Métodos Multirrevolución

Ddays-2010

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Introduction

IVP problem:

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- A Runge-Kutta scheme advances by some convex combinations of the vector field

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i g(Y_i)$$

$$Y_i = y_0 + h \sum_{j=1}^{i-1} a_{ij} g(Y_j), \quad i = 1, \ldots, s$$

Introduction

- For differential problems for which the solution is almost periodic, with (almost) period T, multirevolution methods try to advance the solution N periods at once, $y_0 \mapsto y_N \simeq y(t_0 + NT)$, making use of the information about the period T.

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- They were introduced by Taratynova in 1960
- They are closely related to averaging methods (Krylov-Bogolyubov 1935, Perko 1968, Miranker-Wahba 1972)

Introduction

- In a more general framework, Multirevolution methods provide a numerical approximation of

$$y_N \simeq \varphi^N(y_0) = \varphi \circ \ldots \circ \varphi(y_0)$$

for a near identity map φ, that is, assuming that $\varphi(y) - y$ is small for all y in a neighbourhood of y_0

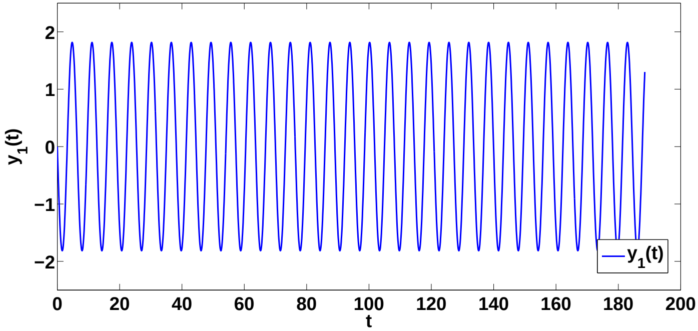
In the case of a differential system with an almost periodic solution $y(t) \simeq y(t+T)$, the near identity map could be the exact flow after the T-period

$$\varphi(y_0) = \varphi_T(y_0) = y(t_0 + T; t_0, y_0)$$

Example:

$$y'' + y = \mu y^3, \quad \mu = 0.01$$

$$y(0) = 0, \quad y'(0) = -1.8.$$



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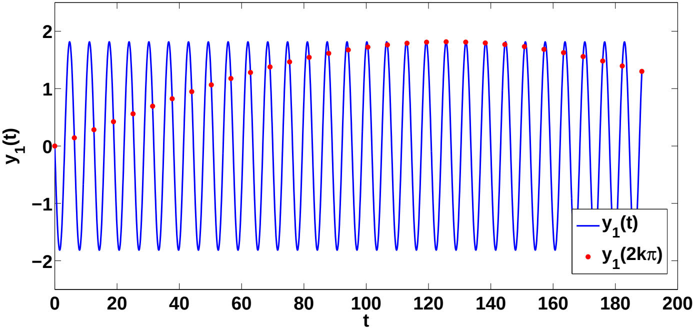
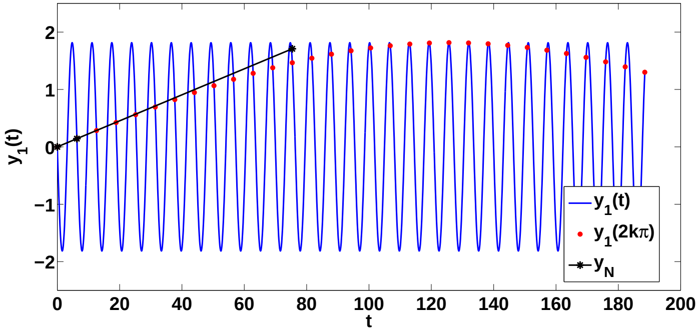


The idea is to follow the points $y(kT)$, with $T = 2\pi$ the period of the simpler problem $y'' + y = 0$



Euler's MRRK method:

$$y_N = y_0 + N\,((\varphi(y_0) - y_0))$$
$$\varphi(y_0) \simeq y(t_0 + T)$$

$T = 2\pi$

An s–stage MRRK method approximates

$$y_N \simeq \varphi^N(y_0) = \varphi \circ \cdots \circ \varphi(y_0)$$

by

$$y_N = y_0 + N \sum_{i=1}^{s} b_i(\varphi(Y_i) - Y_i),$$
$$Y_i = y_0 + N \sum_{j=1}^{s} a_{ij}(\varphi(Y_j) - Y_j), \quad (1 \le i \le s)$$

Specified by the Butcher array

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}, \qquad \mathbf{c} = (c_i),\ \mathbf{A} = (a_{ij}),\ \mathbf{b} = (b_i) \in \mathbb{R}^s$$

$$y_N = \psi(N, y_0)$$

MRRK methods: order

How must the coefficients a_{ij}, b_i be chosen?

