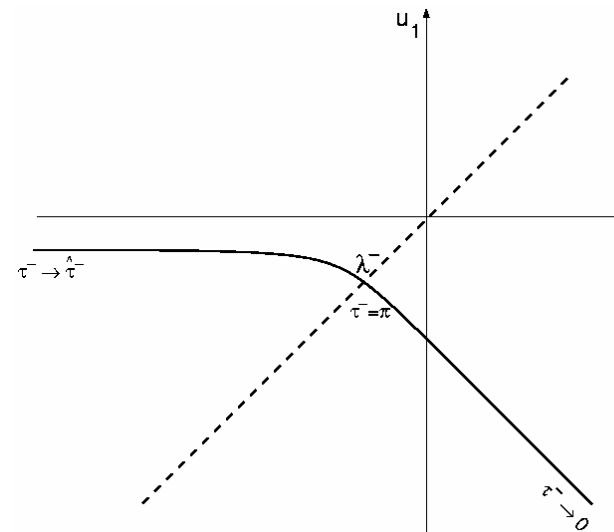
$$u_1(\tau^-) = \frac{z_1}{y_1} = \lambda^- - \beta^- [(\gamma^-)^2 + 1] \frac{e^{-\gamma^- \tau^-} \sin(\tau^-)}{\varphi_{-\gamma^-}(\tau^-)}$$



$\gamma^- < 0$



$\gamma^- = 0$

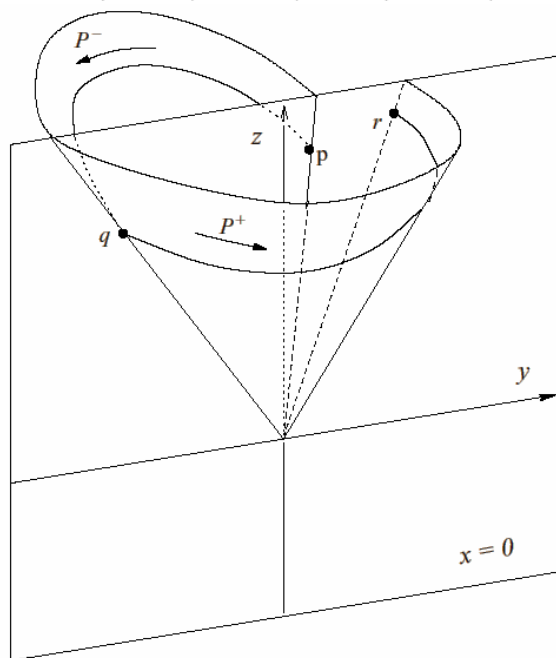


$\gamma^- > 0$

- S^- es decreciente.
- $(S^-)''$ posee signo constante.
- Asíntota oblicua $u_1 = -u_0 + \frac{2}{3} (2\alpha^- + \lambda^-)$

Semiaplicaciones de Poincaré. Existencia de Conos Invariantes

$$p = (y_0, z_0), q = (y_1, z_1), r = (y_2, z_2)$$



$$u_0(\tau^-) = \frac{z_0}{y_0} = \lambda^- + \beta^- [(\gamma^-)^2 + 1] \frac{e^{\gamma^- \tau^-} \sin(\tau^-)}{\varphi_{\gamma^-}(\tau^-)}$$

$$\downarrow S^-$$

$$u_1(\tau^-) = \frac{z_1}{y_1} = \lambda^- - \beta^- [(\gamma^-)^2 + 1] \frac{e^{-\gamma^- \tau^-} \sin(\tau^-)}{\varphi_{-\gamma^-}(\tau^-)}$$

$$u_1(\tau^+) = \frac{z_1}{y_1} = \lambda^+ + \beta^+ [(\gamma^+)^2 + 1] \frac{e^{\gamma^+ \tau^+} \sin(\tau^+)}{\varphi_{\gamma^+}(\tau^+)}$$

$$\downarrow S^+$$

$$u_2(\tau^+) = \frac{z_2}{y_2} = \lambda^+ - \beta^+ [(\gamma^+)^2 + 1] \frac{e^{-\gamma^+ \tau^+} \sin(\tau^+)}{\varphi_{-\gamma^+}(\tau^+)}$$

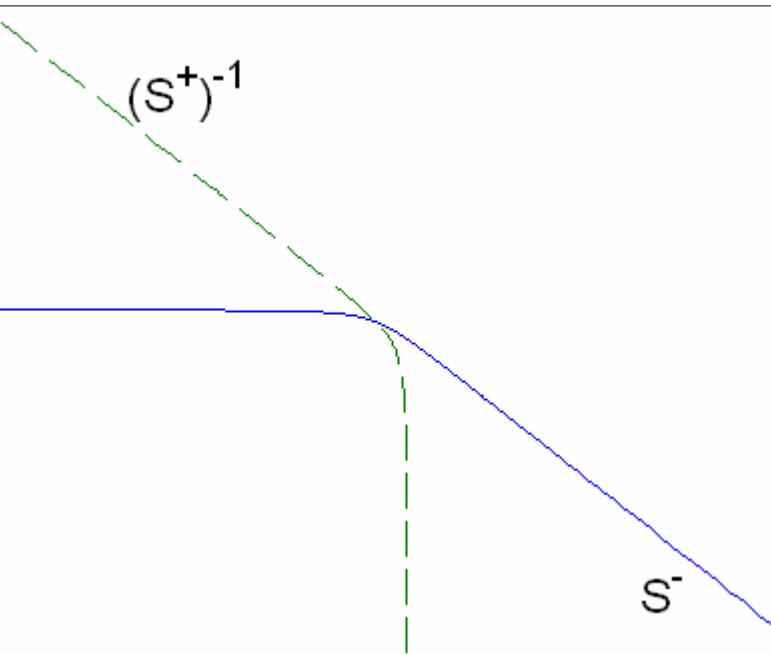
Existe cono invariante bizonal $\iff \exists \bar{u}_0 \in \mathbb{R} : (S^+ \circ S^-)(\bar{u}_0) = \bar{u}_0$

$S^-(\bar{u}_0) = (S^+)^{-1}(\bar{u}_0)$ (i.e., S^- y $(S^+)^{-1}$ se cortan)

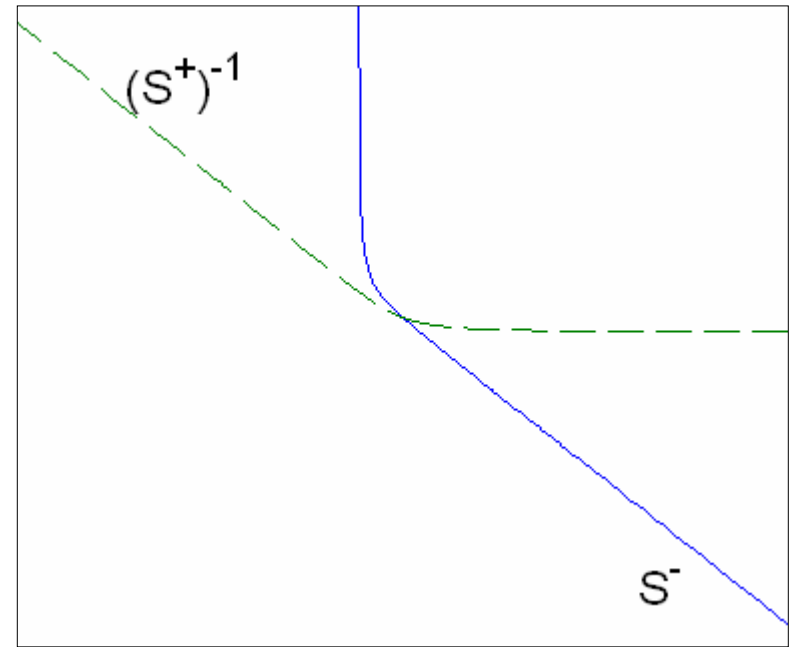
Semiaplicaciones de Poincaré. Existencia de Conos Invariantes