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Defining $f(y) \equiv (\varphi(y) - y)/\varepsilon = \mathcal{O}(1)$

$$\begin{aligned} y_N &= y_0 + \varepsilon N \sum_{i=1}^{s} b_i f(Y_i), \\ Y_i &= y_0 + \varepsilon N \sum_{j=1}^{s} a_{ij} f(Y_j) \quad (1 \le i \le s) \end{aligned}$$

We can apply the B-series theory (assuming $h = \varepsilon N$ small).

MRRK methods: order

$$y_N = \psi(N, y_0) = y_0 + \sum_{\rho(\tau)\geq 1} h^{\rho(\tau)} \frac{\alpha(\tau)\gamma(\tau)}{\rho(\tau)!} \mathrm{b}^T \phi_{\mathbf{A}}(\tau)\ F(\tau)(y_0).$$

- $\alpha(\tau)$, $\gamma(\tau)$, $\rho(\tau)$, the same as for standard RK
- $F(\tau)$ the elementary differentials associated to function $f(y)$
- $\phi_{\mathbf{A}}(\tau)$ the vectors associated to the matrix $\mathbf{A}$

What is the B-serie of the true solution $\varphi^N(y_0)$?

$$
\begin{array}{ll}
Y_1 = y_0 & \\
Y_2 = y_0 + \varepsilon f(Y_1) = y_0 + \varphi(Y_1) - Y_1 & = \varphi(Y_1) \\
Y_3 = y_0 + \varepsilon(f(Y_1) + f(Y_2)) = y_0 + \varphi(Y_1) - Y_1 + \varphi(Y_2) - Y_2 & = \varphi(Y_2) \\
\vdots & \\
Y_{N-1} = y_0 + \varepsilon \sum_{j=1}^{N-2} f(Y_j) & = \varphi(Y_{N-2}) \\
y_N = y_0 + \varepsilon(f(Y_1) + \ldots + f(Y_{N-1})) = \varphi(Y_{N-1}) = \varphi^N(y_0) &
\end{array}
$$

$\varphi^N(y_0)$ is the solution of an N–stage MRRK method defined by the Butcher array (with step size ε)

$$
\begin{array}{c|c} \mathbf{c} & \Lambda \\ \hline & \beta^T \end{array}, \qquad \Lambda = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 1 & 1 & 0 & & & \\ \vdots & \vdots & \ddots & & & \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}, \; \beta = (1, \ldots, 1)^T \in \mathbb{R}^N
$$

MRRK methods: order

B-serie for the true solution

$$\varphi^N(y_0) = y_0 + \sum_{\rho(\tau)\geq 1} \varepsilon^{\rho(\tau)} \frac{\alpha(\tau)\gamma(\tau)}{\rho(\tau)!} \beta^T \, \phi_\Lambda(\tau) \, F(\tau)(y_0)$$

$$= y_0 + \sum_{\rho(\tau)\geq 1} h^{\rho(\tau)} \frac{\alpha(\tau)\gamma(\tau)}{\rho(\tau)!} \, \frac{\beta^T \, \phi_\Lambda(\tau)}{N^{\rho(\tau)}} \, F(\tau)(y_0)$$

$h = \varepsilon N$.

MRRK methods: order

Error of the MRRK

$$\varphi^N(y_0) - y_N = \sum_{\rho(\tau)\geq 1} h^{\rho(\tau)} \frac{\alpha(\tau)\gamma(\tau)}{\rho(\tau)!} \left(\frac{\beta^T \phi_\Lambda(\tau)}{N^{\rho(\tau)}} - \mathrm{b}^T \phi_\mathbf{A}(\tau) \right) F(\tau)(y_0)$$

Order p

$$\mathrm{b}^T \phi_\mathbf{A}(\tau) = \frac{\beta^T \phi_\Lambda(\tau)}{N^{\rho(\tau)}}, \quad \forall \rho(\tau) \leq p$$