Un único cono invariante (bizonal e hiperbólico)

$$\underline{\gamma^+ \cdot \gamma^- > 0}$$



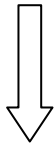
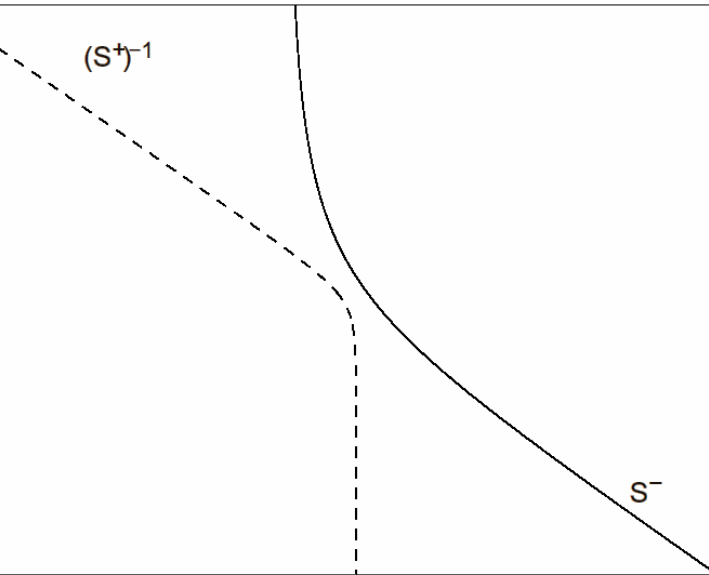
$\gamma^+ > 0, \gamma^- > 0$. Cono atractivo



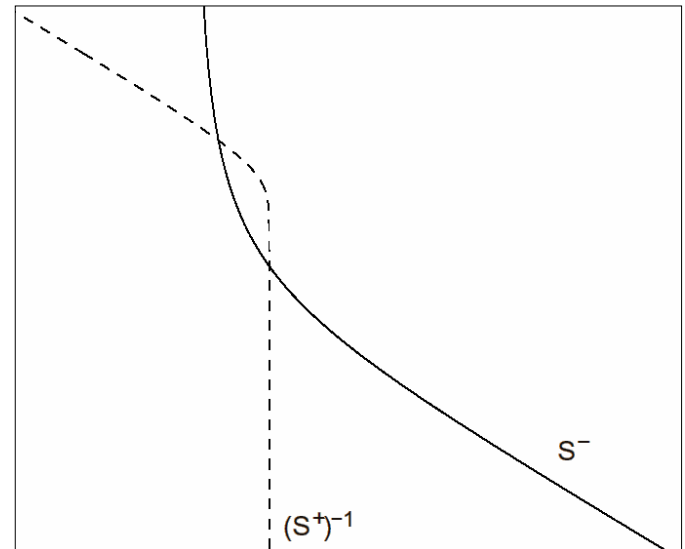
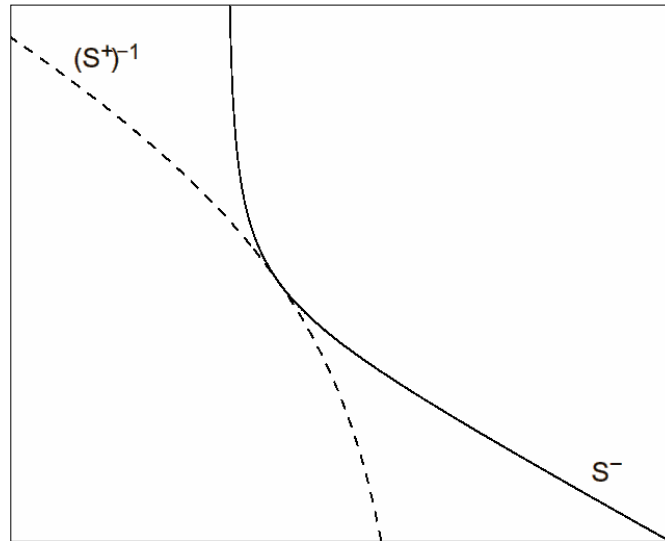
$\gamma^+ < 0, \gamma^- < 0$. Cono repulsivo

Semiaplicaciones de Poincaré. Existencia de Conos Invariantes

$$\gamma^+ \cdot \gamma^- < 0$$

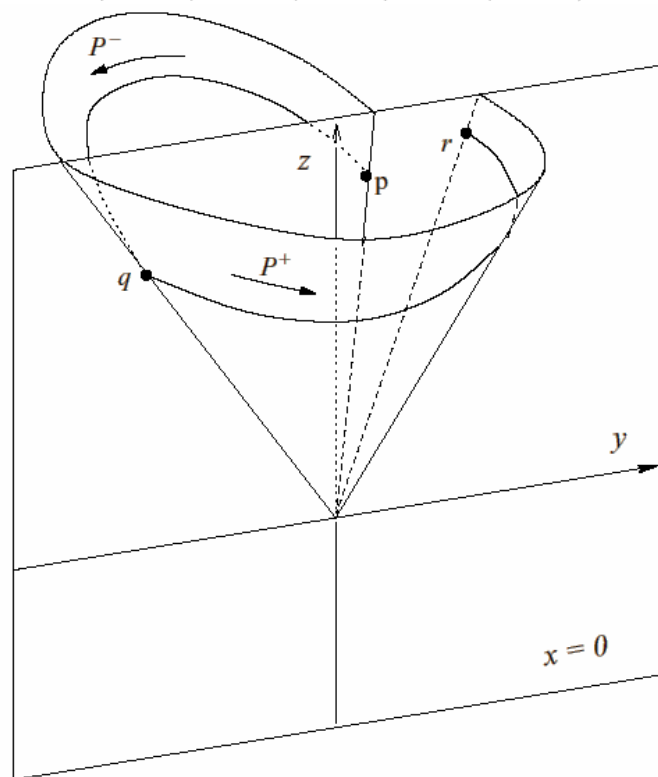


El origen es globalmente
asintóticamente estable



Estabilidad

$$p = (y_0, z_0), q = (y_1, z_1), r = (y_2, z_2)$$



$$\begin{cases} \frac{z_1}{y_1} = S^- \left(\frac{z_0}{y_0} \right) \\ \frac{y_1}{y_0} = \frac{\varphi_{-\gamma^-}(\tau^-)}{\varphi_{\gamma^-}(\tau^-)} e^{\left(\gamma^- + \frac{\alpha^-}{\beta^-} \right) \tau^-} \end{cases}$$

$$\begin{cases} \frac{z_2}{y_2} = S^+ \left(\frac{z_1}{y_1} \right) \\ \frac{y_2}{y_1} = \frac{\varphi_{-\gamma^+}(\tau^+)}{\varphi_{\gamma^+}(\tau^+)} e^{\left(\gamma^+ + \frac{\alpha^+}{\beta^+} \right) \tau^+} \end{cases}$$

Dinámica sobre el cono

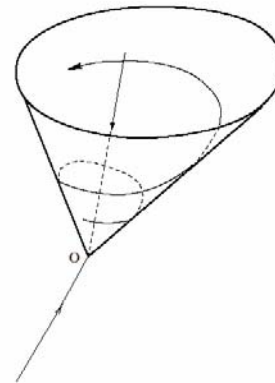
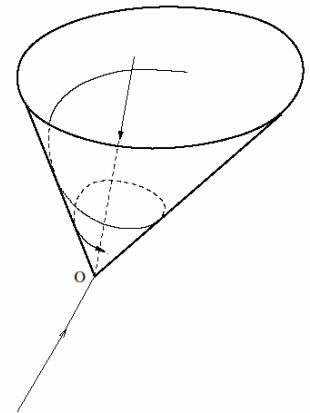
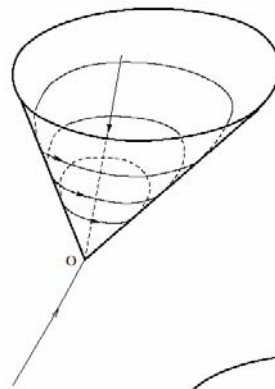
$$\frac{y_2}{y_0} = \frac{y_2}{y_1} \cdot \frac{y_1}{y_0} = \frac{\varphi_{-\gamma^+}(\tau^+)}{\varphi_{\gamma^+}(\tau^+)} e^{\left(\gamma^+ + \frac{\alpha^+}{\beta^+} \right) \tau^+} \cdot \frac{\varphi_{-\gamma^-}(\tau^-)}{\varphi_{\gamma^-}(\tau^-)} e^{\left(\gamma^- + \frac{\alpha^-}{\beta^-} \right) \tau^-}$$