$\mathbf{A}$, and b will depend on N

MRRK methods: order

Order conditions:

$$
\begin{aligned}
\mathbf{b}^T\mathbf{e} &= 1,\\
\mathbf{b}^T\mathbf{c} &= N_2/N^2,\\
\mathbf{b}^T\mathbf{c}^2 &= (2N_3+N_2)/N^3,\\
\mathbf{b}^T\,\mathbf{A}\,\mathbf{c} &= N_3/N^3,\\
\mathbf{b}^T\,\mathbf{c}^3 &= (6N_4+6N_3+N_2)/N^4,\\
\mathbf{b}^T\,(\mathbf{c}\cdot\mathbf{A}\,\mathbf{c}) &= (3N_4+2N_3)/N^4,\\
\mathbf{b}^T\,\mathbf{A}\,\mathbf{c}^2 &= (2N_4+N_3)/N^4,\\
\mathbf{b}^T\,\mathbf{A}^2\,\mathbf{c} &= N_4/N^4,\\
&\vdots
\end{aligned}
$$

$N_j = \binom{N}{j}$

Explicit MRRK methods

Simplifying conditions (for explicit methods)

$$\mathbf{b}^T\,\mathbf{A} = \frac{(N-1)}{N}\,\mathbf{b}^T - (\mathbf{b}\cdot\mathbf{c})^T,$$

$$\mathbf{A}\,\mathbf{c} = \frac{1}{2}\,\mathbf{c}^2 - \frac{1}{2N}\,\mathbf{c} - \left(\frac{c_2^2}{2} - \frac{c_2}{2N}\right)\,\mathbf{e}_2,$$

$$\mathbf{A}\mathbf{c}^2 = \frac{1}{3}\,\mathbf{c}^2 - \frac{c_2^3}{3}\,\mathbf{e}_2 - \frac{1}{2N}\,\mathbf{c}^2 + \frac{1}{6N^2}\,\mathbf{c},$$

$$b_2 = 0.$$

Explicit MRRK methods

Method:

$$c_3 = 3N/(10(N-1)),$$

$$c_4 = \frac{4N^2 - 15N + 1}{5N(N-3)},$$

$$c_5 = \frac{24N^4 - 271N^3 + 887N^2 - 1077N + 237}{3N(9N^3 - 97N^2 + 307N - 419)},$$

$$c_6 = \frac{N-1}{N},$$

$$\ldots$$

Numerical examples

Example 1:

$$\frac{d}{dt}y(t) = \begin{pmatrix} 0 & 1/\mu & 0.1 & 0 \\ -1/\mu & 0 & 0 & 0.1 \\ 0.05 & 0 & 0 & \mu \\ 0 & 0.05 & -\mu & 0 \end{pmatrix} y(t),$$

$$y(0) = (1,0,1,0)^T$$

$$0 < \mu \ll 1$$

$$t \in [0, 32\pi] = [0, 1600T]$$

Frequencies:

$$w_{1,2} = \frac{1}{\mu} \pm \frac{\mu}{200}, \qquad w_{3,4} = \mu \pm \frac{\mu}{200}, \qquad \mu = 10^{-2}$$

Numerical examples

MMRK:

- We use DOPRI54, variable stepsize
- $T = 2\pi\mu = 0.02\pi$

Source of errors:

- Computation of $\varphi(y)$ (with some error tolerance Tol)
- MRRK: Depends on the value of N ($h = \varepsilon N$)

Numerical examples

MRRK with no error in the computation of $\varphi(y)$:

N	error
16	$5.20e-11$
32	$5.35e-11$
200	$5.51e-9$
400	$2.28e-7$
800	$9.51e-6$
1600	$3.16e-4$

Errors computed at the end point $t_f = 1600T$ if the integration interval.
$||\varphi(y) - y|| \simeq 6.3 \times 10^{-4}$

Numerical examples

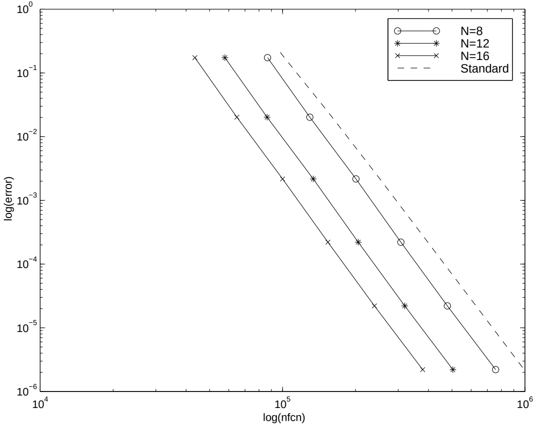
Small value of $N(\geq 6)$, $\varphi(y)$ computed with error tolerance Tol:

Tol	error $N=8$	nfcn $N=8$	error DOPRI5(4)	nfcn DOPRI5(4)
10^{-4}	$1.69e-1$	87600	$2.00e-1$	98329
10^{-5}	$1.72e-2$	130800	$1.92e-2$	157543
10^{-6}	$2.02e-3$	202800	$2.13e-3$	250093
10^{-7}	$2.14e-4$	310800	$2.23e-4$	396463
10^{-8}	$2.21e-5$	483600	$2.26e-5$	628399
10^{-9}	$2.25e-6$	764400	$2.27e-6$	995983

Errors computed at the end point $t_f = 1600T$ if the integration interval.

Numerical examples

$\log(\text{nfcn}) - \log(\text{error})$ plot to compare the relative efficiency of the DOPRI5(4) with MRRK for different values of N (varying Tol and N):



Numerical examples

Example 2:

$$y'' + y = \mu y^3,$$
$$y(0) = 1, \quad y'(0) = 0$$
$$t \in [0, 384\pi] = [0, 192T].$$

- φ is here the one period map of the unperturbed problem, $y'' + y = 0$
- $T = 2\pi$.

$\mu = 5 \times 10^{-3}$:

Tol	error DOPRI5(4)	error $N = 12$	error $N = 16$	error $N = 24$	error $N = 32$
10^{-4}	$6.25e-2$	$5.32e-2$	$5.32e-2$	$5.32e-2$	$5.32e-2$
10^{-5}	$6.98e-3$	$6.30e-3$	$6.30e-3$	$6.29e-3$	$6.20e-3$
10^{-6}	$7.08e-4$	$6.71e-4$	$6.71e-4$	$6.64e-4$	$5.78e-4$
10^{-7}	$6.99e-5$	$6.74e-5$	$6.74e-5$	$6.00e-5$	$2.50e-5$
10^{-8}	$6.88e-6$	$6.73e-6$	$6.72e-6$	$3.00e-7$	$8.65e-5$
10^{-9}	$6.78e-7$	$6.78e-7$	$6.66e-7$	$6.35e-6$	$9.25e-5$

$$\|\varphi(y) - y\| \simeq 0.0118$$

Numerical examples

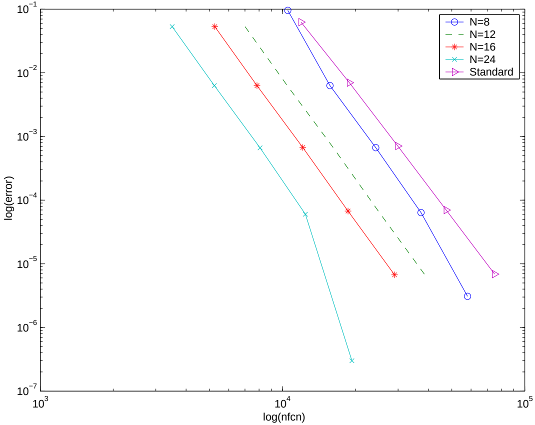
$\mu = 10^{-3}$:

Tol	error DOPRI5(4)	error $N = 16$	error $N = 32$	error $N = 48$	error $N = 96$
10^{-4}	$9.48e-4$	$9.58e-4$	$9.59e-4$	$9.58e-4$	$9.58e-4$
10^{-5}	$8.62e-4$	$7.78e-4$	$7.78e-4$	$7.78e-4$	$7.77e-4$
10^{-6}	$1.31e-4$	$1.22e-4$	$1.23e-4$	$1.22e-4$	$1.21e-4$
10^{-7}	$1.56e-5$	$1.50e-5$	$1.50e-5$	$1.50e-5$	$1.32e-5$
10^{-8}	$1.71e-6$	$1.66e-6$	$1.66e-6$	$1.65e-6$	$7.17e-6$
10^{-9}	$1.79e-7$	$1.77e-7$	$1.77e-7$	$1.64e-7$	$1.56e-6$

$$||\varphi(y) - y|| \simeq 0.0024$$

Numerical examples

$\mathbf{log}(\text{nfcn}) - \mathbf{log}(\text{error})$ plot to compare the relative efficiency of the DOPRI5(4) with MRRK for different values of N (varying Tol and N):



Thank you
for
your attention

Ddays - Calatayud

November 2010