Dinámica sobre el cono

$\frac{y_2}{y_0} < 1 \implies$ **Origen asintóticamente estable**

$\frac{y_2}{y_0} = 1 \implies$ **Centro**

$\frac{y_2}{y_0} > 1 \implies$ **Origen inestable**

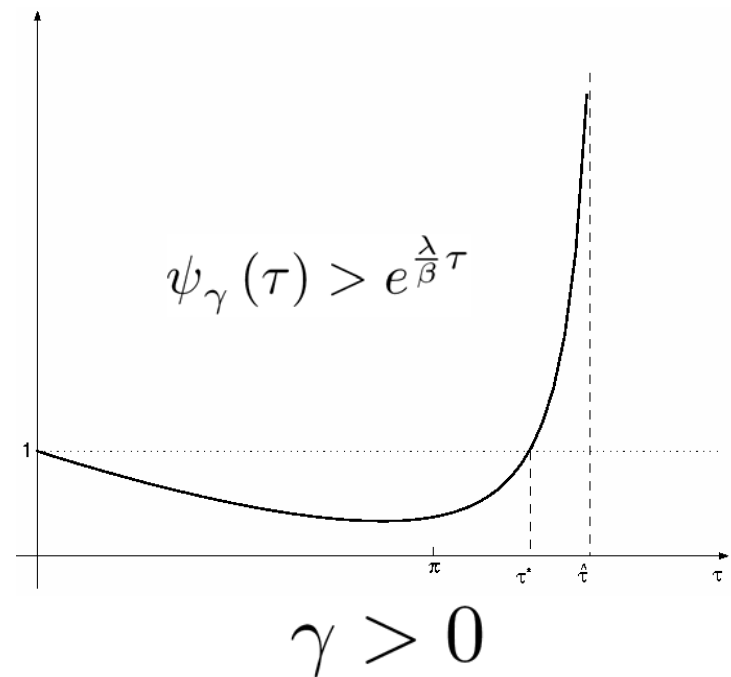
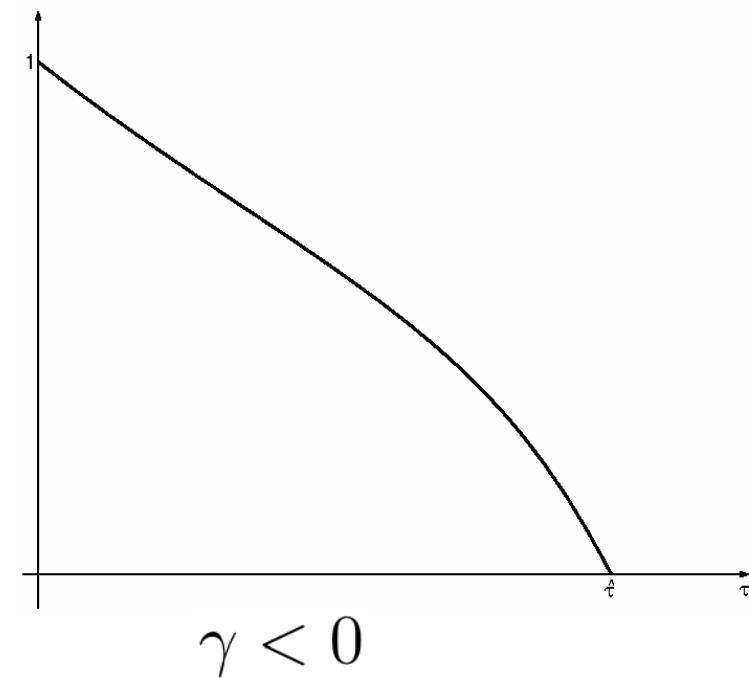


Dinámica sobre el cono

$$\frac{y_2}{y_0} = \frac{y_2}{y_1} \cdot \frac{y_1}{y_0} = \frac{\varphi_{-\gamma^+}(\tau^+)}{\varphi_{\gamma^+}(\tau^+)} e^{\left(\gamma^+ + \frac{\alpha^+}{\beta^+}\right)\tau^+} \cdot \frac{\varphi_{-\gamma^-}(\tau^-)}{\varphi_{\gamma^-}(\tau^-)} e^{\left(\gamma^- + \frac{\alpha^-}{\beta^-}\right)\tau^-}$$

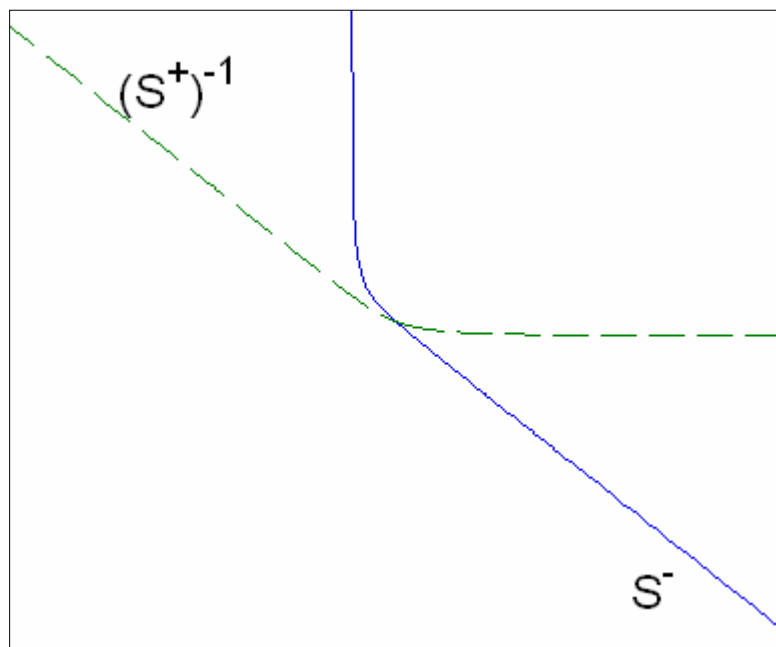
$$\psi_{\gamma}(\tau) = \frac{\varphi_{-\gamma}(\tau)}{\varphi_{\gamma}(\tau)} e^{(\gamma + \frac{\alpha}{\beta})\tau}$$

$$\alpha < 0$$

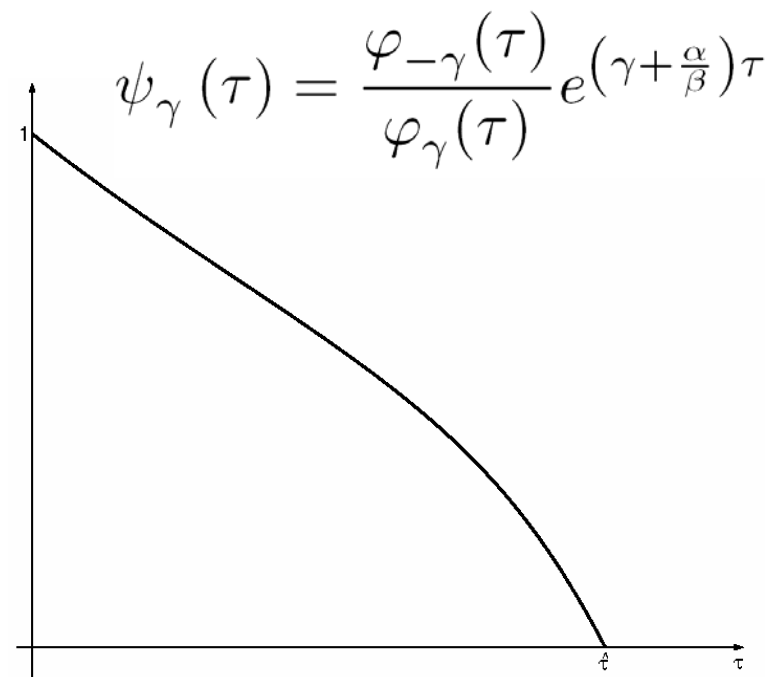


Casos de Estabilidad Asintótica

$$1. \gamma^+ < 0, \gamma^- < 0$$



Cono repulsivo

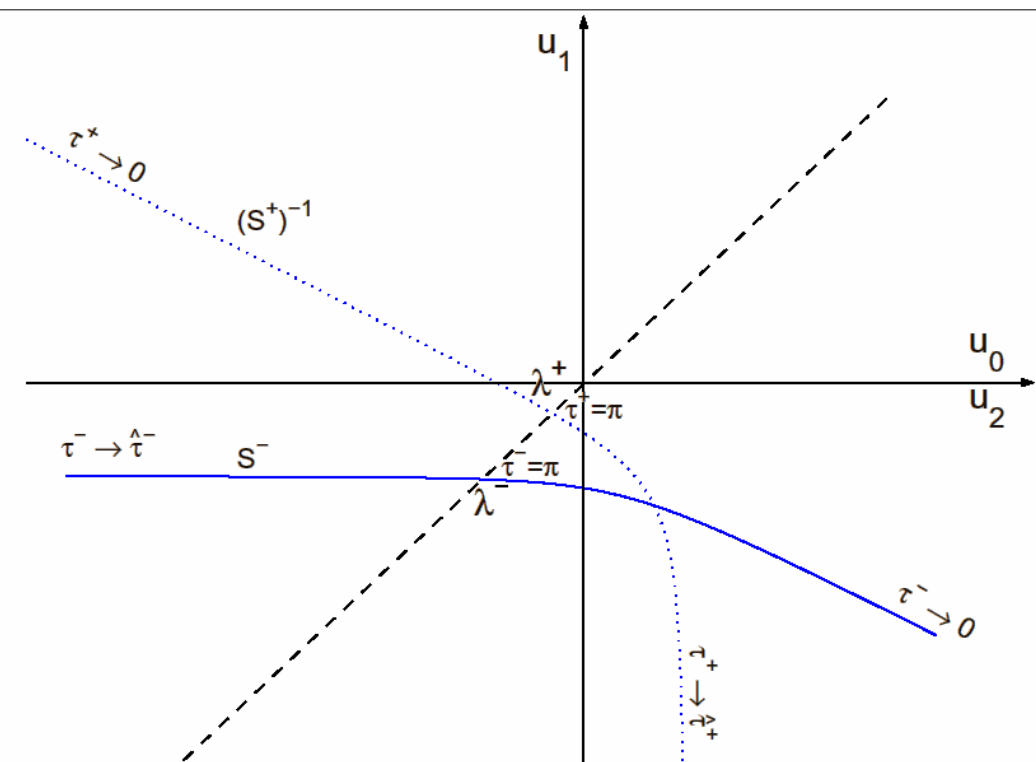


$$\frac{y_2}{y_0} = \frac{y_2}{y_1} \cdot \frac{y_1}{y_0} = \psi_{\gamma^+}(\tau^+) \cdot \psi_{\gamma^-}(\tau^-) < 1$$

$$2. \lambda^+ = \lambda^-$$

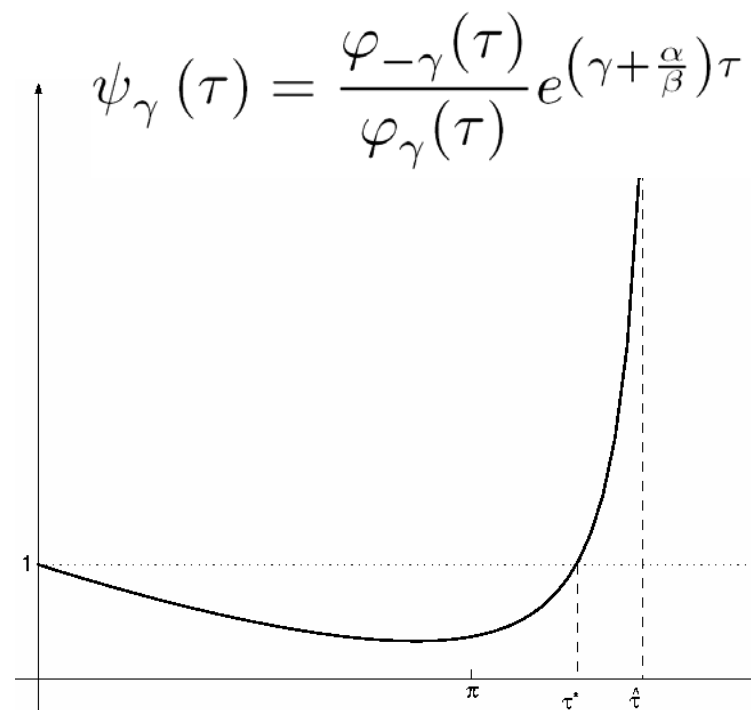
Esquema de la prueba de inestabilidad

- Fijamos $\alpha^+ > \lambda^+, \beta^+$ y α^-
- Tomamos $\alpha^- > \lambda^- = -\beta^- = \lambda^+ - k, \quad k > 0$



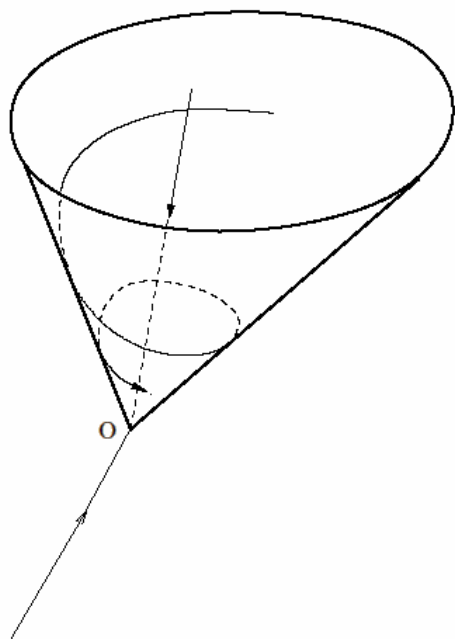
Cono atractivo

$$\frac{y_2}{y_0} = \frac{y_2}{y_1} \cdot \frac{y_1}{y_0} = \underbrace{\psi_{\gamma^+}(\tau^+)}_{\downarrow +\infty} \cdot \underbrace{\psi_{\gamma^-}(\tau^-)}_{> e^{-\pi}} > 1$$

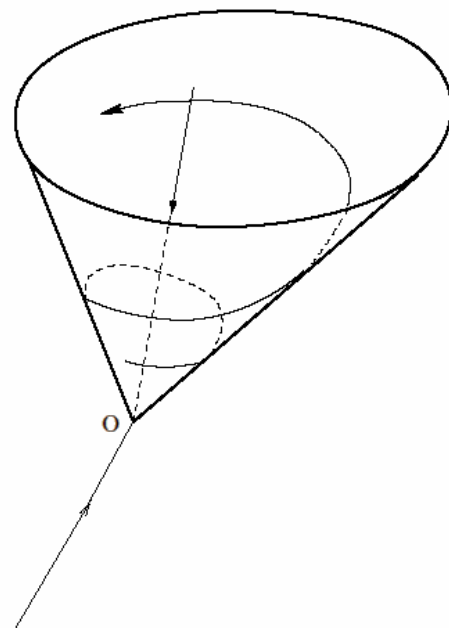
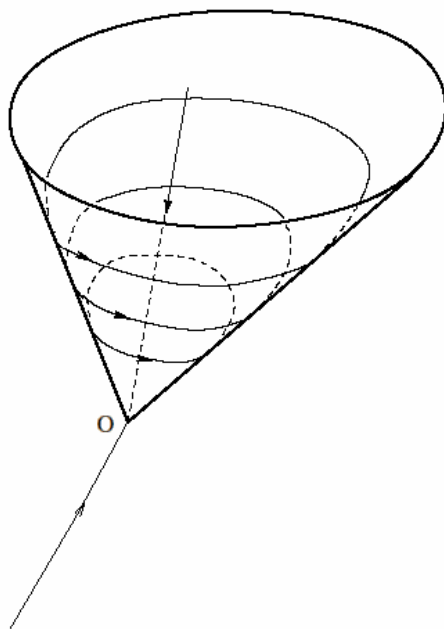


Dinámica sobre el cono

$$\gamma^+ > 0, \gamma^- > 0$$



$$\lambda^+ - \lambda^- \approx 0$$



$$\lambda^+ - \lambda^- \gg 1$$

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